CHAPTER III

REGULAR GENERALIZED CONTINUOUS FUNCTIONS

In this chapter, we study relations among continuous, g-continuous, gc-irresolute, r-g-continuous and r-g-irresolute functions.

3.1 Continuous Functions and Generalized Continuous Functions

Definition 3.1.1 A map $f: X \to Y$ from a topological space X into a topological space Y is called *g-continuous* if the inverse image of closed set in Y is g-closed in X.

Theorem 3.1.2 *Let X and Y be topological spaces and $f: X \to Y$ is g-continuous if and only if the inverse image of every open in Y is g-open in X.*

Proof. ($\Rightarrow$) Let G be any open set in Y. Then $Y-G$ is closed in Y. Since f is g-continuous, $f^{-1}(Y-G)$ is g-closed in X. But $f^{-1}(Y-G) = X - f^{-1}(G)$. This implies that $f^{-1}(G)$ is g-open in X.

($\Leftarrow$) Let F be any closed set in Y. Then $Y-F$ is open in Y. By assumption, $f^{-1}(Y-F)$ is g-open in X. But $f^{-1}(Y-F) = X - f^{-1}(F)$. This implies that $f^{-1}(F)$ is g-closed in X. Hence f is g-continuous.

Theorem 3.1.3 *Let X and Y be topological spaces. If $f: X \to Y$ is continuous, then f is g-continuous but not conversely.*

Proof. Let F be a closed set in Y. Since f is continuous, $f^{-1}(F)$ is closed in X which implies that $f^{-1}(F)$ is g-closed in X. Hence f is g-continuous. The converse need not be true as seen by the following example.

Example 3.1.4 Let $X=\{a,b,c\}$, $\Im=\{\varnothing,\{a\},X\}$, $Y=\{p,q\}$ and $\Im'=\{\varnothing,\{q\},Y\}$. Let $f:(X,\Im)\to(Y,\Im')$ be defined by $f(a)=f(c)=q$ and $f(b)=p$ It is easy to see that f is g-continuous but it is not continuous. Since $\{q\}$ is open in Y but $f^{-1}(\{q\})=\{a,c\}$ is not open in X.

Definition 3.1.5 A topological space X is called $T_{\frac{1}{2}}-space$ if every g-closed set in X is closed in X.

Proposition 3.1.6 *A topological space X is a* $T_{\frac{1}{2}}-space$ *if and only if each singleton subset of X is closed or open.*

Proof. ($\Rightarrow$) Let $x\in X$. Suppose that $\{x\}$ is not closed. Therefore $X-\{x\}$ is not open. Since $X-\{x\}\subseteq X$, $\overline{X-\{x\}}\subseteq X$, it implies that $X-\{x\}$ is g-closed. But X is a $T_{\frac{1}{2}}-space$, so $X-\{x\}$ is closed, $\{x\}$ is open.

($\Leftarrow$) Let A be a g-closed in X and let $x\in\overline{A}$. If $\{x\}$ is open, we have that $\{x\}\cap A\neq\varnothing$, so $x\in A$. If $\{x\}$ is closed. We shall shows that $\overline{\{x\}}\cap A\neq\varnothing$. Suppose that $\overline{\{x\}}\cap A=\varnothing$, we have that $A\subseteq X-\overline{\{x\}}$. Since A is g-closed, therefore $\overline{A}\subseteq X-\overline{\{x\}}$. But $X-\overline{\{x\}}\subseteq X-\{x\}$. Hence $\{x\}\cap\overline{A}=\varnothing$, so $x\notin\overline{A}$, which is a contradiction. Therefore $\overline{\{x\}}\cap A\neq\varnothing$, $\{x\}\cap A\neq\varnothing$, $x\in A$. Hence $\overline{A}\subseteq A$. But $A\subseteq\overline{A}$, $A=\overline{A}$, it implies that A is closed. Therefore X is a $T_{\frac{1}{2}}-space$.

Theorem 3.1.7 *Let X be a* $T_{\frac{1}{2}}-space$ *and Y be a topological space. Then* $f:X\to Y$ *is continuous if and only if* f *is g-continuous.*

Proof. ($\Rightarrow$) By Theorem 3.1.3.

($\Leftarrow$) Let F be a closed set in *Y*. Since f is g-continuous, $f^{-1}(F)$ is g-closed in *X*. Since X is a $T_{\frac{1}{2}}-space$, we have $f^{-1}(F)$ is closed in *X*. Hence f is continuous.

