CHAPTER III

GEOMETRY OF NAKANO SEQUENCE SPACES

In this Chapter, we give necessary and sufficient conditions for a Nakano sequence space to have some particular properties.

More notations to be used later :

$$M = \sup \{P_k : k \in \mathbb{N}\}$$

$$x|_i = (x_1, x_2, x_3, \ldots, x_i, 0, 0, 0, \ldots)$$

$$x|_{\mathbb{N} \setminus i} = (0, 0, 0, \ldots 0, x_{i+1}, x_{i+2}, x_{i+3}, \ldots)$$

Lemma 3.1 Let $x = (x_K)$ be an element of ℓ.

(i) If $0 < a < 1$, then $a^M \rho(\frac{x}{a}) \leq \rho(x)$.

(ii) If $0 < a < 1$, then $\rho(ax) \leq a\rho(x) \leq \rho(x)$.

(iii) If $a \geq 1$, then $\rho(x) \leq a^M \rho(\frac{x}{a})$.

(iv) If $a \geq 1$, then $\rho(x) \leq a\rho(x) \leq \rho(ax)$.

Proof. We prove here for (i) and (ii). The ones for (iii) and (iv) are similar to these cases. Let a be a real number such that $0 < a < 1$. We use the fact that $a^M \leq a^{P_K} \leq a$ for all $K \in \mathbb{N}$. We consider

$$\begin{aligned}
\rho(x) &= \sum_K |x_K|^{P_K} \\
&= \sum_K a^{P_K} \left|\frac{x_K}{a}\right|^{P_K} \\
&\geq \sum_K a^M \left|\frac{x_K}{a}\right|^{P_K} \\
&= a^M \sum_K \left|\frac{x_K}{a}\right|^{P_K} \\
&= a^M \rho(\frac{x}{a})
\end{aligned}$$

and

$$\begin{aligned}\rho(ax) &= \sum_K |ax_K|^{P_K} \\ &= \sum_K a^{P_K}|x_K|^{P_K} \\ &\leq \sum_K a|x_K|^{P_K} \\ &= a\sum_K |x_K|^{P_K} \\ &= a\ \rho(x) \\ &\leq \rho(x).\end{aligned}$$

Therefore, $a^M\rho(\frac{x}{a}) \leq \rho(x)$ and $\rho(ax) \leq a\ \rho(x) \leq \rho(x)$ and the proofs of (i) and (ii) are complete. □

Lemma 3.2 Let x be an element of ℓ.

(i) If $\|x\| < 1$, then $\rho(x) \leq \|x\|$.

(ii) If $\|x\| > 1$, then $\rho(x) \geq \|x\|$.

(iii) $\|x\| = 1$ if and only if $\rho(x) = 1$.

(iv) $\|x\| < 1$ if and only if $\rho(x) < 1$.

(v) $\|x\| > 1$ if and only if $\rho(x) > 1$.

Proof. (i) : Let ε be any real number such that $0 < \varepsilon < 1 - \|x\|$. This implies $\|x\| + \varepsilon < 1$. By definition of $\|\bullet\|$, there exists $\lambda > 0$ such that $\|x\| + \varepsilon > \lambda > \|x\|$ and $\rho(\frac{x}{\lambda}) \leq 1$. Then

$$\begin{aligned}\rho(x) &\leq \rho((\frac{\|x\|+\varepsilon}{\lambda})\ x) && \text{(by Lemma 3.1 (iv))} \\ &= \rho((\|x\|+\varepsilon)\frac{x}{\lambda}) \\ &\leq (\|x\|+\varepsilon)\rho(\frac{x}{\lambda}) && \text{(by Lemma 3.1 (ii))} \\ &\leq \|x\| + \varepsilon.\end{aligned}$$

That is

$$\rho(x) \le \|x\| + \varepsilon \quad \text{for all } \varepsilon \in (0,\ 1 - \|x\|)$$

Putting $A = \{\|x\| + \varepsilon \ : 0 < \varepsilon < 1 - \|x\|\}$. We see that $\|x\| = \inf A$. Since $\rho(x)$ is a lower bound of A, $\rho(x) \le \|x\|$.

