CHAPTER IV

REGULAR GENERALIZED CONTINUOUS FUNCTIONS FROM ANY TOPOLOGICAL SPACES INTO A PRODUCT SPACE

In this chapter, we study g-open, g-closed, r-g-open and r-g-closed sets in a product space. Moreover we also study g-continuous, r-g-continuous, gc-irresolute and r-g-irresolute functions from any topological spaces into a product space.

4.1 Generalized Open Sets, Generalized Closed Sets, Regular Generalized Open Sets and Regular Generalized Closed Sets

Theorem 4.1.1 *Let* $\{ X_\alpha \mid \alpha \in I \}$ *be a family of topological spaces and* $G_\alpha \subseteq X_\alpha$ *for each* $\alpha \in I$. *If* G_α *is g-open in* X_α *for each* $\alpha \in I$ *and* $G_\alpha = X_\alpha$ *for all but finitely many* $\alpha \in I$, *then* $\prod_{\alpha \in I} G_\alpha$ *is g-open in* $\prod_{\alpha \in I} X_\alpha$.

***Proof*.** Let F be a closed set in $\prod_{\alpha \in I} X_\alpha$ such that $F \subseteq \prod_{\alpha \in I} G_\alpha$. For each $(x_\alpha)_{\alpha \in I} \in F$, $\prod_{\alpha \in I} \overline{\{x_\alpha\}} = \overline{\prod_{\alpha \in I} \{x_\alpha\}} = \overline{\{(x_\alpha)_{\alpha \in I}\}} \subseteq \overline{F} = F \subseteq \prod_{\alpha \in I} G_\alpha$. Terefore $\overline{\{x_\alpha\}} \subseteq G_\alpha$ for each $\alpha \in I$. Since $\overline{\{x_\alpha\}}$ is closed and G_α is g-open in X_α, by Theorem 2.6.4, $\overline{\{x_\alpha\}} \subseteq Int(G_\alpha)$ for each $\alpha \in I$. Then $\prod_{\alpha \in I} \overline{\{x_\alpha\}} \subseteq \prod_{\alpha \in I} Int(G_\alpha)$. Hence $F \subseteq \prod_{\alpha \in I} Int(G_\alpha)$. But since $\prod_{\alpha \in I} Int(G_\alpha) = Int(\prod_{\alpha \in I} G_\alpha)$, so we have $F \subseteq Int(\prod_{\alpha \in I} G_\alpha)$. Therefore, by Theorem 2.6.4, $\prod_{\alpha \in I} G_\alpha$ is g-open in $\prod_{\alpha \in I} X_\alpha$.

Theorem 4.1.2 *Let* $\{ X_\alpha \mid \alpha \in I \}$ *be a family of* T_1*-spaces and* G_α *a nonempty subset of* X_α *for each* $\alpha \in I$. *If* $\prod_{\alpha \in I} G_\alpha$ *is g-open in* $\prod_{\alpha \in I} X_\alpha$, *then* G_α *is g-open in* X_α *for each* $\alpha \in I$.

Proof. Case 1. $Int(\prod_{\alpha\in I} G_\alpha) \neq \varnothing$. Then there exists a finite subset J of I such that $G_\alpha = X_\alpha$ for all $\alpha \in I - J$. Let $\alpha_0 \in I$ and F_{α_0} be a closed subset of X_{α_0} such that $F_{\alpha_0} \subseteq G_{\alpha_0}$. For each $\alpha \in J - \{\alpha_0\}$, choose $x_\alpha \in G_\alpha$. Since X_α is $T_1 - space$, $\{x_\alpha\}$ is closed in X_α. Therefore $F_{\alpha_0} \times \prod_{\substack{\alpha\in I-J \\ \alpha\neq\alpha_0}} X_\alpha \times \prod_{\substack{\alpha\in J \\ \alpha\neq\alpha_0}} \{x_\alpha\}$ closed in $\prod_{\alpha\in I} X_\alpha$. Since $\prod_{\alpha\in I} G_\alpha$ is g-open in $\prod_{\alpha\in I} X_\alpha$, we have, by Theorem 2.6.4,