Theorem 3.1.8 *Let X, Y and Z be topological spaces. If $f: X \to Y$ is g-continuous and $g: Y \to Z$ is continuous, then the composition $g \circ f: X \to Z$ is g-continuous.*

Proof. Let F be any closed set in *Z*. Since g is continuous, $g^{-1}(F)$ is closed in *Y*. Since f is g-continuous, we have $f^{-1}(g^{-1}(F))$ is g-closed in *X*. Hence $g \circ f$ is g-continuous.

In general, the composition of g-continuous need not be g-continuous as we will see in the next example.

Example 3.1.9 Let $X=Y=Z=\{a,b,c\}, \Im=\{\varnothing,\{a,b\},X\}$, $\Im'=\{\varnothing,\{a\},\{b,c\},Y\}$ and $\Im''=\{\varnothing,\{a,c\},Z\}$. Let $f:(X,\Im)\to(Y,\Im')$ be defined by $f(a)=c$, $f(b)=b$ and $f(c)=c$. Let $g:(Y,\Im')\to(Z,\Im'')$ be the identity map. It is easy to see that both f and g are g-continuous but $g \circ f$ is not g-continuous because $(g\circ f)^{-1}(\{b\})=\{b\}$ is not g-closed in *X*.

Theorem 3.1.10 *Let X and Z be topological spaces and Y be a $T_{\frac{1}{2}}-space$. If $f: X \to Y$ and $g: Y \to Z$ are g-continuous, then $g \circ f: X \to Z$ is g-continuous.*

Proof. Let F be any closed set in *Z*. Since g is g-continuous, $g^{-1}(F)$ is g-closed in *Y*. But *Y* is $T_{\frac{1}{2}}-space$, so $g^{-1}(F)$ is closed in *Y*. Since f is g-continuous, it implies that $f^{-1}(g^{-1}(F))$ is g-closed in *X*. Hence $g \circ f$ is g-continuous.

3.2 Generalized Continuous Functions and gc-irresolute Functions

Definition 3.2.1 A map $f: X \to Y$ from a topological space X into a topological space Y is called *gc-irresolute* if the inverse image of g-closed set in Y is g-closed in X.

Theorem 3.2.2 *Let X and Y be topological spaces and $f: X \to Y$ is gc-irresolute if and only if the inverse image of every g-open in Y is g-open in X.*

Proof. ($\Rightarrow$) Let G be any g-open set in Y. Then $Y - G$ is g-closed in Y. Since f is gc-irresolute, $f^{-1}(Y-G)$ is g-closed in X. But $f^{-1}(Y-G) = X - f^{-1}(G)$. This implies that $f^{-1}(G)$ is g-open in X.

($\Leftarrow$) Let F be any g-closed set in Y. Then $Y - F$ is g-open in Y. By assumption, $f^{-1}(Y-F)$ is g-open in X. But $f^{-1}(Y-F) = X - f^{-1}(F)$. This implies that $f^{-1}(F)$ is g-closed in X. Hence f is gc-irresolute.

Theorem 3.2.3 *Let X and Y be topological spaces. If $f: X \to Y$ is gc-irresolute, then f is g-continuous but not conversely.*

Proof. Let F be a closed set in Y. Then F is g-closed in Y. Since f is gc-irresolute, $f^{-1}(F)$ is g-closed in X. Hence f is g-continuous. The converse need not be true as seen by the following example.

Example 3.2.4 Let $X = Y = \{a,b,c\}$, $\Im = \{\varnothing, \{a\}, \{c\}, \{a,c\}, X\}$ and $\Im' = \{\varnothing, \{a\}, Y\}$. Let $f:(X,\Im) \to (Y,\Im')$ be defined by $f(a) = f(c) = a$ and $f(b) = b$. Then f is g-continuous. However, $\{a,c\}$ is g-closed in Y but $f^{-1}(\{a,c\})$ is not g-closed in X. Therefore f is not gc-irresolute.

Theorem 3.2.5 *Let X, Y and Z be topological spaces. If $f: X \to Y$ is gc-irresolute and $g: Y \to Z$ is g-continuous, then the composition $g \circ f: X \to Z$ is g-continuous.*

Proof. Let F be any closed set in Z. Since g is g-continuous, $g^{-1}(F)$ is g-closed in Y. Since f is gc-irresolute, we have $f^{-1}(g^{-1}(F))$ is g-closed in X. Hence $g \circ f$ is g-continuous.

