CHAPTER II

PRELIMINARIES

We begin with some basic knowledge of module theory as well as some elementary results.

Throughout the thesis, all rings are associative with identity and all right modules are unital.

1. Essential and co-essential submodules.

We now consider some special classes of submodules in a module M.

2.1.1 Definition. Let M be a right R-module. A submodule N of M is called *essential* or *large* in M if it has non-zero intersection with any non-zero submodule of M. If N is essential in M we will write $N \subset^e M$.

Dually, a submodule N is called *co-essential* or *small* in M if for any submodule X of M such that $N + X = M$, then $X = M$. For a co-essential submodule N of M we will write $N \subset^o M$.

2.1.2 Proposition. *A submodule K of M is essential in M if and only if for any $0 \neq m \in M$ there exists an $r \in R$ such that $0 \neq mr \in K$.*

Proof.($\Longrightarrow$) For any $0 \neq m \in M$, we have $0 \neq mR \subset M$. Since K is essential in M, we get $K \cap mR \neq 0$. Hence there exists a $0 \neq x \in K \cap mR$, i.e., $0 \neq x = mr \in K$ for some $r \in R$.

($\Longleftarrow$) Let N be any non-zero submodule of M. Take any $0 \neq x \in N$. Then $K \cap xR \neq 0$, since $0 \neq xr \in K$ for some $r \in R$ by assumption. Hence $K \cap N \neq 0$ and therefore K is essential in M. □

2.1.3 Proposition. *Let M be a right R-module. A submodule K of M is essential in M if and only if for any non-zero element $x \in M$, there is an essential right ideal L of R such that $0 \neq xL \subset K$.*

Proof.($\Longrightarrow$) Let K be an essential submodule of M. Take $x \in K$. Let $L = \{r \in R \mid xr \in K\}$. Then by Proposition 2.1.2, $L \neq 0$. We claim that L is essential in R. For an element $y \in LR$, we can write $y = \Sigma_{i=1}^{n} l_i r_i$, where $l_i \in L, r_i \in R$, $1 \leq i \leq n$ for some $n \in N$.

Consider $xy = x(\Sigma_{i=1}^{n} l_i r_i) = \Sigma_{i=1}^{n} x(l_i r_i) = \Sigma_{i=1}^{n} x(l_i) r_i$. Since $xl_i \in K$, we have $(xl_i)r_i \in K$ for all i= 1, 2, 3,..., n. Thus $xy \in K$, hence $y \in L$. This shows that $LR \subset L$, i.e., L is a right ideal of R. Since for any $a \in eL$, we have $a = eL \in K$. It follows that $xL \subset K$. Since $K \subset^e M$, then there exists an $r \in R$ such that $0 \neq xr \in K$, by Proposition 2.1.2. It follows that $xL \neq 0$. Let P be a non-zero right ideal of R. We want to show that $P \cap L \neq 0$. In fact, if $xP = 0$, then $P \subset L$. Hence $P \cap L = P \neq 0$. If $xP \neq 0$, then $xP \cap K \neq 0$. It means that there exists an element $r \in R$ such that $0 \neq xr \in K$. Hence $0 \neq r \in P \cap L$, showing that $L \subset^e R$.

($\Longleftarrow$) Let N be a non-zero submodule of M. Take any $0 \neq n \in N$. By assumption, $0 \neq nL \subset K$ for some essential right ideal L of R. Hence $0 \neq nL \subset N \cap K$. It follows that K is essential in M. □

2.1.4 Definition. A submodule N of M is called *closed* in M if it has no essential extensions but itself. In other words, a submodule N of M is closed in

M if for any submodule X of M containing N such that N is essential in X we must have $X = N$. We see that every direct summand of M is closed in M.

Let M be a right R-module and N its submodule. A submodule K of M is called a *complement* of N in M if K is maximal among submodules Q of M such that $Q \cap N = 0$. By Zorn's Lemma, every submodule of any module M has at least a complement in M. We define that a submodule K of M is called a *complement in* M if there is a submodule N of M such that K is a complement of N in M.