(ii) : Let ε be any real number such that $0 < \varepsilon < \dfrac{\|x\|-1}{\|x\|}$. This implies $1 < (1-\varepsilon)\|x\| < \|x\|$. By definition of $\|\cdot\|$, we have

$$\begin{aligned} 1 &< \rho\left(\frac{x}{(1-\varepsilon)\|x\|}\right) \\ &\le \frac{1}{(1-\varepsilon)\|x\|}\rho(x) \qquad \text{(by Lemma 3.1 (ii)).} \end{aligned}$$

That is

$$(1-\varepsilon)\|x\| \le \rho(x) \qquad \text{for all } \varepsilon \in \left(0,\ \frac{\|x\|-1}{\|x\|}\right).$$

Putting $A = \{(1-\varepsilon)\|x\| : 0 < \varepsilon < \dfrac{\|x\|-1}{\|x\|}\}$, then $\|x\| = \sup A$. Since $\rho(x)$ is an upper bound of A, $\|x\| \le \rho(x)$.

(iii) : ($\Rightarrow$) Suppose $\|x\| = 1$, and let ε be any positive real number. There exists $\lambda > 0$ such that $1+\varepsilon > \lambda > \|x\| = 1$ and $\rho(\frac{x}{\lambda}) \le 1$. By Lemma 3.1 (iii), we obtain

$$\begin{aligned} \rho(x) &\le \lambda^M \rho\left(\frac{x}{\lambda}\right) \\ &\le \lambda^M \\ &\le (1+\varepsilon)^M . \end{aligned}$$

That is

$$[\rho(x)]^{\frac{1}{M}} \le 1+\varepsilon \qquad \text{for all } \varepsilon > 0 .$$

This implies $\rho(x) \le 1$.

If $\rho(x) < 1$, we choose a real number $a \in (0, 1)$ such that $\rho(x) < a^M < 1$. From Lemma 3.1 (i), we obtain

$$\begin{aligned} \rho\left(\frac{x}{a}\right) &\le \frac{1}{a^M}\rho(x) \\ &< 1 \qquad (\rho(x) < a^M) \end{aligned}$$

which implies $\|x\| \le a$. But $a < 1$, thus, $\|x\| < 1$, a contradiction. Therefore, we conclude that $\rho(x) = 1$.

($\Leftarrow$) Suppose $\rho(x) = 1$. By definition of $\|\cdot\|$, we immediately have $\|x\| \le 1$. If $\|x\| < 1$, then $\rho(x) \le \|x\| < 1$. A contradiction. Therefore, we conclude that $\|x\| = 1$.

(iv) : ($\Rightarrow$) Suppose $\|x\| < 1$. By Lemma 3.2 (i) , we immediately have
$\rho(x) < 1$.

($\Leftarrow$) Suppose $\|x\| \ge 1$. If $\|x\| = 1$, then $\rho(x) = 1$, by Lemma 3.2(iii). If $\|x\| > 1$, then it follows from Lemma 3.2 (ii), that $\rho(x) > 1$.
Therefore, we can conclude that $\rho(x) \ge 1$.

(v) : This follows directly from Lemma 3.2 (iii) and (iv). □

Lemma 3.3 Let x be an element of ℓ .

(i) If $0 < a < 1$ and $\|x\| > a$, then $\rho(x) > a^M$.

(ii) If $a \ge 1$ and $\|x\| < a$, then $\rho(x) < a^M$.

Proof. (i) : Suppose $0 < a < 1$ and $\|x\| > a$.
Then

$$\frac{1}{a}\|x\| > 1 \Rightarrow \left\|\frac{x}{a}\right\| > 1$$

$$\Rightarrow \rho\left(\frac{x}{a}\right) > 1 \quad \text{(by Lemma 3.2 (v))}$$

$$\Rightarrow a^M \rho\left(\frac{x}{a}\right) > a^M$$

$$\Rightarrow \rho(x) > a^M \quad \text{(by Lemma 3.1 (i))}.$$

Therefore , $\|x\| > a$ implies $\rho(x) > a^M$.