Theorem 3.2.6 *Let X and Y be topological spaces. If $f: X \to Y$ is closed and g-continuous, then f is gc-irresolute.*

Proof. Let G be a g-open set in Y. Let F be closed set in X such that $F \subseteq f^{-1}(G)$. Then $f(F) \subseteq G$. Since f is closed, $f(F)$ is closed in Y. By Theorem 2.6.4, we have $f(F) \subseteq Int(G)$. Hence $F \subseteq f^{-1}(Int(G))$. Since f is g-continuous and $Int(G)$ is open in Y, we have $f^{-1}(Int(G))$ is g-open in X. It follows by Theorem 2.6.4 that $F \subseteq Int(f^{-1}(Int(G))) \subseteq Int(f^{-1}(G))$. Therefore by Theorem 2.6.4, $f^{-1}(G)$ is g-open in X. Hence f is gc-irresolute.

Theorem 3.2.7 *Let X, Y and Z be topological spaces. If $f: X \to Y$ and $g: Y \to Z$ are gc-irresolute, then $g \circ f: X \to Z$ is gc-irresolute.*

Proof. It follows directly from the definition.

3.3 Continuous Functions and Regular Generalized Continuous Functions

Definition 3.3.1 A map $f: X \to Y$ from a topological space X into a topological space Y is called *r-g-continuous* if the inverse image of closed set in Y is r-g-closed in X.

Theorem 3.3.2 *Let X and Y be topological spaces and* $f : X \to Y$ *is r-g-continuous if and only if the inverse image of every open in Y is r-g-open in X.*

Proof. ($\Rightarrow$) Let G be any open set in Y. Then $Y - G$ is closed in Y. Since f is r-g-continuous, $f^{-1}(Y - G)$ is r-g-closed in X. But $f^{-1}(Y - G) = X - f^{-1}(G)$. This implies that $f^{-1}(G)$ is r-g-open in X.

($\Leftarrow$) Let F be any closed set in Y. Then $Y - F$ is open in Y. By assumption, $f^{-1}(Y - F)$ is r-g-open in X. But $f^{-1}(Y - F) = X - f^{-1}(F)$. This implies that $f^{-1}(F)$ is r-g-closed in X. Hence f is r-g-continuous.

Theorem 3.3.3 *Let X and Y be topological spaces. If* $f : X \to Y$ *is continuous, then f is r-g-continuous but not conversely.*

Proof. Let F be a closed set in Y. Since f is continuous, $f^{-1}(F)$ is closed in X which implies that $f^{-1}(F)$ is r-g-closed in X. Hence f is r-g-continuous.

The converse is not true as seen by the following example.

Example 3.3.4 Let $X = \{a,b,c\}$, $\Im = \{\varnothing, \{a,b\}, X\}$, $Y = \{p,q\}$ and $\Im' = \{\varnothing, \{p\}, Y\}$. Let $f : (X, \Im) \to (Y, \Im')$ be defined by $f(a) = q$ and $f(b) = f(c) = p$. It is easy to see that f is r-g-continuous but it is not continuous. Since $\{p\}$ is open in Y but $f^{-1}(\{p\}) = \{b,c\}$ is not open in X.

Definition 3.3.5 A topological space X is called $T^*_{\frac{1}{2}}-space$ if every r-g-closed set in X is closed in X.

Theorem 3.3.6 *Let X be a $T^*_{\frac{1}{2}}-space$ and Y be a topological space. Then $f: X \to Y$ is continuous if and only if f is r-g-continuous.*

Proof. $(\Rightarrow)$ By Theorem 3.3.3.

$(\Leftarrow)$ Let F be a closed set in *Y*. Since f is r-g-continuous, $f^{-1}(F)$ is r-g-closed in *X*. Since X is a $T^*_{\frac{1}{2}}-space$, we have $f^{-1}(F)$ is closed in *X*. Hence f is continuous.

Theorem 3.3.7 *Let X, Y and Z be topological spaces. If $f: X \to Y$ is r-g-continuous and $g: Y \to Z$ is continuous, then the composition $g \circ f: X \to Z$ is r-g-continuous.*

Proof. Let F be any closed set in *Z*. Since g is continuous, $g^{-1}(F)$ is closed in *Y*. Since f is r-g-continuous, we have $f^{-1}(g^{-1}(F))$ is r-g-closed in *X*. Hence $g \circ f$ is r-g-continuous.

