CHAPTER V

PRESERVATION THEOREMS ON SOME TOPOLOGICAL SPACES

In this chapter, we define two new concepts of connected topological space called connected* and connected** spaces and study some properties of them and we investigate preservation theorems concerning of connected* and connected** spaces. Finally, we introduced concepts of g-Hausdorff and rg-Hausdorff spaces and we investigate preservation theorems concerning g-Hausdorff and rg-Hausdorff spaces.

5.1 Preservation Theorems Concerning Connected* and Connected** Spaces

In this section, we defined connected* and connected** spaces and study some properties of them. Finally, we investigate theorem concerning of connected* and connected** spaces.

Definition 5.1.1 A topological space X is said to be *connected** if X cannot be written as a disjoint union of two nonempty g-open subsets of X.

Theorem 5.1.2 *Let $(X,\Im)$ be any topological space. Then the following statement are equivalent.*

(a) X is connected.*

(b) X is not a disjoint union of two nonempty g-closed subset of X.

(c) The only subsets of X both g-open and g-closed are $\varnothing$ and X.

(*d*) *Let* $Y=\{0,1\}$ *have the discrete topology. Then there is no g-continuous function from X onto Y.*

Proof. $(a) \Rightarrow (b)$. Suppose $X = A \cup B$ where A and B are disjoint nonempty g-closed subsets of *X*. Then $X - A = B$ and $X - B = A$ are both the complements of g-closed sets, hence they are g-open. Thus $X = A \cup B$ is also the expression of X as a union of two disjoint nonempty g-open subsets of *X*. Hence X is not connected*.

$(b) \Rightarrow (c)$. Suppose A is a subset of X with is both g-open and g-closed but that A is neither X nor $\varnothing$. Then $X - A$ is also g-open and g-closed and nonempty. Thus $X = (X - A) \cup A$ is the expression of X as the union of two disjoint nonempty g-closed subsets, which contradicts to (b).

$(c) \Rightarrow (d)$. Suppose $f : X \to Y$ is a g-continuous onto mapping. Then $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are both nonempty. Thus $U \neq \varnothing$ and $U \neq X$. Since $\{0\}$ and $\{1\}$ are open subsets of Y and f is g-continuous, we have U and V are g-open subsets of *X*. But $U = V^c$, so U is both g-open and g-closed sets, which contradicts to (c).

$(d) \Rightarrow (a)$. Suppose X is not connected*. Then $X = U \cup V$ where U and V are nonempty disjoint g-open subsets of *X*. Define $f : X \to Y$ by $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \in V$. Then $f^{-1}(\{0\}) = U$ and $f^{-1}(\{1\}) = V$ hence f is g-continuous and onto which contradicts to (d).

Remark 5.1.3 It is obvious that every connected* space is connected. The next example shows that the converse is not true.

Example 5.1.4 Let $X=\{a,b,c\}$ and $\Im=\{\varnothing,\{a\},X\}$. Then the topological space $(X,\Im)$ is connected. However, since $\{c\}$ is both g-open and g-closed, $(X,\Im)$ is not connected* by Theorem 5.1.2 .

Theorem 5.1.5 *If X is $T_{\frac{1}{2}}-space$, then X is connected* if and only if X is connected.*

Proof. By Remark 5.1.3 we need only to show that if X is connected implies X is connected*. Suppose that X is not *connected**. Then $X=A\cup B$, where A and B are nonempty disjoint g-closed subsets of X. Since X is a $T_{\frac{1}{2}}-space$, A and B also closed subsets of X, therefore X is not connected.

Theorem 5.1.6 *If $f:X\to Y$ is g-continuous and onto and X is connected*, then Y is connected.*

Proof. Assume that $f:X\to Y$ is g-continuous and onto and X is connected*. Suppose Y is not connected. Let $Y=U\cup V$, where U and V are nonempty disjoint open subsets of Y. Then $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty disjoint g-open subsets of X whose union is X because f is g-continuous and onto. Therefore X is not connected* which contradicts to our assumption. Hence Y must be connected.