2.1.5 Proposition. *Let $K \subset L$ be submodules of M such that K is a complement in L and L is a complement in M. Then K is a complement in M.*

Proof. See [13, Lemma 1.2]. □

2.1.6 Proposition. *Let N, L be submodules of M such that $N \cap L = 0$. Then:*

(1) There is a complement K of N in M such that $L \subset K$;

(2) $K \oplus N$ is essential in M;

(3) K is closed in M.

Proof. See [3, 1.10]. □

2.1.7 Proposition. *Let K be a submodule of M and L a complement of K in M. Then K is closed if and only if K is a complement of L in M. Especially, a submodule K of M is closed if and only if K is a complement in M.*

Proof. See [3, 1.10]. □

2.1.8 Proposition. *Let K, L, N be submodules of M such that $K \subset L$. Then:*

(1) There is a closed submodule H of M such that N is essential in H.

(2) If L is closed in M, then L/K is closed in M/K.

Proof. See [3, 1.10]. □

2.1.9 Proposition. *Let K be a complement in M. Then K is a direct summand of M if and only if there is a complement L of K in M such that every homomorphism $\varphi : K \oplus L \longrightarrow M$ can be lifted to a homomorphism $\theta : M \longrightarrow M$.*

Proof. One direction is clear. Assume that there is a complement L of K in M which satisfies the above property. Let $\varphi : K \oplus L \longrightarrow M$ defined by $\varphi(x+y) = x$ $(x \in K, y \in L)$. By hypothsis, there exists a homomorphism $\theta : M \longrightarrow M$ such that $\theta(x + y) = x(x \in K, y \in L)$. Clearly, $K \subset \mathrm{Im}\theta$ and $L \subset \mathrm{Ker}\theta$.

Let $0 \neq v \in \mathrm{Im}\theta$, then there exists $u \in M$ such that $v = \theta(u), u \notin L$. Therefore $K \cap (L + uR) \neq 0$. Hence there exist $x \in K$, $y \in L$ and $r \in R$ such that $0 \neq x = y + ur$, then $x = \theta(x) = \theta(y + ur) = vr$. It follows that $vR \cap K \neq 0$ for every $v \neq 0, v \in \mathrm{Im}\theta$. This implies that K is essential in $\mathrm{Im}\theta$. Because K is a complement in M, hence $K = \mathrm{Im}\theta$. It is easy to verify that $M = K \oplus \mathrm{Ker}\theta$. Hence K is a direct summand of M. □

2. Socles and radicals of modules

2.2.1 Definition. Let M be a right R-module. The socle of M, denoted by $\mathrm{Soc}(M)$, is the sum of all simple submodules of M. It was shown that the socle of M is the intersection of all essential submodules of M. Dually, the radical of M denote by $\mathrm{rad}(M)$ is the intersection of all maximal submodules of M. This is the sum of all small submodules of M (see [1]).

2.2.2 Proposition. *If* $(M_\alpha)_{\alpha\in A}$ *is a family of submodules of* M *with* $M=\bigoplus_A M_\alpha$, *then* $Soc(M)=\bigoplus_A Soc(M_\alpha)$ *and* $Rad(M)=\bigoplus_A Rad(M_\alpha)$.

Proof. By [1, Proposition 9.19]. □

2.2.3 Lemma. *If* S *is the socle of a direct sum* $\bigoplus_{\alpha\in K} N_\alpha$, *then* $S=\bigoplus_{\alpha\in K} Soc(N_\alpha)$. *Hence*

$$(\bigoplus_{\alpha\in K} N_\alpha)/S \cong \bigoplus_{\alpha\in K}(N_\alpha/soc(N_\alpha)).$$

Proof. See [7, Folgerung 9.1.5]. □

2.2.4 Lemma. *Let* M *be a module and* $S=Soc(M)$, *then:*

(1) If A *and* B *are submodules of* M *with* $A\cap B=0$, *then*

$$((A+S)/S)\cap((B+S)/S)=0;$$

(2) If A *is a direct summand of* M, *then* $(A+S)/S$ *is a direct summand of* M/S;

(3) If $\bigoplus_{i\in I} A_i$ *is a direct sum of submodules of* M, *then* $\bigoplus_{i\in I}((A_i+S)/S)$ *is also a direct sum of submodules in* M/S.