In general the composition of r-g-continuous need not be r-g-continuous as seen in the next example.

Example 3.3.8 Let $X = Y = Z = \{a,b,c\}$, $\Im = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$, $\Im' = \{\varnothing, \{a\}, Y\}$ and $\Im'' = \{\varnothing, \{b,c\}, Z\}$. Let $f: (X, \Im) \to (Y, \Im')$ be defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Let $g: (Y, \Im') \to (Z, \Im'')$ be the identity map. Then f and g are r-g-continuous but $g \circ f$ is not r-g-continuous because $(g \circ f)^{-1}(\{a\}) = \{a\}$ is not r-g-closed in *X*.

Theorem 3.3.9 *Let X and Z be topological spaces and Y be a $T^*_{\frac{1}{2}}-space$. If $f: X \to Y$ and $g: Y \to Z$ are r-g-continuous, then the composition $g \circ f: X \to Z$ is r-g-continuous.*

Proof. Let F be any closed set in *Z*. Since *g* is r-g-continuous, $g^{-1}(F)$ is r-g-closed in *Y*. But *Y* is $T_{\frac{1}{2}}^{*}-space$, so $g^{-1}(F)$ is closed in *Y*. Since f is r-g-continuous, it implies that $f^{-1}(g^{-1}(F))$ is r-g-closed in *X*. Hence $g\circ f$ is r-g-continuous.

Theorem 3.3.10 *Let X and Y be topological spaces. Let $A,B\subseteq X$ be g-closed and open in X such that $X=A\cup B$ and let $f:A\to Y$ and $g:B\to Y$ be r-g-continuous maps such that $f(x)=g(x)$ for every $x\in A\cap B$. Let $h:X\to Y$ be defined by $h(x)=f(x)$ if $x\in A$ and $h(x)=g(x)$ if $x\in B$. Then h is r-g-continuous.*

Proof. Let F be a closed set in *Y*. Clearly $h^{-1}(F)=f^{-1}(F)\cup g^{-1}(F)$. Since $f:A\to Y$ and $g:B\to Y$ are r-g-continuous, we have $f^{-1}(F)$ and $g^{-1}(F)$ are r-g-closed in A and B respectively. Since A is g-closed and open in *X*, we have by Theorem 2.6.7 that $f^{-1}(F)$ is r-g-closed in X. Similarly, $g^{-1}(F)$ is r-g-closed in *X*. By Theorem 2.6.6, $f^{-1}(F)\cup g^{-1}(F)$ is r-g-closed in *X*. Therefore, $h^{-1}(F)$ is r-g-closed in *X*.

3.4 Generalized Continuous and Regular Generalized Continuous Functions

Theorem 3.4.1 *Let X and Y be topological spaces. If $f:X\to Y$ is g-continuous, then f is r-g-continuous but not conversely.*

Proof. Let F be a closed set in *Y*. Since f is g-continuous, $f^{-1}(F)$ is g-closed in *X* which implies that $f^{-1}(F)$ is r-g-closed in *X*. Hence f is r-g-continuous.

The converse is not true as seen by the following example.

Example 3.4.2 Let $X = Y = \{a,b,c\}$, $\Im = \{\varnothing, \{a\}, \{a,b\} X\}$ and $\Im' = \{\varnothing, \{a,c\}, Y\}$. Let $f:(X,\Im) \to (Y,\Im')$ be the identity map. It is easy to see that f is r-g-continuous but f is not g-continuous because $f^{-1}(\{b\}) = \{b\}$ is not g-closed in *X*.

Definition 3.4.3 A topological space *X* is called $T_{rg} - space$ if every r-g-closed set in *X* is g-closed in *X*.

Theorem 3.4.4 *Let X be a $T_{rg} - space$ and Y be a topological space. Then $f: X \to Y$ is g-continuous if and only if f is r-g-continuous.*

Proof. ($\Rightarrow$) By Theorem 3.4.1.

($\Leftarrow$) Let F be a closed set in *Y*. Since f is r-g-continuous, $f^{-1}(F)$ is r-g-closed in *X*. Since *X* is a $T_{rg} - space$, we have $f^{-1}(F)$ is g-closed in *X*. Hence f is g-continuous.