$$F_{\alpha_0} \times \prod_{\substack{\alpha\in I-J \\ \alpha\neq\alpha_0}} X_\alpha \times \prod_{\substack{\alpha\in J \\ \alpha\neq\alpha_0}} \{x_\alpha\} \subseteq Int(\prod_{\alpha\in I} G_\alpha) = \prod_{\alpha\in I} Int(G_\alpha).$$

This implies that $F_{\alpha_0} \subseteq Int(G_{\alpha_0})$. It follows by Theorem 2.6.4 that G_{α_0} is g-open in X_{α_0}. Since α_0 is arbitary, we obtain that G_α is g-open in X_α.

Case 2. $Int(\prod_{\alpha\in I} G_\alpha) = \varnothing$. Let $\alpha_0 \in I$ and F_{α_0} be a closed subset of X_{α_0} such that $F_{\alpha_0} \subseteq G_{\alpha_0}$. For each $\alpha \in I - \{\alpha_0\}$, choose $x_\alpha \in G_\alpha$. It is obvious that $F_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} \{x_\alpha\} \subseteq \prod_{\alpha\in I} G_\alpha$. Since X_α is $T_1 - space$, $\{x_\alpha\}$ is closed in X_α for each $\alpha \in I$. Therefore $F_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} \{x_\alpha\}$ is closed in $\prod_{\alpha\in I} X_\alpha$. By Theorem 2.6.4 we have $F_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} \{x_\alpha\} \subseteq Int(\prod_{\alpha\in I} G_\alpha) = \varnothing$, so $F_{\alpha_0} = \varnothing$, hence $F_{\alpha_0} \subseteq Int(G_{\alpha_0})$. By Theorem 2.6.4, we get that G_{α_0} is g-open in X_{α_0}.

Theorem 4.1.3 *Let* $\{ X_\alpha \mid \alpha \in I \}$ *be a family of* $T_1 - spaces$, G_α *a nonempty subset of* X_α *for each* $\alpha \in I$ *and* $Int(\prod_{\alpha\in I} G_\alpha) \neq \varnothing$. *Then* $\prod_{\alpha\in I} G_\alpha$ *is g-open in* $\prod_{\alpha\in I} X_\alpha$ *if and only if* G_α *is g-open in* X_α *for each* $\alpha \in I$ *and* $G_\alpha = X_\alpha$ *for all but finitely many* $\alpha \in I$.

Proof. $(\Rightarrow)$ By Theorem 4.1.2 Case 1.

$(\Leftarrow)$ By Theorem 4.1.1.

Theorem 4.1.4 *Let* $\{X_\alpha \mid \alpha \in I\}$ *be a family of topological spaces and let* $\alpha_0 \in I$ *be fixed. If* F_{α_0} *is g-closed in* X_{α_0}, *then* $\prod_{\alpha \neq \alpha_0} X_\alpha \times F_{\alpha_0}$ *is g-closed in* $\prod_{\alpha \in I} X_\alpha$.

Proof. Since $X_{\alpha_0} - F_{\alpha_0}$ is g-open in X_{α_0}. By Theorem 4.1.1, $\prod_{\alpha \neq \alpha_0} X_\alpha \times (X_{\alpha_0} - F_{\alpha_0})$ is g-open in $\prod_{\alpha \in I} X_\alpha$. But $\prod_{\alpha \neq \alpha_0} X_\alpha \times (X_{\alpha_0} - F_{\alpha_0}) = \prod_{\alpha \in I} X_\alpha - \prod_{\alpha \neq \alpha_0} X_\alpha \times F_{\alpha_0}$. Therefore $\prod_{\alpha \in I} X_\alpha - \prod_{\alpha \neq \alpha_0} X_\alpha \times F_{\alpha_0}$ is g-open in $\prod_{\alpha \in I} X_\alpha$. Hence $\prod_{\alpha \neq \alpha_0} X_\alpha \times F_{\alpha_0}$ is g-closed in $\prod_{\alpha \in I} X_\alpha$.