Proof. This proof is given by Nguyen Viet Dung [12] and we present it here for the sake of completeness.

Let $f: M\to M/S$ be the canonical map.

(1) Suppose that A and B are submodules of M with $A\cap B=0$. Set $\bar{V}=f(A)\cap f(B)$. There exists a submodule V of A such that $f(V)=\bar{V}$. Clearly $V\subseteqq B+S=B\oplus T$ for some submodule T of S. Since $V\cap B=0$, it follows that V is isomorphic to a submodule of T. Hence $V\subseteqq S$ which implies that $\bar{V}=0$. Therefore we have $f(A)\cap f(B)=0$.

(2) Let A be a direct summand of M. Then $M = A \oplus B$ for some submodule B of M. Clearly $M/S = f(A) + f(B)$. By (1) we have $f(A) \cap f(B) = 0$. Thus $f(A)$ is a direct summand of M/S.

(3) This is an immediate consequence of (1). □

3. Semisimple modules

2.3.1 Definition. A right R-module M is semisimple if and only if Soc(M)=M.

2.3.2 Proposition. *For a right R-module the following statements are equivalent:*

(a) M is semisimple;

(b) M is the sum of simple submodules;

(c) M is the direct sum of its simple submodules;

(d) Every submodule of M is a direct summand;

(e) Every short exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

of right R-modules splits.

Proof. From [1, Theorem 9.6]. □

4. Finitely generated and finitely cogenerated modules

A module M is said to be *finitely generated* if it is generated by a finite number of its elements, i.e., there exist elements m_1,...,m_n of M such that

$M = \Sigma_{i=1}^{n} m_i R$ for some positive integer n. The following property of finitely generated module is useful :

2.4.1 Lemma. *A right R-module M is finitely generated if and only if for any family submodules $\{M_i,\ i \in I\}$ such that $M = \sum_{i \in I} M_i$, then there exits a finite subset I_o of I such that $M = \sum_{i \in I_o} M_i$.*

Proof. Since $M = \sum_{x \in M} xR$, then one direction is clear. Let M be a finitely generated module with spanning set $\{x_1, x_2, ..., x_n\}$. Then each x_i, $(1 \leq i \leq n)$ is contained in a finite sum $\sum_{x_j \in I_i} M_j$. Put $I_o = \bigcup_{1 \leq i \leq n} I_i$, we get the result. □

The above property of finitely generated module leads to the following definition that we say here for the duality.

2.4.2 Definition. A module M is called *finitely cogenerated* if for any family $\{M_i, i \in I\}$ of submodules of M such that $\bigcap_{i \in I} M_i = 0$, there exists a finite subset I_o of I such that $\bigcap_{i \in I_o} M_i = 0$.

2.4.3 Definition. A submodule A of a module M is called a *direct summand* of M if there exists a submodule B of M such that $A + B = M$ and $A \cap B = 0$. We will write $M = A \oplus B$.

2.4.4 Definition. Let M be a right R-module. A family $\{X_i\}_{i \in I}$ of submodules of M is called *independent* if for every $j \in I$ we have $X_j \cap \sum_{i \in I \setminus \{j\}} X_i = 0$. M is a direct sum of the family $\{X_i\}_{i \in I}$ of submodules of M if $M = \sum_{i \in I} X_i$ and the family $\{X_i\}_{i \in I}$ is independent. In case M is a direct sum of the family $\{X_i\}_{i \in I}$ of submodules of M, we say that $\bigoplus_{i \in I} X_i$ is a decomposition of M into direct summands.