(ii) : Suppose $a \geq 1$ and $\|x\| < a$.

Then

$$\left\|\frac{x}{a}\right\| < 1 \Rightarrow \rho(\frac{x}{a}) < 1 \qquad \text{(by Lemma 3.2 (iv))}$$

$$\Rightarrow a^M \rho(\frac{x}{a}) < a^M$$

$$\Rightarrow \rho(x) < a^M \qquad \text{(by Lemma 3.1 (iii))}.$$

Therefore, $\|x\| < a$ implies $\rho(x) < a^M$. □

Lemma 3.4 Let x^n be a sequence of elements of ℓ.

(i) If $\lim_{n \to \infty} \|x^n\| = 1$, then $\lim_{n \to \infty} \rho(x^n) = 1$.

(ii) If $\lim_{n \to \infty} \rho(x^n) = 0$, then $\lim_{n \to \infty} \|x^n\| = 0$.

Proof. (i) : Suppose $\lim_{n \to \infty} \|x^n\| = 1$. Let ε be any positive real number less than one. By definition of a limit, there exists $N \in \mathbb{N}$ such that

$$1 - \varepsilon < \|x^n\| < 1 + \varepsilon \quad \text{for all } n \geq N.$$

By Lemma 3.3, we obtain

$$(1 - \varepsilon)^M < \rho(x^n) < (1 + \varepsilon)^M \qquad \text{for all } n \geq N,$$

which implies $[\rho(x^n)]^{\frac{1}{M}} \to 1$. That is $\lim_{n \to \infty} \rho(x^n) = 1$.

(ii) : Suppose $\|x^n\| \not\to 0$. Thus we may assume that there exists $\varepsilon \in (0, 1)$ such that $\|x^n\| > \varepsilon$ for all $n \in \mathbb{N}$. By Lemma 3.3 (i), we immediately have $\rho(x^n) > \varepsilon^M$ for all $n \in \mathbb{N}$. This implies $\rho(x^n) \not\to 0$. □

Lemma 3.5 For each $k \in \mathbb{N}$, let $\pi_k : \ell \to \mathbb{R}$ be defined by $\pi_k(x) = x_k$ for all $x = (x_m) \in \ell$. Then π_k is a continuous linear functional on ℓ.

Proof. Let $k \in \mathbb{N}$. At first, we shall prove that π_k is a linear operator. Let $x = (x_m)$, $y = (y_m)$ be elements of ℓ, and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned}\pi_k(\alpha x + \beta y) &= \pi_k(\alpha(x_1, x_2, x_3, \dots) + \beta(y_1, y_2, y_3, \dots)) \\ &= \pi_k((\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots)) \\ &= \alpha x_k + \beta y_k \\ &= \alpha \pi_k(x) + \beta \pi_k(y).\end{aligned}$$

Therefore, π_k is a linear operator.

Next, we shall prove π_k is continuous at 0. Given $\varepsilon > 0$, we choose $\delta = \varepsilon$. If for each $x = (x_m) \in \ell$ such that $\|x\| < \delta$, then

$$\begin{aligned}\pi_k(x) &= |x_k| \\ &\le \|x|_k\| \\ &\le \|x\| \\ &< \delta = \varepsilon.\end{aligned}$$

Therefore, π_k is continuous at 0. By Theorem 2.4.2, we immediately have π_k a continuous function. Therefore, π_k is a continuous linear functional on ℓ. □

Corollary 3.6 Let $(x^n) = ((x_k^n))$ be a weakly convergent sequence in ℓ, say, $x^n \xrightarrow{w} x = (x_k)$. Then $x_k^n \to x_k$ for all $k \in \mathbb{N}$.

Proof. By the definition of weak convergence and Lemma 3.5, we have $\lim_{n \to \infty} \pi_k(x^n) = \pi_k(x)$ for all $k \in \mathbb{N}$. This implies $x_k^n \to x_k$ for all $k \in \mathbb{N}$. □

Theorem 3.7 ℓ is a Banach space.

