CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations and theorems that will be used in the later chapters.

2.1 Topological Spaces

Definition 2.1.1 Let X be a set. A topology (or topological structure) in X is a family $\Im$ of subset of X that satisfies :

(i) Each union of members of $\Im$ is also a member of $\Im$.

(ii) Each finite intersection of members of $\Im$ is also a member of $\Im$.

(iii) $\varnothing$ and X are members of $\Im$.

Each members of $\Im$ is called *open*. A subset A of X is *closed* in X if X-A is open.

Definition 2.1.2 A couple $(X,\Im)$ consisting of a set X and a topology $\Im$ in X is called a topological space.

Note that in the set $X \neq \varnothing$, let $\Im = P(X)$. Then $(X,\Im)$ is topological space and it is called the discrete topological spaces.

Definition 2.1.3 Let X be topological space and $E \subseteq X$, the closure of E in X is the set $\overline{E} = \cap \left\{ K \subseteq X \,\middle|\, K \text{ is closed and } E \subseteq K \right\}$.

Definition 2.1.4 Let X be a topological space and $E \subseteq X$, the interior of E in X is the set $Int(E) = \cup \left\{ G \subseteq X \mid G \text{ is open and } G \subseteq E \right\}$.

Definition 2.1.5 Let $(X,\Im)$ be a topological space and $Y \subseteq X$, the collection $\Im' = \left\{ G \cap Y \mid G \in \Im \right\}$ is a topology for Y, called the *relative topology* for Y. The fact that a subset of X is being given this topology is signified by referring to it as a *subspace* of X.

Definition 2.1.6 Let $(X,\Im)$ be a topological space. A family $\mathcal{B} \subseteq \Im$ is called a *basis* for $\Im$ if each open set (that is, member of $\Im$) is the union of members of $\mathcal{B}$.

Definition 2.1.7 Let $(X,\Im)$ be a topological space, a *subbasis* for $\Im$ (or a subbasis for X) is a collection $S \subseteq \Im$ such that the collection of all finite intersections of elements from S forms a basis for $\Im$

Theorem 2.1.8 *Let* $\mathcal{B} = \left\{ U_\mu \mid \mu \in I \right\}$ *be any family of subsets of* X *that satisfies the following condition* :

For each $(\mu,\lambda) \in I \times I$ *and each* $x \in U_\mu \cap U_\lambda$, *there exists some* U_α *with* $x \in U_\alpha \subseteq U_\mu \cap U_\lambda$.

Then the family $\Im(\mathcal{B})$ *consisting of* $\varnothing, X$ *and all unions of members of* $\mathcal{B}$, *is a topology for* X ; *that is,* $\mathcal{B} \cup \left\{\varnothing\right\} \cup \left\{X\right\}$ *is a basis for some topology.* $\Im(\mathcal{B})$ *is unique and is the smallest topology containing* $\mathcal{B}$.

Proof. see [4] page 67.

2.2 Continuous Functions

Definition 2.2.1 Let $(X,\Im_X)$ and $(Y,\Im_Y)$ be a topological spaces. A map $f:X\to Y$ is called *continuous* if the inverse image of each open set in Y is open in X.

Theorem 2.2.2 *Let X, Y be topological spaces and $f:X\to Y$ a map. The following statement are equivalent:*

(i) f is continuous.

(ii) The inverse image of each closed set in Y is closed in X.

(iii) If $E\subseteq X$, then $f(\overline{E})\subseteq\overline{f(E)}$.

Proof. see [2] page 59.

Theorem 2.2.3 *Let X, Y and Z be topological spaces. If $f:X\to Y$ and $g:Y\to Z$ are continuous, then $g\circ f:X\to Z$ is continuous.*

Proof. see [8] page 45.

Definition 2.2.4 Let X and Y be topological spaces. A map $f:X\to Y$ is called *open* if image of each open in X is open in Y.

2.3 Product Spaces

Definition 2.3.1 Let $\{X_\alpha \mid \alpha\in I\}$ be a family of sets. The *Cartesian product* of the sets X_α is the set

$$\prod_{\alpha\in I} X_\alpha = \left\{ f:I\to \bigcup_{\alpha\in I} X_\alpha \;\middle|\; f(\alpha)\in X_\alpha \text{ for each } \alpha\in I \right\}.$$

In practice the value of $f \in \prod_{\alpha \in I} X_\alpha$ at α is usualy denoted by f_α rather than $f(\alpha)$ and f_α is refered to as the *αth coordinate* of f and f is denoted by $(x_\alpha)_{\alpha \in I}$. The space X_α is the *αth factor space.*

The map $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ defined by $\pi_\beta((x_\alpha)_{\alpha \in I}) = x_\beta$ is called the *projection map* of $\prod_{\alpha \in I} X_\alpha$ on X_β or more simply the *α th projection map.*

Definition 2.3.2 Let $\{ X_\alpha \mid \alpha \in I \}$ be any family of topological spaces. For each $\alpha \in I$, let $\Im_\alpha$ be the topology for X_α. The cartesian product topology in $\prod_{\alpha \in I} X_\alpha$ is that having $S = \{ \pi_\beta(U_\beta) \mid U_\beta \in \Im_\beta , \beta \in I \}$ as a subbasis.