Theorem 4.1.5 *Let* $\{X_\alpha \mid \alpha \in I\}$ *be a family of topological spaces and* $F_\alpha \subseteq X_\alpha$ *for each* $\alpha \in I$. *Then* $\prod_{\alpha \in I} F_\alpha$ *is g-closed in* $\prod_{\alpha \in I} X_\alpha$ *if and only if* F_α *is g-closed in* X_α *for each* $\alpha \in I$.

Proof. $(\Rightarrow)$ Let $\alpha \in I$ and G_α be open in X_α such that $F_\alpha \subseteq G_\alpha$. Since $\prod_{\beta \in I} F_\beta \subseteq \prod_{\beta \neq \alpha} X_\beta \times G_\alpha$, $\prod_{\beta \neq \alpha} X_\beta \times G_\alpha$ is open in $\prod_{\beta \in I} X_\beta$ and $\prod_{\beta \in I} F_\beta$ is g-closed in $\prod_{\beta \in I} X_\beta$, we obtain that $\overline{\prod_{\beta \in I} F_\beta} \subseteq \prod_{\beta \neq \alpha} X_\beta \times G_\alpha$. But $\overline{\prod_{\beta \in I} F_\beta} = \prod_{\beta \in I} \overline{F_\beta}$, it follows that $\overline{F_\alpha} \subseteq G_\alpha$. Hence F_α is g-closed in X_α.

$(\Leftarrow)$ Suppose that $\prod_{\alpha \in I} F_\alpha$ is not g-closed in $\prod_{\alpha \in I} X_\alpha$. There exists a open set G in $\prod_{\alpha \in I} X_\alpha$ such that $\prod_{\alpha \in I} F_\alpha \subseteq G$ but $\overline{\prod_{\alpha \in I} F_\alpha} \not\subseteq G$. Therefore there exists $\beta \in I$ such that $\overline{F_\beta} \not\subseteq \pi_\beta(G)$. Since $\pi_\beta(G)$ is open and F_β is a g-closed containing $\pi_\beta(G)$, we get that $\overline{F} \subseteq \pi_\beta(G)$, this is a contradiction.

The next theorem shows that with some conditions if a product of sets in a product space is r-g-open, then each coordinate factor must also be r-g-open. To shows this we need the following lemma.

Lemma 4.1.6 *Let* $\{ X_\alpha | \alpha \in I \}$ *be a family of topological spaces and* $F_\alpha \subseteq X_\alpha$ *for each* $\alpha \in I$ *. If* F_α *is regular closed in* X_α *for each* $\alpha \in I$ *and* $F_\alpha = X_\alpha$ *for all but finitely many* $\alpha \in I$ *, then* $\prod_{\alpha \in I} F_\alpha$ *is regular closed in* $\prod_{\alpha \in I} X_\alpha$ *.*

Proof. Since F_α is regular closed in X_α for each $\alpha \in I$, $F_\alpha = \overline{Int(F_\alpha)}$, hence $\prod_{\alpha \in I} F_\alpha = \prod \overline{Int(F_\alpha)} = \overline{Int(\prod_{\alpha \in I} F_\alpha)}$. Therefore $\prod_{\alpha \in I} F_\alpha$ is regular closed in $\prod_{\alpha \in I} X_\alpha$.