Proof. Let $(x^n) = ((x_k^n))$ be a Cauchy sequence in ℓ. Given $\varepsilon \in (0, 1)$. Thus, there exists $N \in \mathbb{N}$ such that $\|x^n - x^m\| < \varepsilon^M$ for all $n, m \ge N$. By Lemma 3.2 (i), we obtain

(3.1) $$\rho(x^n - x^m) < \varepsilon^M \text{ for all } n, m \geq N.$$

This implies $|x_k^n - x_k^m| < \varepsilon$ for all $k \in \mathbb{N}$, for all $n, m \geq N$. Thus (x_k^n) is a Cauchy sequence in $\mathbb{R}$, for all $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, for each $k \in \mathbb{N}$, there exists $x_k \in \mathbb{R}$ such that $x_k^n \to x_k$. That is $x_k^n \to x_k$ for all $k \in \mathbb{N}$. Putting $x = (x_k)$. We shall show that $x^n \to x$ and $x \in \ell$. For each $r \in \mathbb{N}$, by (3.1), we have

(3.2) $$\rho([x^n - x^m]|_r) < \varepsilon^M \text{ for all } n, m \geq N.$$

Since $x_k^m \to x_k$ for all $k = 1, 2, \ldots, r$, $\rho([x^n - x^m]|_r) \to \rho([x^n - x]|_r)$, as $m \to \infty$, for all $n \geq N$. From (3.2), we have $\rho([x^n - x]|_r) \leq \varepsilon^M$ for all $n \geq N$. That is $\rho([x^n - x]|_r) \leq \varepsilon^M$ for all $r \in \mathbb{N}$, for all $n \geq N$. This implies

(3.3) $$\rho(x^n - x) \leq \varepsilon^M \text{ for all } n \geq N.$$

By Lemma 3.3(i), we immediately have $\|x^n - x\| \leq \varepsilon$ for all $n \geq N$. This means that $x^n \to x$. From (3.3), we see that $x^N - x \in \ell$. Since ℓ is a linear space, $x = x^N - (x^N - x) \in \ell$. Therefore, ℓ is complete. □

Lemma 3.8 Assume $x = (x_k)$, $x^n = (x_k^n) \in \ell$ for all $n \in \mathbb{N}$, if $\rho(x^n) \to \rho(x)$ and $x_k^n \to x_k$ for all $k \in \mathbb{N}$, then $x^n \to x$.

Proof. Suppose that $x^n \not\to x$. Thus, by Lemma 3.4(ii), we have $\rho(\frac{x^n - x}{2}) \not\to 0$. Without loss of generality, we may assume that there exists $\varepsilon \in (0, 1)$ such that $\rho(\frac{x^n - x}{2}) > \varepsilon$ for all $n \in \mathbb{N}$. Since $(\rho(\frac{x^n - x}{2}))$ is a bounded sequence, it has a convergent subsequence. Passing through a subsequence, if necessary, we can assume that

(3.1) $$\rho\left(\frac{x^n - x}{2}\right) \to \varepsilon_0 \text{ for some } \varepsilon_0 \geq \varepsilon .$$

Since $\rho(x) = \lim\limits_{n \to \infty} \rho(x|_n)$ and $(\rho(x|_n))$ is an increasing sequence, $\rho(x) = \sup\{\rho(x|_n) : n \in \mathbb{N}\}$. So there exists $i \in \mathbb{N}$ such that $\rho(x|_i) > \rho(x) - \frac{\varepsilon}{2}$. This implies $\rho(x|_{\mathbb{N}\setminus i}) < \frac{\varepsilon}{2}$. Since $x^n_k \to x_k$ for all $k \in \mathbb{N}$, $\rho(x^n|_i) \to \rho(x|_i)$ and $\rho\left(\frac{x^n - x}{2}\Big|_i\right) \to 0$. We consider

$$\lim_{n \to \infty} \rho\left(\frac{x^n - x}{2}\right) = \lim_{n \to \infty} \left[\rho\left(\frac{x^n - x}{2}\Big|_i\right) + \rho\left(\frac{x^n - x}{2}\Big|_{\mathbb{N}\setminus i}\right)\right]$$

