CHAPTER II

PRELIMINARIES

In this chapter, we begin with some basic knowledge of module theory that will be used in the later chapters.

Throughout the thesis, all rings are associative with identity and all right modules are unital.

1. Homomorphisms and Annihilators

2.1.1 The Factor Theorem. *Let* M, $M^{/}$ *and* N *be right R-modules and* $f: M \rightarrow N$ *be an R-homomorphism. If* $g: M \rightarrow M^{/}$ *is an epimorphism with* $Ker\, g \subseteq Ker\, f$, *there exists a unique homomorphism* $h: M^{/} \rightarrow N$ *such that* $hg = f$.

Proof. See [1] page 45.

2.1.2 Definition. *Let M be a right R-module. For each subset X of M, the set* $r_R(X) = \{ r \in R \,/\, xr = 0 \text{ for all } x \in X \}$ *is called a right annihilator of X in R. For each subset A of R, the set* $l_M(A) = \{ x \in M \,/\, xa = 0 \text{ for all } a \in A \}$ *is called a left annihilator of A in M.*

2.1.3 Proposition. *Let M be a right R-module, let* $X \subset M$, *and let* $A \subset R$. *Then*

(1) $l_M(A)$ *is a submodule of M.*

(2) $r_R(X)$ *is a right ideal of R.*

Proof. See [1] page 37.

2.1.4 Proposition. *Let M be a right R-module, let X,Y be subsets of M and let A, B be subsets of R. Then*

(1) $X \subset Y$ *implies* $r_R(Y) \subset r_R(X)$ *and* $A \subset B$ *implies* $l_M(B) \subset l_M(A)$.

(2) $A \subset r_R(l_M(A))$ *and* $X \subset l_M(r_R(X))$.

(3) $l_M(A) = l_M(r_R(l_M(A)))$ *and* $r_R(X) = r_R(l_M(r_R(X)))$.

Proof. See [1] page 38.

2.1.5 Corollary. *A right R-module M is cyclic if and only if it is isomorphic to a factor module of R. If* $M = xR$, *then* $M \cong R/r_R(x)$ *and M is simple if and only if* $r_R(x)$ *is a maximal right ideal.*

Proof. See [1] page 47.

2. Cogenerated , Reject and Radical

2.2.1 Definition. *Let* $\mathcal{U}$ *be a class of modules. A module M is cogenerated by* $\mathcal{U}$ *(or* $\mathcal{U}$ *cogenerates M) in case there is an indexes set* $(U_\alpha)_{\alpha \in A}$ *in* $\mathcal{U}$ *and a monomorphism*

$$0 \longrightarrow M \longrightarrow \prod_{\alpha \in A} U_\alpha$$

2.2.2 Definition. *Let* $\mathcal{U}$ *be a class of modules and M be a right R-module. The reject of* $\mathcal{U}$ *in M are defined by*

$$Rej_M(\mathcal{U}) = \bigcap \{ Ker\, h \,/\, h : M \longrightarrow U \text{ for some } U \in \mathcal{U} \}$$

In the particular case where $\mathcal{U} = \{U\}$ *is a singaton, these assume the simpler form.*

$$Rej_M(U) = \bigcap \{ Ker\, h \;/\; h \in Hom\,(M, U) \}$$

2.2.3 Proposition. *Let M and U be right R-modules. Then U cogenerates M if and only if* $Rej_M(U) = 0$

Proof. See [1] page 109.

2.2.4 Proposition. *Let M be a right R-module. Then*

$$Rad\,(M) = \cap\{\text{ submodule } K \text{ of } M \,/\, K \text{ is maximal in } M\ \}$$

Proof. See [1] page 120.

2.2.5 Proposition. *Let M be a right R-module. Then Rad M = 0 if and only if M is cogenerated by the class of simple modules.*

Proof. See [1] page 121.

3. Projective and Injective Modules

2.3.1 Definition. *Let P be right R-modules. P is called projective if given any R-homomorphism* $f : P \to B$ *and any epimorphism* $g : A \to B$ *then there exists a homomorphism* $h : P \to A$ *such that* $gh = f$.

2.3.2 Definition. *Let M be right R-modules. M is called injective if given any R-homomorphism* $f : A \to M$ *and any monomorphism* $g : A \to B$ *then there exists a homomorphism* $h : B \to M$ *such that* $hg = f$.