2.4.5 Lemma. *Any direct summand of a finitely generated modules is again finitely generated. Especially, if M is finitely generated, then any decomposition of M into direct summands is finite.*

Proof. It follows from Lemma 2.4.1. □

5. Noetherian and Artinian modules

An ascending chains of submodules of a module M is a family $\{A_i \mid i \in I\!N\}$ of submodules such that

$$A_1 \subset A_2 \subset ... \subset A_n \subset A_{n+1} \subset ...$$

An ascending chain $\{A_i \mid i \in I\!N\}$ of submodules of M is said to be *stationary* if there is an $n_0 \in I\!N$ such that $A_n = A_{n+1}$ for all $n \geq n_0$.

Similarly, a descending chains of submodules of a module M is a family $\{A_i \mid i \in I\!N\}$ of submodules such that

$$A_1 \supset A_2 \supset A_3 \supset ... \supset A_n \supset A_{n+1} \supset ...$$

The descending chain $\{A_i \mid i \in I\!N\}$ of submodules of M is called *stationary* if there is an $n_0 \in N$ such that $A_n = A_{n+1}$ for all $n \geq n_0$.

2.5.1 Definition. A right R-module M is called *Noetherian* if any ascending chain of submodules of M is stationary. A ring R is called *right Noetherian* if it is noetherian as a right R-module. Dually, a module M_R is called *Artinian* if every descending chain of submodules is stationary. A ring R is *right Artinian* if R_R is Artinian.

2.5.2 Proposition. *For a module M the following statements are equivalent:*

(a) M is Noetherian;

(b) Every submodule of M is finitely generated;

(c) Every non-empty set of submodules of M has a maximal element.

Proof. See [1, Proposition 10.9]. □

2.5.3 Proposition. *For a module M the following statements are equivalent:*

(a) M is Artinian;

(b) Every factormodule of M is finitely cogenerated;

(c) Every non-empty set of submodules of M has a minimal element.

Proof. See [1, Proposition 10.10]. □

Let $\mathcal{C}$ be a class of submodules of M. We say that M satisfies acc (resp. dcc) on $\mathcal{C}$ if every ascending (descending) chain of elements of $\mathcal{C}$ (ordered by inclusion) is stationary. The following proposition gives us an image.

2.5.4 Proposition. *Let M be a non-zero module that satisfies acc or dcc on direct summands. Then M is the direct sum*

$$M = M_1 \oplus \ldots \oplus M_n,$$

for a finite set of indecomposable submodules.

Proof. See [1, Proposition 10.14]. □

2.5.5 Proposition. *For a semisimple module M the following statements are equivalent:*

(a) M is Artinian;

(b) M is Noetherian;

(c) M is finitely generated;

(d) M is finitely cogenerated.

Proof. See [1, Corollary 10.16]. □

2.5.6 Lemma. *If M satisfies acc (resp. dcc) on direct summands, then every direct summand of M also has acc (resp. dcc) on direct summands.*

Proof. Let K be a direct summand of M and $\{N_i \mid i \in \mathbb{N}\}$ be an ascending (descending) chain of direct summands in K. Then $\{N_i \mid i \in \mathbb{N}\}$ is the ascending (descending) chain of direct summands in M and the lemma follows. □

6. UC-modules

Recall that a submodule K of M is closed (in M) provided K has no proper essential extension in M. We will denote such a module K of M by $K \subset_C M$.

By Zorn's Lemma, every submodule N of M is essential in a closed submodule K of M. We call K a *closure* of N in M. It is clear that a submodule N of M may have many closures in M. From this it leads to the following definition.

2.6.1 Definition. Let M be a right R-module. A module M is called *UC-modules* if every submodule has a unique closure.

2.6.2 Proposition. *The following statments are equivalent for a module M.*

(1) M is a UC-module;

(2) If $K \subset_C M$, $N \subset M$ then $K \cap N \subset_C N$;

(3) If $K \subset_C M$, $L \subset_C M$ then $K \cap L \subset_C M$;

(4) M/K is a UC-module for any $K \subset_C M$.

Proof. See [14, Lemma 6]. □

7. Injective and projective modules

It is well-known that the following conditions are equivalent for a map $f : A \to B$, where A, B are sets.

(1) f is an injective map, i.e., f is 1-1;

(2) for any set C and map $h, g : C \to A$, we have $fh = fg$ implies $h = g$;

(3) there exists $f' : B \to A$ such that $f'f = 1_A$.