Theorem 4.1.7 *Let* $\{ X_\alpha | \alpha \in I \}$ *be a family of topological spaces,* G_α *a nonempty subset of* X_α *such that* G_α *contains a nonempty regular closed subset* F_α *of* X_α *for each* $\alpha \in I$*. If* $\prod_{\alpha \in I} G_\alpha$ *is r-g-open in* $\prod_{\alpha \in I} X_\alpha$ *and* $Int(\prod_{\alpha \in I} G_\alpha) \neq \varnothing$*, then* G_α *is r-g-open in* X_α *for each* $\alpha \in I$*.*

Proof. Since $Int(\prod_{\beta \in I} G_\beta) \neq \varnothing$, there exists a finite subset J of I such that $G_\beta = X_\beta$ for all $\beta \in I - J$. Let $\alpha \in I$ and E_α be a regular closed in X_α such that $E_\alpha \subseteq G_\alpha$. Therefore $E_\alpha \times \prod_{\substack{\beta \in J \\ \beta \neq \alpha}} F_\beta \times \prod_{\substack{\beta \in I-J \\ \beta \neq \alpha}} X_\beta \subseteq \prod_{\beta \in I} G_\beta$. By Lemm 4.1.6, $E_\alpha \times \prod_{\substack{\beta \in J \\ \beta \neq \alpha}} F_\beta \times \prod_{\substack{\beta \in I-J \\ \beta \neq \alpha}} X_\beta$ is regular closed in $\prod_{\beta \in I} X_\beta$ and by Theorem 2.6.8, we have $E_\alpha \times \prod_{\substack{\beta \in J \\ \beta \neq \alpha}} F_\beta \times \prod_{\substack{\beta \in I-J \\ \beta \neq \alpha}} X_\beta \subseteq Int(\prod_{\beta \in I} G_\beta)$. This implies that $E_\alpha \subseteq Int(G_\alpha)$. It follows from Theorem 2.6.8 that G_α is r-g-open in X_α.

We shows in the next theorem that the under some conditions the product of r-g-open sets is also r-g-open.

Theorem 4.1.8 *Let $\{ X_\alpha \mid \alpha \in I \}$ be a family of topological spaces and $G_\alpha \subseteq X_\alpha$ for each $\alpha \in I$ and $G_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$. If π_α is a regular closed map and G_α is r-g-open in X_α for each $\alpha \in I$, then $\prod_{\alpha \in I} G_\alpha$ is r-g-open in $\prod_{\alpha \in I} X_\alpha$.*

Proof. Let F be a regular closed in $\prod_{\alpha \in I} X_\alpha$ such that $F \subseteq \prod_{\alpha \in I} G_\alpha$. Then $\pi_\alpha(F) \subseteq G_\alpha$. Since $\pi_\alpha(F)$ is regular closed in X_α, it follows by Theorem 2.6.8 $\pi_\alpha(F) \subseteq Int(G_\alpha)$. Therefore $F \subseteq \prod_{\alpha \in I} \pi_\alpha(F) \subseteq \prod_{\alpha \in I} Int(G_\alpha) = Int(\prod_{\alpha \in I} G_\alpha)$.

By Theorem 2.6.8, we get that $\prod_{\alpha \in I} G_\alpha$ is r-g-open in $\prod_{\alpha \in I} X_\alpha$.

The next theorem shows that if a product of sets in a product space is r-g-closed, then each coordinate factor must also be r-g-closed. To shows this we need the following lemma.

Lemma 4.1.9 *Let $\{ X_\alpha \mid \alpha \in I \}$ be a family of topological spaces and $G_\alpha \subseteq X_\alpha$ for each $\alpha \in I$. If G_α is regular open in X_α for each $\alpha \in I$ and $G_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$, then $\prod_{\alpha \in I} G_\alpha$ is regular open in $\prod_{\alpha \in I} X_\alpha$.*

Proof. Since G_α is regular open in X_α for each $\alpha \in I$, $G_\alpha = Int(\overline{G_\alpha})$, hence $\prod_{\alpha \in I} G_\alpha = \prod_{\alpha \in I} Int(\overline{G_\alpha}) = Int(\overline{\prod_{\alpha \in I} G_\alpha})$. Therefore $\prod_{\alpha \in I} G_\alpha$ is regular open in $\prod_{\alpha \in I} X_\alpha$.