Theorem 5.1.7 *If $f:X\to Y$ is gc-irresolute and onto and X is connected*, then Y is connected*.*

Proof. Assume that $f: X \to Y$ is gc-irresolute and onto and X is *connected**. Suppose Y is not *connected**. Then there is a nonempty proper subset A of Y such that A is both g-open and g-closed . Since f is gc-irresolute and onto, it follows that $f^{-1}(A)$ is a nonempty proper subset of X which both g-open and g-closed. Hence X is not connected* which is a contradiction. Therefore Y is connected*.

Definition 5.1.8 A topological space X is said to be *connected*** if X cannot be written as a disjoint union of two nonempty r-g-open subsets of X.

Theorem 5.1.9 *Let* $(X, \Im)$ *be any topological space. Then the following statement are equivalent.*

*(a) X is connected**.*

(b) X is not a disjoint union of two nonempty r-g-closed subset of X.

(c) The only subset of X both r-g-open and r-g-closed are ∅ and X.

(d) Let $Y = \{0,1\}$ *have the discrete topology. Then there is no r-g-continuous function from X onto Y.*

Proof. $(a) \Rightarrow (b)$. Suppose $X = A \cup B$ where A and B are disjoint nonempty r-g-closed subsets of X. Then $X - A = B$ and $X - B = A$ are both the complements of r-g-closed sets, hence they are r-g-open. Thus $X = A \cup B$ is also the expression of X as a union of two disjoint nonempty r-g-open subsets of X. Hence X is not connected**.

$(b) \Rightarrow (c)$. Suppose A is a subset of X with is both r-g-open and r-g-closed but that A is neither X nor $\varnothing$. Then $X - A$ is also r-g-open and r-g-closed and nonempty. Thus $X = (X - A) \cup A$ is the expression of X as a union of two disjoint nonempty r-g-closed subsets, which contradicts to (b).

$(c) \Rightarrow (d)$. Suppose $f: X \to Y$ is a r-g-continuous onto mapping. Then $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are both nonempty. Thus $U \neq \varnothing$ and $U \neq X$. Since $\{0\}$ and $\{1\}$ are open subsets of Y and f is r-g-continuous, we have U and V are r-g-open subsets of X. But $U = V^c$, so U is both r-g-open and r-g-closed sets, which contradicts to (c).

$(d) \Rightarrow (a)$. Suppose X is not connected**. Then $X = U \cup V$ where U and V are nonempty disjoint r-g-open subsets of X. Define $f: X \to Y$ by $f(x) = 0$ if $x \in U$ and $f(x) = 1$ if $x \in V$. Then $f^{-1}(\{0\}) = U$ and $f^{-1}(\{1\}) = V$ hence f is r-g-continuous and onto which contradicts to (d).

Remark 5.1.10 It is obvious that every connected** space is connected*. The next example shows that the converse is not true.

Example 5.1.11 Let $X = \{a,b,c\}$ and $\Im = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$. Then the topological space $(X, \Im)$ is connected*. However, since $\{c\}$ is both r-g-open and r-g-closed, $(X, \Im)$ is not connected** by Theorem 5.1.9 .

Theorem 5.1.12 *If X is T_{rg}- space, then X is connected ** if only if is X connected*.*

Proof. By remark 5.1.10 we need only to show that if X is connected* implies X is connected**. Suppose that X is not connected**. Then $X = A \cup B$, where A and B are nonempty disjoint r-g-closed subsets of X. Since X is a $T_{rg}-space$, A and B also g-closed subsets of X, therefore X is not connected*.

Theorem 5.1.13 *If $f: X \to Y$ is r-g-continuous and onto and X is connected**, then Y is connected.*

Proof. Assume that $f: X \to Y$ is r-g-continuous and onto and X is connected**. Suppose Y is not connected. Let $Y = U \cup V$, where U and V are nonempty disjoint open subsets of Y. Then $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty disjoint r-g-open subsets of X whose union is X because f is r-g-continuous and onto. Therefore X is not connected** which contradicts to our assumption. Hence Y must be connected.

Theorem 5.1.14 *If $f: X \to Y$ is r-g-irresolute and onto and X is connected**, then Y is connected**.*

Proof. Assume that $f: X \to Y$ is r-g-irresolute and onto and X is connected**. Suppose Y is not connected**. Then there is a nonempty proper subset A of Y such that A is both r-g-open and r-g-closed. Since f is r-g-irresolute and onto, it follows that $f^{-1}(A)$ is a nonempty proper subset of X which both r-g-open and r-g-closed. Hence X is not connected** which a contradiction. Therefore Y is connected**.