$$= \lim_{n \to \infty} \rho\left(\frac{x^n - x}{2}\Big|_i\right) + \lim_{n \to \infty} \rho\left(\frac{x^n - x}{2}\Big|_{\mathbb{N}\setminus i}\right)$$

$$= 0 + \lim_{n \to \infty} \rho\left(\frac{x^n - x}{2}\Big|_{\mathbb{N}\setminus i}\right)$$

$$\leq \lim_{n \to \infty} \left[\frac{1}{2}\rho(x^n|_{\mathbb{N}\setminus i}) + \frac{1}{2}\rho(x|_{\mathbb{N}\setminus i})\right]$$

$$= \frac{1}{2}\lim_{n \to \infty} \rho(x^n|_{\mathbb{N}\setminus i}) + \frac{1}{2}\rho(x|_{\mathbb{N}\setminus i})$$

$$= \frac{1}{2}\lim_{n \to \infty} [\rho(x^n) - \rho(x^n|_i)] + \frac{1}{2}\rho(x|_{\mathbb{N}\setminus i})$$

$$= \frac{1}{2}\left[\lim_{n \to \infty} \rho(x^n) - \lim_{n \to \infty} \rho(x^n|_i)\right] + \frac{1}{2}\rho(x|_{\mathbb{N}\setminus i})$$

$$= \frac{1}{2}[\rho(x) - \rho(x|_i)] + \frac{1}{2}\rho(x|_{\mathbb{N}\setminus i})$$

$$= \frac{1}{2}\,\rho(x|_{\mathbb{N}\setminus i}) + \frac{1}{2}\,\rho(x|_{\mathbb{N}\setminus i})$$

$$= \rho(x|_{\mathbb{N}\setminus i})$$

$$< \frac{\varepsilon}{2}$$

$$< \varepsilon_o .$$

Therefore, $\rho(\frac{x^n - x}{2}) \not\to \varepsilon_o$. This is contradicting to (3.1). Hence $x^n \to x$. □

Theorem 3.9 ℓ has property (H).

Proof. Let $x \in S(\ell)$, $x^n \in B(\ell)$ for all $n \in \mathbb{N}$ be such that $x^n \xrightarrow{w} x$ and $\|x^n\| \to 1 = \|x\|$. This implies $\rho(x) = 1$, $x_k^n \to x_k$ for all $k \in \mathbb{N}$, by Corollary 3.6, and $\rho(x^n) \to 1$, by Lemma 3.4(i). By Lemma 3.8, it is obvious that $x^n \to x$. Hence ℓ has property (H). □

Theorem 3.10 Ext $B(\ell) \subset S(\ell)$

Proof. Let $x = (x_k) \in$ Ext $B(\ell)$, then $\|x\| \le 1$. Suppose that $\|x\| < 1$, by Lemma 3.2 (iv), we obtain $\rho(x) < 1$. Putting $t = 1 - \rho(x|_{\mathbb{N}\setminus 1})$. Thus, $|x_1|^{p_1} < t$. We choose a number r such that $|x_1|^{p_1} < r < t$. Since $\lim\limits_{\alpha \to x_1} |\alpha|^{p_1} = |x_1|^{p_1}$, there exists $\delta > 0$ such that

$$\text{(3.1)} \quad \text{if } \alpha \in \mathbb{R} \text{ and } |\alpha - x_1| < \delta \text{, then } \left|\,|\alpha|^{p_1} - |x_1|^{p_1}\right| < \frac{t-r}{2}.$$

Since $|(x_1 \pm \frac{\delta}{2}) - x_1| < \delta$ and by (3.1), we have $|x_1 \pm \frac{\delta}{2}|^{p_1} - |x_1|^{p_1} < \frac{t-r}{2}$, which implies

$$|x_1 \pm \frac{\delta}{2}|^{P_1} < |x_1|^{P_1} + \frac{t-r}{2}$$

$$< r + \frac{t-r}{2}$$

$$= \frac{t+r}{2}$$

$$< \frac{t+t}{2}$$

$$= t .$$

Let $y = (y_k)$ and $z = (z_k)$ where

$$(y_k, z_k) = \begin{cases} (x_1 + \frac{\delta}{2}, x_1 - \frac{\delta}{2}) &, \quad k = 1 \\ (x_k, x_k) &, \quad k > 1. \end{cases}$$