Theorem 3.4.5 *Let X and Z be topological spaces and Y be a $T_{\frac{1}{2}} - space$. If $f: X \to Y$ is r- g-continuous and $g: Y \to Z$ is g-continuous, then $g \circ f: X \to Z$ is r-g-continuous.*

Proof. Let F be any closed set in *Z*. Since *g* is g-continuous, $g^{-1}(F)$ is g-closed in *Y*. But *Y* is $T_{\frac{1}{2}} - space$, so $g^{-1}(F)$ is closed in *Y*. Since f is r-g-continuous, we have $f^{-1}(g^{-1}(F))$ is r-g-closed in *X*. Hence $g \circ f$ is r-g-continuous.

3.5 Regular Generalized Continuous Functions and r-g-irresolute Functions

Definition 3.5.1 A map $f : X \to Y$ from a topological space X into a topological space Y is called *r-g-irresolute* if the inverse image of r-g-closed set in Y is r-g-closed in X.

Theorem 3.5.2 *Let X and Y be topological spaces and $f : X \to Y$ is r-g-irresolute if and only if the inverse image of every r-g-open in Y is r-g-open in X.*

Proof. ($\Rightarrow$) Let G be any r-g-open set in Y. Then $Y - G$ is r-g-closed in Y. Since f is r-g-irresolute, $f^{-1}(Y - G)$ is r-g-closed in X. But $f^{-1}(Y - G) = X - f^{-1}(G)$. This implies that $f^{-1}(G)$ is r-g-open in X.

($\Leftarrow$) Let F be any r-g-closed set in Y. Then $Y - F$ is r-g-open in Y. By assumption, $f^{-1}(Y - F)$ is r-g-open in X. But $f^{-1}(Y - F) = X - f^{-1}(F)$. This implies that $f^{-1}(F)$ is r-g-closed in X. Hence f is r-g-irresolute.

Theorem 3.5.3 *Let X and Y be topological spaces. If $f : X \to Y$ is r-g-irresolute, then f is r-g-continuous but not conversely.*

Proof. Let F be a closed set in Y. Then F is r-g-closed in Y. Since f is r-g-irresolute, $f^{-1}(F)$ is r-g-closed in X. Hence f is r-g-continuous.

The converse need not be true as seen by the following example.

Example 3.5.4 Let $X = Y = \{a,b,c\}$, $\Im = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$ and $\Im' = \{\varnothing, \{a,b\}, Y\}$. Let $f:(X,\Im) \to (Y,\Im')$ be defined by $f(a) = f(c) = c$ and $f(b) = b$. Then f is r-g-continuous. However, $\{b\}$ is r-g-closed in Y but $f^{-1}(\{b\}) = \{b\}$ is not r-g-closed in X. Hence f is not r-g-irresolute.

Theorem 3.5.5 *Let X, Y and Z be topological spaces. If $f : X \to Y$ is r-g-irresolute and $g : Y \to Z$ is r-g-continuous, then the composition $g \circ f : X \to Z$ is r-g-continuous.*

Proof. Let F be any closed set in *Z*. Since g is r-g-continuous, $g^{-1}(F)$ is r-g-closed in *Y*. Since f is r-g-irresolute, we have $f^{-1}(g^{-1}(F))$ is r-g-closed in *X*. Hence $g \circ f$ is r-g-continuous.

Definition 3.5.6 A map $f : X \to Y$ from a topological space X into a topological space Y is called *regular closed* if $f(F)$ is regular closed in Y for every closed set F in *X*.

Theorem 3.5.7 *Let X and Y be topological spaces. If $f : X \to Y$ is regular closed and r-g-continuous, then f is r-g-irresolute.*

Proof. Let G be a r-g-open set in *Y*. Let F be regular closed set in X such that $F \subseteq f^{-1}(G)$. Then $f(F) \subseteq G$. Since f is regular closed, $f(F)$ is regular closed in *Y*. By Theorem 2.6.8, we have $f(F) \subseteq Int(G)$. Hence $F \subseteq f^{-1}(Int(G))$. Since f is r-g-continuous and $Int(G)$ is open in *Y*, we have $f^{-1}(Int(G))$ is r-g-open in *X*. It follows by Theorem 2.6.8 that $F \subseteq Int(f^{-1}(Int(G))) \subseteq Int(f^{-1}(G))$. Therefore by Theorem 2.6.8, $f^{-1}(G)$ is r-g-open in *X*. Hence f is r-g-irresolute.