5.2 Preservation Theorems Concerning g-Hausdorff and rg-Hausdorff Spaces

In this section, we define new concepts of Hausdorff spaces called g-Hausdorff and rg-Hausdorff spaces and investigate preservation theorems concerning g-Hausdorff and rg-Hausdorff spaces.

Definition 5.2.1 A space X is said to be *g-Hausdorff* if whenever x and y are distinct points of X, there are disjoint g-open sets U and V in X with $x \in U$ and $y \in V$.

It is obvious that every Hausdorff space is g-Hausdorff space. The following example shows that the converse is not true.

Example 5.2.2 Let $X = \{a,b,c\}$ and $\Im = \{\varnothing, \{a\}, X\}$. It clear that X is not Hausdorff. Since $\{a\}$, $\{b\}$ and $\{c\}$ are all g-open, it follow that $(X, \Im)$ is g-Hausdorff.

Theorem 5.2.3 *Let X be a topological space and Y be Hausdorff. If $f : X \to Y$ is injective and g-continuous, then X is g-Hausdorff.*

Proof. Let x and y be any two distinct points of X. Then $f(x)$ and $f(y)$ are distinct points of Y because f injective. Since Y is Hausdorff, there are disjoint open sets U, V in Y containing $f(x)$ and $f(y)$ respectively. Since f is g-continuous and $U \cap V = \varnothing$, we have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint g-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is a g-Hausdorff.

Theorem 5.2.4 *Let X be a topological space and Y be g-Hausdorff. If $f : X \to Y$ is injective and gc-irresolute, then X is g-Hausdorff.*

Proof. Let x and y be any two distinct points of X. Then $f(x)$ and $f(y)$ are distinct points of Y because f injective. Since Y is g-Hausdorff, there are disjoint g-open sets U, V in Y containing $f(x)$ and $f(y)$ respectively. Since f is gc-irresolute and $U \cap V = \varnothing$, we have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint g-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is a g-Hausdorff.

Definition 5.2.5 A space X is said to be *rg-Hausdorff* iff whenever x and y are distinct points of X, there are disjoint r-g-open sets U and V in X with $x \in U$ and $y \in V$.

It is obvious that every g-Hausdorff space is rg-Hausdorff space. The following example shows that the converse is not true.

Example 5.2.6 Let $X = \{a,b,c\}$ and $\Im = \{\varnothing, \{a\}, \{a,b\} X\}$. Since $\{a\},\{b\}$ and $\{c\}$ are all r-g-open sets, it implies that $(X,\Im)$ is rg-Hausdorff. Since $\{c\}$ and $\{b,c\}$ are not g-open in *X*, it follows that a and c cannot be separated by any two g-open sets in X. Hence $(X,\Im)$ is not g-Hausdorff.

Theorem 5.2.7 *Let X be a topological space and Y be Hausdorff. If $f: X \to Y$ is injective and r-g-continuous, then X is rg-Hausdorff.*

Proof. Let x and y be any two distinct points of *X*. Then $f(x)$ and $f(y)$ are distinct points of Y because f injective. Since Y is Hausdorff, there are disjoint open sets U, V in Y containing $f(x)$ and $f(y)$ respectively. Since f is r-g-continuous and $U \cap V = \varnothing$, we have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint r-g-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is a rg-Hausdorff.

Theorem 5.2.8 *Let X be a topological space and Y be rg-Hausdorff. If $f: X \to Y$ is injective and r-g-irresolute, then X is rg-Hausdorff.*

Proof. Let x and y be any two distinct points of *X*. Then $f(x)$ and $f(y)$ are distinct points of Y because f injective. Since Y is rg-Hausdorff, there are disjoint r-g-open sets *U, V* in Y containing $f(x)$ and $f(y)$ respectively. Since f is r-g-irresolute and $U \cap V = \varnothing$, we have $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint r-g-open sets in X such that $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Hence X is a rg-Hausdorff.