Theorem 4.1.10 *Let $\{ X_\alpha \mid \alpha \in I \}$ be a family of topological spaces and $F_\alpha \subseteq X_\alpha$ for each $\alpha \in I$. If $\prod_{\alpha \in I} F_\alpha$ is r-g-closed in $\prod_{\alpha \in I} X_\alpha$, then F_α is r-g-closed in X_α for each $\alpha \in I$.*

Proof. Let $\alpha \in I$ and G_α is regular open in X_α such that $F_\alpha \subseteq G_\alpha$. Then $\prod_{\beta \in I} F_\beta \subseteq \prod_{\beta \neq \alpha} X_\beta \times G_\alpha$. By Lemma 4.1.9, $\prod_{\beta \neq \alpha} X_\beta \times G_\alpha$ is regular open in $\prod_{\beta \in I} X_\beta$. Then $\overline{\prod_{\beta \in I} F_\beta} \subseteq \prod_{\beta \neq \alpha} X_\beta \times G_\alpha$. This implies that $\overline{F_\alpha} \subseteq G_\alpha$. Hence F_α is r-g-closed in X_α.

4.2 Generalized Continuous, Regular Generalized Continuous, gc-irresolute and r-g-irresolute Functions

In this section, we study g-continuous, r-g-continuous, gc-irresolute and r-g-irresolute functions from any topological spaces into a product space. We can completely give characterizations of some type of these continuity.

Theorem 4.2.1 *Let Y be a topological space and let $\{ X_\alpha \mid \alpha \in I \}$ be a family of topological spaces. Let $f : Y \to \prod_{\alpha \in I} X_\alpha$ be a function. If f is g-continuous, then the composite function $\pi_\alpha \circ f : Y \to X_\alpha$ is g-continuous for each $\alpha \in I$.*

Proof. Suppose that f is g-continuous. Since $\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha$ is continuous for each $\alpha \in I$, it follows by Theorem 3.1.7 that $\pi_\alpha \circ f$ is g-continuous for each $\alpha \in I$.

The converse of Theorem 4.2.1 is not true by the following example.

Example 4.2.2 Let $X=\{1,2,3,4\}$, $\Im_X=\{\varnothing,\{1\},\{2\},\{1,2\},X\}$, $Y_1 = Y_2 = \{a,b\}$, $\Im_{Y_1} = \{\varnothing,\{a\},Y_1\}$, $\Im_{Y_2} = \{\varnothing,\{a\},Y_2\}$, $Y = Y_1 \times Y_2 = \{(a,a),(a,b),(b,a),(b,b)\}$ and $\Im_Y = \{\varnothing, Y_1 \times Y_2, \{(a,a),(a,b)\}, \{(a,a),(b,a)\}, \{(a,a)\}, \{(a,a),(a,b),(b,a)\}\}$.

Define $f: X \to Y$ by $f(1)=(a,a), f(2)=(b,b), f(3)=(a,b)$ and $f(4)=(b,a)$. It is easy to see that $\pi_1 \circ f$ and $\pi_2 \circ f$ are g-continuous. However, $\{(b,b)\}$ is closed in Y but $f^{-1}(\{(b,b)\})=\{2\}$ is not g-closed in X. Therefore f is not g-continuous.

Theorem 4.2.3 *Let* Y *be a* $T_{\frac{1}{2}}$*-space and let* $\{X_\alpha \mid \alpha \in I\}$ *be a family of topological spaces. Let* $f: Y \to \prod_{\alpha \in I} X_\alpha$ *be a function. Then* f *is g-continuous if and only if the composite function* $\pi_\alpha \circ f: Y \to X_\alpha$ *is g-continuous for each* $\alpha \in I$.