Theorem 3.5.8 *Let X, Y and Z be topological spaces. If* $f: X \to Y$ *and* $g: Y \to Z$ *are r-g-irresolute, then* $g \circ f: X \to Z$ *is r-g-irresolute.*

Proof. It follows directly from the definition.

3.6 Regular Generalized Continuous Functions and gc-irresolute Functions

Theorem 3.6.1 *Let X and Y be topological spaces. If* $f: X \to Y$ *is gc-irresolute, then* f *is r-g-continuous but not conversely.*

Proof. Let F be a closed set in *Y*. Then F is g-closed in *Y*. Since f is gc-irresolute, $f^{-1}(F)$ is g-closed in *X* which implies that $f^{-1}(F)$ is r-g-closed in *X*. Hence f is r-g-continuous.

The converse need not be true as seen by the following example.

Example 3.6.2 Let $X = \{a,b,c\}$, $\Im = \{\varnothing,\{a\},X\}$, $Y = \{p,q\}$ and $\Im' = \{\varnothing,\{p\},Y\}$. Let $f:(X,\Im) \to (Y,\Im')$ be defined by $f(a) = q$ and $f(b) = f(c) = p$. Then f is r-g-continuous but f is not gc-irresolute because $f^{-1}(\{q\}) = \{a\}$ is not g-closed in *X*.

Theorem 3.6.3 *Let X be a* T_{rg} *− space and Y be topological spaces. If* $f: X \to Y$ *is bijective, open and r-g-continuous, then* f *is gc-irresolute.*

Proof. Let F be any g-closed set in *Y*. Let U be open set in X such that $f^{-1}(F) \subseteq U$. Then $F \subseteq f(U)$ (because f is onto). Since $f(U)$ is open and F is g-closed in *Y*, we have $\overline{F} \subseteq f(U)$ which implies that $f^{-1}(\overline{F}) \subseteq U$

(because f is injective). Since f is r-g-continuous, $f^{-1}(\overline{F})$ is r-g-closed in X. But X is $T_{rg}-space$, so $f^{-1}(\overline{F})$ is g-closed in X, it implies that $\overline{f^{-1}(\overline{F})} \subseteq U$, hence $\overline{f^{-1}(F)} \subseteq U$. This shows that f is gc-irresolute.

Example 3.6.4 Let $X = Y = \{a, b, c\}$, $\Im_X = \{\varnothing, \{a\}, \{b, c\}, X\}$ and $\Im_Y = \{\varnothing, \{a\}, \{b, c\}, Y\}$. Let $f : (X, \Im_X) \to (Y, \Im_Y)$ be identity map. It is easy to see that X is $T_{rg}-space$ and f is bijective, open and r-g-continuous. It is obvious that f is gc-irresolute.

Theorem 3.6.5 *Let X and Z be topological spaces and Y be a $T_{\frac{1}{2}}-space$. If $f : X \to Y$ is r-g-continuous and $g : Y \to Z$ is gc-irresolute, then $g \circ f : X \to Z$ is r-g-continuous.*

Proof. Let F be any closed set in Z. Then F is g-closed in Z. Since g is gc-irresolute, $g^{-1}(F)$ is g-closed in Y. But Y is $T_{\frac{1}{2}}-space$, so $g^{-1}(F)$ is closed in Y. Since f is r-g-continuous, we have $f^{-1}(g^{-1}(F))$ is r-g-closed in X. Hence $g \circ f$ is r-g-continuous.

Theorem 3.6.6 *Let X and Z be topological spaces and Y be a $T_{rg}-space$. If $f : X \to Y$ is gc-irresolute and $g : Y \to Z$ is r-g-contiuous, then $g \circ f : X \to Z$ is r-g-continuous.*

Proof. Let F be any closed set in Z. Since g is r-g-continuous, $g^{-1}(F)$ is r-g-closed in Y. But Y is $T_{rg}-space$, so $g^{-1}(F)$ is g-closed in Y. Since f is gc-irresolute, we have $f^{-1}(g^{-1}(F))$ is g-closed in X which implies that $f^{-1}(g^{-1}(F))$ is r-g-closed in X. Hence $g \circ f$ is r-g-continuous.