When we equip a structure for A and B, one can ask the question that "When are the above conditions still equivalent ?", for example, when A, B are abelian groups and f is a group homomorphism. Clearly (1) $\Leftrightarrow$ (2) and (3) $\Rightarrow$ (1), but the converse (1) $\Rightarrow$ (3) is not true. An abelian group A such that (1) $\Rightarrow$ (3) is true for all abelian group B is called a ***divisible group***. An abelian group A is divisible if $Az = A$ for all $0 \neq z \in Z$. Now we consider the same properties for right R-modules A, B and R-homomorphism $f : A \to B$. This leads to the notion of injective modules as follows.

2.7.1 Definiton. Let M be a right R-module. A module N is said to be M-*injective* if for any monomorphism $f : X \longrightarrow M$ and homomorphism $\varphi : X \longrightarrow N$ there exists a homomorphism $\bar{\varphi} : M \longrightarrow N$ such that $\varphi = \bar{\varphi} f$.

A module N is called an *injective* module if it is R - injective. It is well-known that a module N is injective(=R-injective) if it is M-injective for every module M of Mod-R (Baer's Criteria).

A module M is called *self-injective or quasi-injective* if it is M-injective. We note that a submodule N of a module M is called an *invariant* submodule of M if $f(N) \subset N$ for any endomorphism $f : M \longrightarrow M$. Any invariant submodule of a quasi-injective module is again quasi-injective (see [10]).

2.7.2 Proposition. *Let $M = M_1 \oplus M_2$, then the following conditions are equivalent:*

(1) M_2 is M_1-injective;

(2) For every submodule N of M which satisfies $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \subset M'$.

Proof. See [3, Lemma 7.5]. □

Similarly to the above definition of injective, we see that for a map $f : A \rightarrow B$ where A, B are sets, the following conditions are equivalent:

(1) f is surjective (onto);

(2) for any set C and map $h, g : B \rightarrow C$, we have $hf = gf$ implies $h = g$;

(3) there exists a map $f' : B \rightarrow A$ such that $ff' = 1_B$.

It is also clear that (3) $\Rightarrow$ (1) $\Leftrightarrow$ (2) in case that A, B are right R-module and f is an R-homomorphism. The question is that "what is the property of B such that (1) $\Rightarrow$ (3) ?". The structure of projective modules answers this question.

2.7.3 Definition. Let M be a right R-module. A module N is said to be *M-projective* if for any epimorphism $g : M \longrightarrow X$ and any homomorphism $\psi : N \longrightarrow X$, there exists a homomorphism $\bar{\psi} : N \longrightarrow M$ such that $\psi = g\bar{\psi}$.

If M is M-projective, we will say that M is *quasi-projective.* Unlike injectivity, if N is R-projective, it needs not to be M-projective for all M in Mod-R. Therefore we define that a module P is called a *projective* module if it is M-projective for any M in Mod-R. For a given module M, $\sigma[M]$ is denoted for the full subcategory of Mod-R whose object are submodules of M-generated modules. A module M is said to be projective in $\sigma[M]$ if it is X-projective for any $X \in \sigma[M]$. If M is projective in $\sigma[M]$, then it is quasi-projective. Conversely, if M is finitely generated and quasi-projective, then it is projective in $\sigma[M]$ (see [3]).

It is interesting to see that if a module P is quasi-projective and K is an invariant submodule of P, then P/K is also quasi-projective.

8. CS-modules, quasi-continuous and continuous modules

Let M be a right R-module. It is well-knowm that there exists an injective right R-module, denoted by $E(M)$, such that there is a monomorphism $\xi : M \rightarrow E(M)$ with $\xi(M) \subset^e E(M)$. Such a module is called the *injective hull* of M in Mod-R. The existence of M-injective hull of $N \in \sigma[M]$ shows that there exist quasi-injective modules in Mod-R. It is interesting to see that for a quasi-injective module, the following conditions are satisfied:

(C_1) Every submodule of M is essential in a direct summand of M;

(C_2) Every submodule of M which is isomorphic to a direct summand of

M is again a direct summand of M;

(C_3) If M_1 and M_2 are direct summands of M such that $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

A module M is called *continuous* if it satisfies the conditions (C_1) and (C_2). It is called *quasi-continuous* if it satisfies (C_1) and (C_3); and CS-module if it satisfies (C_1) only.