It is clear that $y \neq z$, $x = \frac{y+z}{2}$, and $\rho(y) = |x_1 + \frac{\delta}{2}|^{P_1} + \rho(x|_{\mathbb{N}\setminus 1}) < t + \rho(x|_{\mathbb{N}\setminus 1}) = 1$. Similarly, $\rho(z) < 1$. Therefore, $\|y\| \leq 1$ and $\|z\| \leq 1$. This is contradicting to $x \in \text{Ext } B(\ell)$. We conclude that $\|x\| = 1$, i.e. $x \in S(\ell)$. □

Lemma 3.11 If $x = (x_k) \in S(\ell)$, $x^n = (x_k^n) \in B(\ell)$ for all $n \in \mathbb{N}$, and $\|x^n + x\| \to 2$, then $x_k^n \to x_k$ for each $k \in \mathbb{N}$ where $P_k > 1$.

Proof. Suppose $x_k^n \not\to x_k$ for some $k \in \mathbb{N}$ where $P_k > 1$. Without loss of generality we may assume that $k = 1$, and then assume that, for some $\varepsilon > 0$, $\rho\left(\frac{x^n - x}{2}\Big|_1\right) > \varepsilon$ for all $n \in \mathbb{N}$, since $x_1^n \not\to x_1$. Since $f(x) = |x|^{P_1}$ defines a uniform convex function on $[-1, 1]$, there exists $\delta > 0$ such that

$$\rho\left(\frac{x^n+x}{2}\Big|_1\right) \le (1-\delta)\left[\frac{\rho(x^n|_1)+\rho(x|_1)}{2}\right] \text{ for all } n \in \mathbb{N}.$$

Thus, for each $n \in \mathbb{N}$, we have

$$\rho\left(\frac{x^n+x}{2}\right) = \rho\left(\frac{x^n+x}{2}\Big|_1\right) + \rho\left(\frac{x^n+x}{2}\Big|_{\mathbb{N}\setminus 1}\right)$$

$$\le (1-\delta)\left[\frac{\rho(x^n|_1)+\rho(x|_1)}{2}\right] + \frac{\rho(x^n|_{\mathbb{N}\setminus 1})+\rho(x|_{\mathbb{N}\setminus 1})}{2}$$

$$= \frac{\rho(x^n)+\rho(x)}{2} - \delta\left[\frac{\rho(x^n|_1)+\rho(x|_1)}{2}\right]$$

$$\le \frac{\rho(x^n)+\rho(x)}{2} - \delta\,\rho\left(\frac{x^n-x}{2}\Big|_1\right)$$

$$< \frac{1+1}{2} - \delta\varepsilon$$

$$= 1 - \delta\varepsilon.$$

This implies $1 - \rho\left(\frac{x^n+x}{2}\right) \ge \delta\varepsilon$ for all $n \in \mathbb{N}$, and thus $\rho\left(\frac{x^n+x}{2}\right) \not\to 1$. By Lemma 3.4, we have $\left\|\frac{x^n+x}{2}\right\| \not\to 1$, a contradiction. □

Corollary 3.12 If $x = (x_k) \in S(\ell)$, $y = (y_k) \in B(\ell)$, and $\| y+x\| = 2$, then $y_k = x_k$ for each $k \in \mathbb{N}$ where $P_k > 1$.

Proof. Put $y^n = y$ for all $n \in \mathbb{N}$. It is obvious that $\|y^n + x\| \to 2$. Thus, by Lemma 3.11, we have $y_k^n \to x_k$ for all $k \in \mathbb{N}$. Therefore, $y_k = \lim_{n \to \infty} y_k^n = x_k$ for all $k \in \mathbb{N}$. That is $y_k = x_k$ for all $k \in \mathbb{N}$. □

Theorem 3.13 ℓ is rotund if and only if $P_k = 1$ for at most one k.