Proof. (⇒) By Theorem 4.2.1.

(⇐) Since Y is $T_{\frac{1}{2}}$*-space* and by Theorem 3.1.7, $\pi_\alpha \circ f$ is continuous for each $\alpha \in I$. By Theorem 2.3.4, f is continuous. Hence f is g-continuous.

Corollary 4.2.4 *Let* $\{X_\alpha \mid \alpha \in I\}$ *and* $\{Y_\alpha \mid \alpha \in I\}$ *be family of topological spaces. For each* $\alpha \in I$*, let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a function and* $f: \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ *be defined by* $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. *If* f *is g-continuous, then* f_α *is g-continuous for each* $\alpha \in I$.

Proof. Let π_β and π'_β be the projection of $\prod_{\alpha \in I} X_\alpha$ and $\prod_{\alpha \in I} Y_\alpha$ onto X_β and Y_β, respectively. By Theorem 4.2.1, $\pi'_\beta \circ f$ is g-continuous for each $\beta \in I$. Since $\pi'_\beta \circ f = f_\beta \circ \pi_\beta$, we have that $f_\beta \circ \pi_\beta$ is g-continuous. Let F be closed in Y_β. Then $\pi_\beta^{-1}(f_\beta^{-1}(F)) = f_\beta^{-1}(F) \times \prod_{\alpha \in I} X_\alpha$ is g-closed in $\prod_{\alpha \in I} X_\alpha$. By Theorem 4.1.5, $f_\beta^{-1}(F)$ is g-closed in Y_β. Hence f_β is g-continuous for each $\beta \in I$.

Corollary 4.2.5 *Let* $\{ X_\alpha \mid \alpha \in I \}$ *be a famjly of* $T_{\frac{1}{2}}-$*spaces and* $\{ Y_\alpha \mid \alpha \in I \}$ *be a family of topological spaces. For each* $\alpha \in I$, *let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a function and* $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ *be defined by* $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. *Then* f *is g-continuous if and only if* f_α *is g-continuous for each* $\alpha \in I$.

Proof. ($\Rightarrow$) By Corollary 4.2.4.

($\Leftarrow$) Suppose that f_α is g-continuous for each $\alpha \in I$. Since X_α is a $T_{\frac{1}{2}}-$*space* for each $\alpha \in I$, it follows from Theorem 3.1.7 that f_α is continuous for each $\alpha \in I$. Therefore by Theorem 2.3.5, f is continuous. Hence f is g-continuous.

Theorem 4.2.6 *Let* Y *be a topological space and let* $\{ X_\alpha \mid \alpha \in I \}$ *be a family of topological spaces. Let* $f : Y \to \prod_{\alpha \in I} X_\alpha$ *be a function. If* f *is gc-irresolute, then the composite function* $\pi_\alpha \circ f : Y \to X_\alpha$ *is g-continuous for each* $\alpha \in I$.

Proof. Suppose f is gc-irresolute. Since $\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha$ is continuous for each $\alpha \in I$, we have that π_α is g-continuous for each $\alpha \in I$. It follows from Theorem 3.2.5 that $\pi_\alpha \circ f$ is g-continuous for each $\alpha \in I$.

The converse of Theorem 4.2.6 is not true as seen in Example 4.2.2.

Proposition 4.2.7 *Let* $\{ X_\alpha \mid \alpha \in I \}$ *be a family of topological spaces. Then for each* $\beta \in I$, *the projection mapping* $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ *is gc-irresolute.*

Proof. Let $\beta \in I$ and F_β is g-closed in X_β. Since $\pi_\beta^{-1}(F_\beta) = \prod_{\alpha \neq \beta} X_\alpha \times F_\beta$ and by Theorem 4.1.5, $\prod_{\alpha \neq \beta} X_\alpha \times F_\beta$ is g-closed in $\prod_{\alpha \in I} X_\alpha$, we obtain that π_β is gc-irresolute.

