CHAPTER II

PREMILINARIES

In this Chapter, we give some definitions, notations and theorems that will be used in later chapters.

Thoughout this thesis, our scalar field is the field of real numbers $\mathbb{R}$ and we let $\mathbb{N}$ denote the set of all natural numbers. We note in passing that all results apply to the complex field $\mathbb{C}$ as well.

2.1 Suprema and infima.

Definition 2.1.1 Let S be a subset of $\mathbb{R}$.

(i) An element $u \in \mathbb{R}$ is said to be an *upper bound* of S if $s \leq u$ for all $s \in S$. In this case, we may say that S is *bounded above.*

(ii) An element $w \in \mathbb{R}$ is said to be a *lower bound* of S if $w \leq s$ for all $s \in S$.

Similarly, in this case, we may say that S is *bounded below.*

(iii) S is said to be *bounded* if it is bounded above as well as bounded below.

Definition 2.1.2 Let S be a subset of $\mathbb{R}$

(i) If S is bounded above, then an upper bound is said to be a *supremum* (or a *least upper bound*) of S if it is not greater than any other upper bounds of S.

(ii) If S is bounded below, then a lower bound is said to be *infimum* (or a *greatest lower bound*) of S if it is greater than every other lower bound of S.

2.2 Sequence and series

Definition 2.2.1 Let (x_n) be a sequence of real numbers. We say that (x_n) *approaches the limit* x in $\mathbb{R}$ if for any $\varepsilon > 0$, there is a positive integer N such that

$$|x_n - x| < \varepsilon \qquad \text{for all } n \geq N.$$

Chiang Mai University

We write $\lim_{n \to \infty} x_n = x$ or $\lim (x_n) = x$ or $x_n \to x$. Note that, whenever the limit exists, it is unique.

Definition 2.2.2 If (x_n) is a sequence of real numbers having the limit x, we say that (x_n) is a *convergent* sequence. If (x_n) does not have a limit, we say that (x_n) is a *divergent* sequence.

Definition 2.2.3 A sequence (x_n) of real numbers is said to be *bounded* if its range $\{ x_n : n \in \mathbb{N} \}$ is bounded.

Remark 2.2.4 Let (x_n) and (y_n) be two sequences of real numbers such that $\lim (x_n) = x$ and $\lim (y_n) = y$.

(i) If $\alpha, \beta \in \mathbb{R}$, then $\lim (\alpha x_n + \beta y_n) = \alpha x + \beta y$.

(ii) If $x_n \le y_n$ for all $n \in \mathbb{N}$, then $x \le y$. See [1], [8] for proofs.

Definition 2.2.5 Let (x_n) be a sequence of real numbers. We say that (x_n) is *increasing* if it satisfies the inequalities

$$x_1 \le x_2 \le \dots \le x_n \le x_{n+1} \le \dots .$$

We say that (x_n) is *decreasing* if it satisfies the inequalities

$$x_1 \ge x_2 \ge \dots \ge x_n \ge x_{n+1} \ge \dots .$$

We say that (x_n) is *monotone* if it is either increasing, decreasing.

Theorem 2.2.6 (See [1], page 89) Let (x_n) be a bounded sequence of real numbers.

(i) If (x_n) is an increasing sequence, then $\lim (x_n) = \sup\{x_n : n \in \mathbb{N}\}$

(ii) If (x_n) is a decreasing sequence, then $\lim (x_n) = \inf\{x_n : n \in \mathbb{N}\}$

Definition 2.2.7 Let (x_n) be a sequence of real numbers and let $r_1 < r_2 < \dots < r_n < \dots$ be a strictly increasing sequence of natural numbers. Then the sequence in $\mathbb{R}$ given by

$$(x_{r_1}, x_{r_2}, x_{r_3}, \dots, x_{r_n}, \dots)$$

is called a *subsequence* of (x_n) .

Theorem 2.2.8 (See [1], page 97) Let (x_n) be a sequence of real numbers. Then the following statements are equivalent :

(i) The sequence (x_n) does not converge to $x \in \mathbb{R}$.

(ii) There exists an $\varepsilon_o > 0$ such that for any $k \in \mathbb{N}$, there exists $r_k \in \mathbb{N}$ such that $r_k \geq k$ and $|x_{r_k} - x| \geq \varepsilon_o$.

(iii) There exists an $\varepsilon_o > 0$ and a subsequence (x_{r_n}) of (x_n) such that $|x_{r_n} - x| \geq \varepsilon_o$ for all $n \in \mathbb{N}$.