Proof. ($\Rightarrow$) Suppose on the contrary that $P_k = 1$ for at least 2 k's. Without loss of generality, we may assume that $P_1 = P_2 = 1$. We choose $x = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \ldots)$, $y = (\frac{3}{4}, \frac{1}{4}, 0, 0, 0, \ldots)$ and $z = (\frac{1}{4}, \frac{3}{4}, 0, 0, 0, \ldots)$. It is seen that $\|x\| = \|y\| = \|z\| = 1$, $x = \frac{y+z}{2}$, but $y \neq z$. Thus we have $x \notin \text{Ext } B(\ell)$. Therefore, $S(\ell) \neq \text{Ext } B(\ell)$. That is ℓ is not rotund.

($\Leftarrow$) Without loss of generality, we may assume $P_1 = 1$ and $P_k > 1$ for all $k \geq 2$. It suffices to show $S(\ell) \subset \text{Ext } B(\ell)$. Let $x = (x_k) \in S(\ell)$, $y = (y_k)$ and $z = (z_k) \in B(\ell)$ be such that $x = \frac{y+z}{2}$. Then

$$\begin{aligned} 4 &= \|2x + y + z\| \\ &= \|(x + y) + (x + z)\| \\ &\leq \|x + y\| + \|x + z\| \\ &\leq \|x\| + \|y\| + \|x\| + \|z\| \\ &\leq 1 + 1 + 1 + 1 \\ &= 4, \end{aligned}$$

which implies $\|y\| = 1$, $\|z\| = 1$, $\|x + y\| = 2$ and $\|x + z\| = 2$. By Corollary 3.12, we have $x_k = y_k = z_k$ for all $k \geq 2$. Since $1 = \rho(x) = \rho(y) = \rho(z)$, $|x_1| = |y_1| = |z_1|$. Next, we shall show that $y_1 = z_1$. Suppose $y_1 \neq z_1$. Thus, $y_1 \neq 0$ or $z_1 \neq 0$. If $y_1 \neq 0$, then $y_1 = -z_1$, since $|y_1| = |z_1|$. Therefore $x_1 = \frac{y_1 + z_1}{2} = 0$. That is $|y_1| \neq |x_1|$, a contradiction. Similarly, for case $z_1 \neq 0$. Hence $y_1 = z_1$. We deduce that $y_k = z_k$ for all $k \in \mathbb{N}$, i.e. $y = z$.

□

Lemma 3.14 Assume $x^n = (x_k^n)$, $x = (x_k) \in \ell$ for all $n \in \mathbb{N}$, if $\rho(x^n) \to 1$ and $x_k^n \to x_k$ for all $k \in \mathbb{N}$, then $\rho(x) \leq 1$.

Proof. Suppose $\rho(x) > 1$, we choose a small number ε such that $\rho(x) - \varepsilon > 1$. Since $\rho(x) = \sup \{\rho(x\, l_n) : n \in \mathbb{N}\}$, there exists $i \in \mathbb{N}$ such that $\rho(x\, l_i) > \rho(x) - \varepsilon$. We consider

$$\begin{aligned} \lim_{n \to \infty} \rho(x^n) &\geq \lim_{n \to \infty} \rho(x^n l_i) \\ &= \rho(x\, l_i) \qquad \text{(since } x^n \to x \text{ componentwise)} \\ &> \rho(x) - \varepsilon \\ &> 1. \end{aligned}$$

That is $\rho(x^n) \not\to 1$, a contradiction. □

Theorem 3.15 ℓ is MLUR if and only if $P_k = 1$ for at most one k.