Theorem 2.3.3 *Let $\{ X_\alpha \mid \alpha \in I \}$ be any family of topological spaces. Then for each fixed $\beta \in I$, the projection $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ is a continuous open onto.*

Proof. see [4] page 101.

Theorem 2.3.4 *Let $\{ Y_\alpha \mid \alpha \in I \}$ be any family of topological spaces, and $f : X \to \prod_{\alpha \in I} Y_\alpha$ a map. Then f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in I$.*

Proof. see [4] page 101.

Theorem 2.3.5 *Let $\{ X_\alpha \mid \alpha \in I \}$ and $\{ Y_\alpha \mid \alpha \in I \}$ be any family of topological spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a map. Define $\prod f_\alpha : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ by $(x_\alpha)_{\alpha \in I} \to (f_\alpha(x_\alpha))_{\alpha \in I}$. Then :*

(i) If each f_α is continuous, so also is $\prod f_\alpha$.

(ii) If each f_α is an open map, and all but at most finitely many are surjective, then $\prod f_\alpha$ is also an open map.

Proof. see [4] page 102.

2.4 Separation Axioms

Definition 2.4.1 A topological space X is a T_o *− space* if whenever x and y are distinct points in X, there is an open set containing one and not the other.

Definition 2.4.2 A topological space X is a T_1 *− space* if whenever x and y are distinct points in X, there is a neighborhood of each not containing the other.

Theorem 2.4.3 *A topological space X is a T_1 − space if and only if each singleton subset of X is closed.*

***Proof*.** see [2] page 80.

Definition 2.4.4 A topological space X is a T_2 *− space* (Hausdorff) if whenever x and y are distinct points of X, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

2.5 Connected Spaces

Definition 2.5.1 A topological space Y is *connected* if it is not the union of two nonempty disjoint open sets.

Theorem 2.5.2 *Let* $(X, \Im)$ *be any topological space. Then the following statement are equivalent.*

(i) X *is connected.*

(ii) X *cannot be expressed as the union of two disjoint nonempty closed subsets.*

(iii) *The only subsets of* X *which are open and closed are* X *and* $\varnothing$.

(iv) *Let* $Y = \{0, 1\}$ *have the discrete topology. Then there is no continuous function from* X *onto* Y.

Proof. see [5] page 185.

Theorem 2.5.3 *Let* X *be connected and* Y *be topological space. If* $f : X \to Y$ *is continuous and onto, then* Y *is connected.*

***Proof*.** see [2] page 100.

2.6 Regular Generalized Closed Sets

Definition 2.6.1 A subset A of a topological space X is said to be *regular open* if $A = Int(\overline{A})$.

Definition 2.6.2 A subset A of a topological space X is said to be *regular closed* if $A = \overline{Int(A)}$.

Definition 2.6.3 A subset A of a topological space X is called *g-closed* in X if $\overline{A} \subseteq G$ whenever $A \subseteq G$ and G is open in X. A subset A is called *g-open* in X if its complement, A^c, is *g-closed*.

Theorem 2.6.4 *Let X be topological space and $A \subseteq X$. Then A is g-open if and only if $F \subseteq Int(A)$ whenever F is closed and $F \subseteq A$.*

***Proof*.** see [6] page 206.

Definition 2.6.5 A subset A of a topological space X is called *r-g-closed* in X if $\overline{A} \subseteq G$ whenever $A \subseteq G$ and G is regular open in X. A subset A is called *r-g-open* in X if its complement, A^c, is r-g-closed.

Theorem 2.6.6 *Let X be a topological space and $A, B \subseteq X$. If A and B are r-g-closed sets, then $A \cup B$ is r-g-closed.*

***Proof*.** see [7] page 212.

Theorem 2.6.7 *Let X be topological space and $B \subseteq A \subseteq X$. If B is r-g-closed set relative to A and A is a g-closed open subset in X, then B is r-g-closed in X.*

***Proof*.** see [7] page 212.

Theorem 2.6.8 *Let X be topological space and $A \subseteq X$. Then A is r-g-open if and only if $F \subseteq Int(A)$ whenever F is regular closed and $F \subseteq A$.*

***Proof*.** see [7] page 215.