Theorem 2.2.9 (See [1], page 98) A bounded sequence of real numbers has a convergent subsequence.

Definition 2.2.10 A sequence (x_n) of real numbers is said to be a *Cauchy* sequence if for every $\varepsilon > 0$ there is a natural number N such that for all natural number $n, m \geq N$, we have $|x_n - x_m| < \varepsilon$.

Theorem 2.2.11 (See [1], page 102) A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Definition 2.2.12 Let (x_n) be a real sequence. Let $S_n = x_1 + x_2 + \dots + x_n$. Then (S_n) is called the sequence of *partial sums* of the infinite *series* $\sum_{n=1}^{\infty} x_n$ (or we write $\Sigma_n x_n$).

Definition 2.2.13 The infinite series $\sum_{n=1}^{\infty} x_n$ is said to be *convergent* if the sequence (S_n) of its partial sums is convergent. If $\lim (S_n) = S$ then S is called the sum of the series $\sum_{n=1}^{\infty} x_n$ and we write $S = \sum_{n=1}^{\infty} x_n$. The series $\sum_{n=1}^{\infty} x_n$ is said to be *divergent* as the sequence (S_n) of its partial sums is divergent.

Remark 2.2.14 If $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge respectively to x and y, and α and β are two real numbers, then $\sum_{n=1}^{\infty} (\alpha x_n + \beta y_n) = \alpha x + \beta y$. See [8] for proofs.

2.3 Metric spaces and normed linear spaces.

Definition 2.3.1 Let X be a set and let $d : X \times X \to \mathbb{R}$ be a function. If d satisfies the following conditions, then we say that d is a *metric* on X and call the pair (X, d) a *metric space*.

(i) $d(x, y) \geq 0$ for all $x, y \in X$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 2.3.2 Let (X, d) be a metric space. A sequence (x_n) of members of X converges to $x \in X$ if $\lim_{n \to \infty} d(x_n, x) = 0$. When (x_n) converges to x, we write $\lim_{n \to \infty} x_n = x$ or $x_n \to x$. Note that, whenever the limit exists, it is unique.

Definition 2.3.3 A sequence (x_n) in a metric space is called a *Cauchy* sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that, if $m \geq N$ and $n \geq N$, then $d(x_n, x_m) < \varepsilon$.

Definition 2.3.4 A metric space said to be *complete* if every Cauchy sequence in X converges. Note that, $\mathbb{R}$ is complete.

Definition 2.3.5 Let X = (X, d) and $Y = (Y, \bar{d})$ be metric spaces. A mapping $T : X \to Y$ is said to be *continuous* at a point $x_o \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\bar{d}(Tx, Tx_o) < \varepsilon$ for all x satisfying $d(x, x_o) < \delta$. T is said to be *continuous* if it is continuous at every point of X.

Theorem 2.3.6 (See [5], page 30) A mapping $T : X \to Y$ of a metric space (X, d) into a metric space $(Y, \bar{d})$ is continuous at a point $x_o \in X$ if and only if $x_n \to x_o$ implies $Tx_n \to Tx_o$.

Definition 2.3.7 Let X be a linear space (or a vector space). A *norm* on X is a non-negative real - valved function, written $\|\cdot\|$, such that

(i) $\|x\| = 0$ if and only if $x = 0$

(ii) $\|ax\| = |a| \, \|x\|$ for all $a \in \mathbb{R}$ and $x \in X$, and

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

A linear space X furnished with a norm $\|\cdot\|$ is called a *normed linear* space. Every normed linear space gives rise to the metric $d(x, y) = \|x - y\|$. The norm properties easily show that this is a metric on X. It is called the metric induced by norm.

Definition 2.3.8 A *Banach* space is a complete normed linear space.

2.4 Linear Operators.

Definition 2.4.1 Let X and Y be linear spaces. Let $T : X \to Y$. If

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty \quad \text{for all } \alpha, \beta \in \mathbb{R} \text{ and } x, y \in X,$$

we say T is a *linear operator* or a *linear transformation* from X into Y. When $Y = \mathbb{R}$, we say that T is a *linear functional* on X.

Theorem 2.4.2 (See [2], page 519) Let X and Y be normed linear spaces, and let $T : X \to Y$ be a linear operator. If T is continuous at a point, then T is continuous everywhere, and the continuity is uniform.

