CHAPTER III

CYCLICALLY INJECTIVE RINGS

In this chapter, we will study principally N-injective modules, cyclically injective rings and their applications to V-rings, P-V-rings and regular rings.

1. Principally N-Injective Modules

Definition. *Let R be a ring, M and N be right R-modules. M is called principally N-injective (P-N-injective) if every R-homomorphism $f: nR \longrightarrow M$, $n \in N$, extends to N. In case M is principally R-injective, we call M a principally injective (P-injective) module.*

3.1.1 Examples.

(1). *Every injective module is P-N-injective for all right R-modules N.*

(2). *If every cyclic submodule of M is a direct summand of M, then M is P-M-injective.*

(3). *Q_Z is P-nZ-injective where n positive integer.*

(4). *If R is a regular ring, then every right R-module is P-injective.*

3.1.2 Some properties of principally N-injective modules.

Let R be a ring. Then we have :

(1) If M_R is P-N-injective and $A_R \cong M_R$, then A_R is P-N-injective.

(2) If M_R is P-N-injective and $A_R \cong N_R$, then M_R is P-A-injective.

(3) M_R is P-N-injective if and only if M_R is P-A-injective for all submodules A of N.

(4) If M_R is P-N-injective and every cyclic submodule of N_R is projective, then M/K is P-N-injective for all submodules K of M.

Proof. (1). Let M_R be *P-N*-injective and $A_R \cong M_R$. We will show that A_R is *P-N*-injective. Let $n \in N, f: nR \to A$ be an *R*- homomorphism and $i: nR \to N$ be an inclusion map. Since $A_R \cong M_R$, there exists $g: A \to M$ an *R*- isomorphism. Because M_R is *P-N*-injective , so there exists $h: N \to M$ an *R*- homomorphism such that $hi = gf$. Let $f^* = g^{-1}h$. We get that $f^*i = g^{-1}hi = g^{-1}gf = f$. This means that A_R is *P-N*-injective.

(2). Let M_R be *P-N*-injective and $A_R \cong N_R$. We will show that M_R is *P-A*-injective. Let $a \in A, f: aR \to M$ be an *R*- homomorphism and $i: aR \to A$ be an inclusion map. Since $A_R \cong N_R$, there exists $g: N \to A$ an *R*- isomorphism. Because $a \in A$, so $g(n) = a$ for some $n \in N$. Let $\bar{g} = g|_{nR}$ and $i': nR \to N$ be an inclusion map. We get that $gi' = i\bar{g}$. Since M_R is *P-N*-injective, there exists $h: N \to M$ an *R*- homomorphism such that $hi' = f\bar{g}$. Let $f^* = hg^{-1}$. We get that $f^*i\bar{g} = hg^{-1}i\bar{g}$ $= hg^{-1}gi' = hi' = f\bar{g}$. Because $\bar{g}$ is onto , so $f^*i = f$. This means that M_R *is P-A*-injective.

(3). ($\Rightarrow$) Let M_R be *P-N*-injective and A be a submodule of N. We will show that M_R is *P-A*-injective. Let $a \in A$, $f: aR \to M$ be an *R*-homomorphism and $i: aR \to A$ be an inclusion map. Since $A \subset_{>} N$, aR is a cyclic submodule of N. Let $i': A \to N$ be an inclusion map. Because M_R is *P-N*-injective , so there exists $h: N \to M$ an *R*- homomorphism such that $hi'i = f$. It follows that M_R is *P-A*-injective.

($\Leftarrow$) Assume that M_R is *P-A*-injective for all submodules A of N. Because $N \subset_{>} N$, by assumption we get that M_R is *P-N*-injective.

(4). Assume that M_R is *P-N*-injective and every cyclic submodule of N is projective. Let $n \in N$, $f: nR \to M/K$ be an *R*- homomorphism and $i: nR \to N$ be an inclusion map. We have $\eta: M \to M/K$ is a natural homomorphism. By assumption, we get that nR is projective. Then there exists $g : nR \longrightarrow M$ an R-homomorphism such that $\eta g = f$. Because M_R is P-N-injective, there exists $h : N \longrightarrow M$ an R-homomorphism such that $hi = g$. Let $f^* = \eta h$. Hence we get that $f^* i = \eta hi = \eta g = f$. Therefore M/K is P-N-injective. □

3.1.3 Direct product and direct sum of principally N-injective modules.

(1) If M_R is P-N-injective and A_R is a direct summand of M_R, then A_R is P-N-injective.

(2) A product $\prod_{i\in I} M_i$ is P-N-injective if and only if each M_i is P-N-injective.

(3) A sum $\bigoplus_{i\in I} M_i$ is P-N-injective if and only if each M_i is P-N-injective.