From the above definitions, we infer the implications:

injective $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ CS.

It is well-known that every direct summand of a quasi-injective module is again quasi-injective. Here, the condition $C_i (i = 1, 2, 3)$ is also inherited to any direct summand and we have the following proposition.

2.8.1 Proposition. *The conditions (C_i) i = 1, 2, 3, are closed under direct summands. In other words:*

Every direct summand of a CS-module is again CS;

Every direct summand of a quasi-continuous module is again quasi- continuous;

Every direct summand of a continuous module is again continuous.

Proof. See [8, Prop. 2.7]. □

2.8.2 Proposition. *Module M satisfies(C_3) if and only if for every direct sum $K = K_1 \oplus K_2$ of direct summands K_1 and K_2 of M, every homomorphism $\varphi : K \longrightarrow M$ can be lifted to a homomorphism $\theta : M \longrightarrow M$.*

Proof. See [15, Lemma 3.3.2]. □

9. CESS-modules

2.9.1 Definition. A module M is called a *CESS-module* if every complement with essential socle is a direct summand, equivalently, every submodule with essential socle is essential in a direct summand of M, or again every closure of every semisimple submodule is a direct summand of M.

By [2], CS-modules are CESS-modulus. Modules with zero socle are CESS. From [2], we can find some examples which shows that CESS-module needs not to be CS.

10. Annihilators and singular modules

Let M be a right R-module. For an element $m \in M$, we denote $r_R(m) = \{r \in R \mid mr = 0\}$, the right annihilator of m, and $r_R(M) = \{r \in R \mid mr = 0$ for all $m \in M\}$, the right annihilator of M.

Let I be a two-sided ideal of R. Then R/I is a ring with identity. Clearly, if M is a right $\bar{R}$-module, where $\bar{R} = R/I$, then M can be considered as a right R-module. The converse requires the condition that $I \subset r_R(M)$, i.e., if M is a right R-module and I is a two-sided ideal of R such that $I \subset r_R(M)$, then M can be considered as a right $\bar{R}$-module, where $\bar{R} = R/I$.

Let M be a right R-module. Consider the set of all elements $x \in M$ such that $r_R(x)$ is essential in R. We denote this set by $Z_R(M)$. This is a submodule of M and we called it the *singular submodule* of M. When $Z_R(M) = M$, we will say that M is a singular module. M is called a *non-singular module* if $Z_R(M) = 0$. The following properties are given in [5] :

2.10.1 Proposition. *Let A, B, C be submodules of a non-singular module M. Suppose that A is a submodule of $B \cap C$ and C is a direct summand of M. If A is essential in C and essential in B, then B must be contained in C and therefore B is essential in C.*

Proof. See [10, p. 693]. □

Modifying the properties of singular modules, Wisbauer transfered this notion to the category $\sigma[M]$, for a given right R-module M. Let M and N be right R-module. N is called singular in $\sigma[M]$ or M-singular if there exists a module L in $\sigma[M]$ containing an essential submodule K such that $N \cong L/K$ (see [3]). By definition, every M-singular module belongs to $\sigma[M]$. For $M = R_R$ the notion R-singular is identical to the usual definition of singular right R-module.

The class of all M-singular modules is closed under submodules, homomorphic images and direct sums (see [3]). Hence every module $N \in \sigma[M]$ contains a largest submodule which we denote by $Z_M(N)$. The following properties of M-singular modules are well-known :

2.10.2 Proposition. *Let M be an R-module.*

(1) A simple R-module E is M-singular or M-projective.

(2) If $Soc(M) = 0$, then every simple module in $\sigma[M]$ is M-singular.

Proof. See [3, Proposition 4.2]. □