Definition 2.4.3 A linear operator $T : X \to Y$ is *bounded* if there exists $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$. The operator norm for a bounded linear operator T is defined as $\|T\| = \sup\limits_{\substack{x \in X \\ x \neq 0}} \dfrac{\|Tx\|}{\|x\|}$

Theorem 2.4.4 (See [2], page 520) A linear operator is bounded if and only if it is continuous.

Definition 2.4.5 Let X be a normed linear space. Then the set of all bounded linear functional on X constitutes a normed linear space with norm defined by $\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$ which is called the *dual space of X* and is denoted by X'.

2.5 Strong and weak convergence.

Definition 2.5.1 A sequence (x_n) in a normed linear space X is said to be *strongly convergent* (or *convergent in the norm*) if there is an $x \in X$ such that $\lim_{n \to \infty} x_n = x$ (or $x_n \to x$).

Definition 2.5.2 A sequence (x_n) in a normed linear space X is said to be *weakly convergent* if there is an $x \in X$ such that for every $T \in X'$, $\lim_{n \to \infty} Tx_n = Tx$. This is written as $x_n \xrightarrow{w} x$.

Remark 2.5.3 (See [5], page 259) Strong limits imply weak limits

2.6 Convex functions.

Definition 2.6.1 A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \tag{2.1}$$

for all $x, y \in \mathbb{R}$. If, in addition, the two sides of (2.1) are not equal for all $x \neq y$, then we call f *strictly convex*.

Definition 2.6.2 A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be *uniformly convex* if for any $\varepsilon > 0$ and $x_o > 0$, there exists $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \le (1-\delta)\,\frac{f(x)+f(y)}{2}$$

for all $x, y \in \mathbb{R}$ satisfying $|x - y| \ge \varepsilon \max\{|x|, |y|\} \ge \varepsilon x_o$.

Theorem 2.6.3 (See [4], page 6) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then f is convex if and only if for any $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y \le \alpha f(x) + (1-\alpha) f(y)$.

Theorem 2.6.4 (See [4], page 6) If f is a strictly convex, then f is uniformly convex on any bounded interval.

Theorem 2.6.5 (See [1], page 226) Let I be an open interval and suppose that $f : I \to \mathbb{R}$ has a second derivative on I. Then f is convex on I if and only if $f''(x) \ge 0$ for all $x \in I$.

2.7 Property (R), property (MLUR), property (H), and property (G)

For any Banach space X, we denote $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \le 1\}$.

Definition 2.7.1 Let X be a Banach space. An element x in B(X) is called an *extreme point* if for every y, z in B(X) the equality $2x = y+z$ implies $y = z$. We write *Ext B(X)* for the set of all extreme points in B(X). If Ext B(X) = S(X), then X is called a *rotund* (R) space, or X has *property* (R).

Definition 2.7.2 A Banach space X is said to have the *property* (MLUR) (or X is *midpoint locally uniformly rotund*) if for any $x \in S(X)$ and x_n , $y_n \in B(X)$ with $x_n + y_n \to 2x$ imply $x_n - y_n \to 0$.

Definition 2.7.3 A Banach space X is said to have the *property* (H) if each point of S(X) is an H - point of B(X), that is, every weak convergence of point x_n in B(X) to a point in S(X) with $\|x_n\| \to 1$ is a convergence in norm.

Definition 2.7.4 A Banach space X is said to have the *property* (G) if every point of S(X) is a denting point of B(X), that is,

$$x \notin \overline{co}\,(B(X) \setminus (x + \varepsilon B(X)))$$

for all $x \in S(X)$ and all $\varepsilon > 0$.

Here, co(A) is the convex hull of A, $\overline{A}$ is the closure of A.

2.8 Property K

Definition 2.8.1 A Banach space X is said to have the *property* (K) if the weak topology and norm topology on S(X) are equivalent.

Theorem 2.8.2 (See [4], page 126) If X is a Banach spaces, X has property (G) if and only if X is rotund and has the property (K).

2.9 Sequence spaces.

Definition 2.9.1 A sequence space is a linear space whose elements are sequences in $\mathbb{R}$ under the usual addition and usual scalar multiplication.

Definition 2.9.2 Let (P_k) be a bounded sequence of positive real numbers larger than or equal to one. Let $\ell = \ell^{(P_k)}$ be a space of all real sequences $x = (x_k)$ such that the modular

$$\rho(x) = \sum_{k=1}^{\infty} |x_k|^{P_k} < \infty ,$$

with norm $\|x\| = \inf \{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1 \}$.

This space is called a *Nakano sequence space*.

ลิขสิทธิ์มหาวิทยาลัยเชียงใหม่ Chiang Mai University