Theorem 4.2.8 *Let Y be a topological space and let $\{ X_\alpha \mid \alpha \in I \}$ be a family of topological spaces. Let $f : Y \to \prod_{\alpha \in I} X_\alpha$ be a function. If f is gc-irresolute, then the composite function $\pi_\alpha \circ f : Y \to X_\alpha$ is gc-irresolute for each $\alpha \in I$.*

Proof. By Proposition 4.2.7, $\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha$ is gc-irresolute for each $\alpha \in I$. It follows by Theorem 3.2.7 that $\pi_\alpha \circ f$ is gc-irresolute for each $\alpha \in I$.

The converse of this Theorem 4.2.8 is not true as we have in Example 4.2.2.

Corollary 4.2.9 *Let $\{ X_\alpha \mid \alpha \in I \}$ and $\{ Y_\alpha \mid \alpha \in I \}$ be family of topological spaces. For each $\alpha \in I$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function and $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. If f is gc-irresolute, then f_α is gc-irresolute for each $\alpha \in I$.*

Proof. Let π_β and π'_β be the projection of $\prod_{\alpha \in I} X_\alpha$ and $\prod_{\alpha \in I} Y_\alpha$ onto X_β and Y_β, respectively. By Theorem 4.2.8, $\pi'_\beta \circ f$ is gc-irresolute for each $\beta \in I$. Since $\pi'_\beta \circ f = f_\beta \circ \pi_\beta$, we have that $f_\beta \circ \pi_\beta$ is gc-irresolute. Let F be g-closed in Y_β. Then $\pi_\beta^{-1}(f_\beta^{-1}(F)) = f_\beta^{-1}(F) \times \prod_{\alpha \in I} X_\alpha$ is g-closed in $\prod_{\alpha \in I} X_\alpha$. By Theorem 4.1.5, $f_\beta^{-1}(F)$ is g-closed in Y_β. Hence f_β is gc-irresolute for each $\beta \in I$.

Theorem 4.2.10 *Let Y be a topological space and let $\{ X_\alpha \mid \alpha \in I \}$ be a family of topological spaces. Let $f : Y \to \prod_{\alpha \in I} X_\alpha$ be a function. If f is r-g-continuous, then the composite function $\pi_\alpha \circ f : Y \to X_\alpha$ is r-g-continuous for each $\alpha \in I$.*

Proof. Suppose that f is r-g-continuous. Since $\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha$ is continuous for each $\alpha \in I$, it follows by Theorem 3.3.7 that $\pi_\alpha \circ f$ is r-g-continuous for each $\alpha \in I$.

The converse of this theorem is not true as we will see in the following example.

Example 4.2.11 Let $X=\{1,2,3,4\}$, $\Im_X=\{\varnothing,\{1\},\{2\},\{1,2\},X\}$, $Y_1 = Y_2 = \{a,b\}$, $\Im_{Y_1} = \{\varnothing,\{a\},Y_1\}$, $\Im_{Y_2} = \{\varnothing,\{a\},Y_2\}$, $Y = Y_1 \times Y_2 = \{(a,a),(a,b),(b,a),(b,b)\}$ and $\Im_Y = \{\varnothing, Y_1 \times Y_2, \{(a,a),(a,b)\},\{(a,a),(b,a)\},\{(a,a)\},\{(a,a),(a,b),(b,a)\}\}$. Define $f : X \to Y$ by $f(1)=(b,a), f(2)=(b,b), f(3)=(a,b)$ and $f(4)=(a,a)$. It is easy to see that $\pi_1 \circ f$ and $\pi_2 \circ f$ are r-g-continuous. However, $\{(b,b)\}$ is closed in Y but $f^{-1}(\{(b,b)\}) = \{2\}$ is not r-g-closed in X. Therefore f is not r-g-continuous.