Proof. ($\Rightarrow$) Without loss of generality, we may assume that $P_1 = P_2 = 1$, and let $x = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0)$, $y^n = (\frac{3}{4}, \frac{1}{4}, 0, 0, 0, \ldots)$ and $z^n = (\frac{1}{4}, \frac{3}{4}, 0, 0, 0, \ldots)$ for all $n \in \mathbb{N}$. It is clear that $\|x\| = 1$, $\|y^n\| = \|z^n\| = 1$ for all $n \in \mathbb{N}$, $y^n + z^n \to 2x$ and $\|y^n - z^n\| = 1 \not\to 0$. This implies ℓ is not MLUR.

($\Leftarrow$) We may assume $P_1 = 1$ and $P_k > 1$ for all $k \geq 2$. Let $x = (x_k) \in S(\ell)$, $y^n = (y_k^n)$ and $z^n = (z_k^n) \in B(\ell)$ for all $n \in \mathbb{N}$ be such that $y^n + z^n \to 2x$. Then $\|y^n + z^n + 2x\| \to 4$ and

(3.1) $\quad y_1^n + z_1^n \to 2x_1$

At first, we shall show that $\|y^n + x\| \to 2$ and $\|z^n + x\| \to 2$. Suppose $\|y^n + x\| \not\to 2$. Thus, we may assume that there exists $\varepsilon \in (0, 1)$ such that $\|y^n + x\| < 2 - \varepsilon$ for all $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$,

$$\begin{aligned} \|y^n + z^n + 2x\| &\leq \|y^n + x\| + \|z^n + x\| \\ &< 2 - \varepsilon + \|z^n\| + \|x\| \\ &\leq 4 - \varepsilon . \end{aligned}$$

This implies $4 - \|y^n + z^n + 2x\| > \varepsilon$ for all $n \in \mathbb{N}$. That is $\|y^n + z^n + 2x\| \not\to 4$, a contradiction. Therefore $\|y^n + x\| \to 2$. Similarly, $\|z^n + x\| \to 2$. By Lemma 3.11, We have $y_k^n \to x_k$ and $z_k^n \to x_k$ for all $k \geq 2$. Since $\|y^n + x\| \to 2$ and $\|x\| = 1$, $\|y^n\| \to 1$. Thus, by Lemma 3.4 (i), we have $\rho(y^n) \to 1$. Similarly, $\rho(z^n) \to 1$. Next, we shall show that $y_1^n \to x_1$ and $z_1^n \to x_1$. Let $(y_1^{n'})$ be any subsequence of (y_1^n). Then $(y_1^{n'})$ is a bounded sequence, which then posseses a subsequence $(y_1^{n''})$ such that $y_1^{n''} \to y_1$ for some $y_1 \in \mathbb{R}$. Thus, from (3.1) we have $z_1^{n''} \to 2x_1 - y_1 = z_1$. Let $y = (y_k)$, $z = (z_k)$ be defined by

$$(y_k , z_k) = \begin{cases} (y_1, z_1) & , \quad k = 1 \\ (x_k, x_k) & , \quad k \geq 2 . \end{cases}$$

It is clear that $x = \dfrac{y+z}{2}$. Since $\rho(y^{n''}) \to 1$, $y_k^{n''} \to y_k$ for all $k \in \mathbb{N}$, we have $\rho(y) \leq 1$, by Lemma 3.14. Thus, $\|y\| \leq 1$. Similarly, $\|z\| \leq 1$. By assumption, we know that ℓ is rotund, which implies $x = y = z$, that is $x_1 = y_1 = z_1$. Hence $y_1^{n''} \to x_1$ and therefore $y_1^n \to x_1$. Similarly, $z_1^n \to x_1$. We know now that $y^n \to x$ componentwise and $\rho(y^n) \to 1 = \rho(x)$. Therefore, by Lemma 3.8, we must have $y^n \to x$. Similarly, $z^n \to x$. Hence we deduce that $y^n \to z^n$. □

Theorem 3.16 ℓ has property (G) if and only if $P_k = 1$ for at most one k.

Proof. From the relation

$$(G) \Leftrightarrow (K) + (R),$$

and the property (H) of ℓ, we see that property (G) and property (R) are equivalent on ℓ. Hence, by Theorem 3.13, we deduce that ℓ has property (G) if and only if $P_k = 1$ for at most one k. □