Theorem 4.2.12 *Let Y be a $T^*_{\frac{1}{2}}$-space and let $\{X_\alpha \mid \alpha \in I\}$ be a family of topological spaces. Let $f : Y \to \prod_{\alpha \in I} X_\alpha$ be a function. Then f is r-g-continuous if and only if the composite function $\pi_\alpha \circ f : Y \to X_\alpha$ is r-g-continuous for each $\alpha \in I$.*

Proof. ($\Rightarrow$) By Theorem 4.2.10.

($\Leftarrow$) Since Y is $T^*_{\frac{1}{2}}$-*space* and by Theorem 3.3.6, $\pi_\alpha \circ f$ is continuous for each $\alpha \in I$. By Theorem 2.3.4, f is continuous. Hence f is r-g-continuous.

Theorem 4.2.13 *Let* Y *be a topological space and let* $\{ X_\alpha \mid \alpha \in I \}$ *be a family of topological spaces. Let* $f : Y \to \prod_{\alpha \in I} X_\alpha$ *be a function. If* f *is r-g-irresolute, then the composite function* $\pi_\alpha \circ f : Y \to X_\alpha$ *is r-g-continuous for each* $\alpha \in I$.

Proof. Suppose f is r-g-irresolute. Since $\pi_\alpha : \prod_{\beta \in I} X_\beta \to X_\alpha$ is continuous for each $\alpha \in I$, we have that π_α is r-g-continuous for each $\alpha \in I$. It follows from Theorem 3.3.7 that $\pi_\alpha \circ f$ is r-g-continuous for each $\alpha \in I$.

Corollary 4.2.14 *Let* $\{ X_\alpha \mid \alpha \in I \}$ *and* $\{ Y_\alpha \mid \alpha \in I \}$ *be family of topological spaces. For each* $\alpha \in I$, *let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a function and* $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ *be defined by* $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. *If* f *is r-g-continuous, then* f_α *is r-g-continuous for each* $\alpha \in I$.

Proof. Let π_β and π'_β be the projection of $\prod_{\alpha \in I} X_\alpha$ and $\prod_{\alpha \in I} Y_\alpha$ onto X_β and Y_β, respectively. By Theorem 4.2.13, $\pi'_\beta \circ f$ is r-g-continuous for each $\beta \in I$. Since $\pi'_\beta \circ f = f_\beta \circ \pi_\beta$, we have that $f_\beta \circ \pi_\beta$ is r-g-continuous. Let F be closed in Y_β. Then $\pi_\beta^{-1}(f_\beta^{-1}(F)) = f_\beta^{-1}(F) \times \prod_{\alpha \in I} X_\alpha$ is r-g-closed in $\prod_{\alpha \in I} X_\alpha$. By Theorem 4.1.10, $f_\beta^{-1}(F)$ is r-g-closed in Y_β. Hence f_β is r-g-continuous for each $\beta \in I$.

Corollary 4.2.15 *Let* $\{ X_\alpha \mid \alpha \in I \}$ *be a famjly of* $T_{\frac{1}{2}}^*$ *− spaces and* $\{ Y_\alpha \mid \alpha \in I \}$ *be a family of topological spaces. For each* $\alpha \in I$, *let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a function and* $f : \prod_{\alpha \in I} X_\alpha \to \prod_{\alpha \in I} Y_\alpha$ *be defined by* $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. *Then* f *is r-g-continuous if and only if* f_α *is r-g-continuous for each* $\alpha \in I$.

Proof. ($\Rightarrow$) By Corollary 4.2.14.

($\Leftarrow$) Suppose that f_α is r-g-continuous for each $\alpha \in I$. Since X_α is a $T_{\frac{1}{2}}^{*}$ *– space* for each $\alpha \in I$, it follows from Theorem 3.3.6 that f_α is continuous for each $\alpha \in I$. Therefore by Theorem 2.3.5, f is continuous. Hence f is r-g-continuous.