Proof. (1). Let M_R be P-N-injective and A be a direct summand of M. Then $M = A \oplus K$ for some submodule K of M. We will show that A is P-N-injective. Let $n \in N, f : nR \longrightarrow A$ be an R- homomorphism and $i : nR \longrightarrow N$ be an inclusion map. We have $\eta_A : A \longrightarrow A \oplus K$ is an injection map. Since M_R is P-N-injective, there exists $g : N \longrightarrow M$ an R- homomorphism such that $gi = \eta_A f$. In fact, if $\pi_A : M \longrightarrow A$ is a projection map, then $\pi_A \eta_A = 1_A$. Let $h = \pi_A g$. We get that $hi = \pi_A gi = \pi_A \eta_A f = f$. It follows that A is P-N-injective.

(2). ($\Rightarrow$) Assume that $\prod_{i\in I} M_i$ is P-N-injective. We will show that M_i is P-N-injective for all $i \in I$. Let $n \in N, f : nR \longrightarrow M_i$ be an R- homomorphism and $\iota : nR \longrightarrow N$ be an inclusion map. We have $\eta_i : M_i \longrightarrow \prod_{i\in I} M_i$ is an injection map. Since $\prod_{i\in I} M_i$ is P-N-injective, there exists $g : N \longrightarrow \prod_{i\in I} M_i$ an R- homomorphism such that $g\iota = \eta_i f$. Let $\pi_i : \prod_{i\in I} M_i \longrightarrow M_i$ be a projection map. Putting $h = \pi_i g$, we get that $h\iota = \pi_i g\iota = \pi_i \eta_i f = f$. It follows that M_i is P-N-injective.

($\Leftarrow$) Assume that M_i is P-N-injective for all $i \in I$. We will show that $\prod_{i\in I} M_i$ is P-N-injective. Let $n \in N, f : nR \longrightarrow \prod_{i\in I} M_i$ be an R- homomorphism and $\iota : nR \longrightarrow N$

be an inclusion map. We have $\pi_i : \prod_{i\in I} M_i \longrightarrow M_i$ be a projection map. Since M_i is *P*-*N*-injective, there exists $h_i : N \longrightarrow M_i$ an *R*- homomorphism such that $h_i\iota = \pi_i f$. For each $x \in N$, define

$$h : N \longrightarrow \prod_{i\in I} M_i \text{ by } \pi_i h(x) = h_i(x) \text{ for all } i \in I$$

Since the π_i and the h_i are *R*- homomorphisms, it follows that h defines an *R*-homomorphism. Moreover $\pi_i h = h_i$ for all $i \in I$. We have $\pi_i h\iota = h_i\iota = \pi_i f$ for all $i \in I$. So $h\iota = f$. It follows that $\prod_{i\in I} M_i$ is *P*-*N*-injective.

(3) ($\Rightarrow$) Since M_i is a direct summand of $\bigoplus_{i\in I} M_i$ for all $i \in I$, by (1) we get that M_i is *P*-*N*-injective for all $i \in I$.

($\Leftarrow$) Assume that M_i is *P*-*N*-injective for all $i \in I$. We will show that $\bigoplus_{i\in I} M_i$ is *P*-*N*-injective. Let $n \in N, f : nR \longrightarrow \bigoplus_{i\in I} M_i$ be an *R*- homomorphism and $\iota : nR \longrightarrow N$ be an inclusion map. Put $f(n) = (m_i)_{i\in I}$, then m_i is zero for almost all $i \in I$. We get that $f(nR) = f(n)R \subseteq \bigoplus_{i\in F} M_i$ for some finite subset F of I. Since $\bigoplus_{i\in F} M_i = \prod_{i\in F} M_i$ is *P*-*N*-injective by (2), there exists $h : N \longrightarrow \bigoplus_{i\in F} M_i$ an *R*-homo morphism such that $h\iota = f$. Because $\bigoplus_{i\in F} M_i \subseteq \bigoplus_{i\in I} M_i$, so $h : N \longrightarrow \bigoplus_{i\in I} M_i$ an *R*-homomorphism such that $h\iota = f$. Therefore $\bigoplus_{i\in I} M_i$ is *P*-*N*-injective. □

2. Cyclically Injective Rings

Definition. *Let R be a ring. R is a cyclically injective ring (C-ring) if every simple right R-module is P-N-injective for all cyclic right R-modules N. Equivalently, R is a C-ring if and only if every simple right R-module is P-*R/I*-injective for all right ideals I of R.*

3.2.1 Examples.

(1). *Every V-ring is a C-ring.*

(2). *Every division ring is a C-ring.*

Proof. Let R be a division ring. We want to prove that R is a C-ring. Let S be a simple right R-module and I be a right ideal of R. Since R is a division ring, R and 0 are the only right ideals of R. We will show that S is P-R/I-injective. If $I = 0$, let $0 \neq a \in R$, $f: aR \rightarrow S$ be an R-homomorphism and $i : aR \rightarrow R$ be an inclusion map. Thus $aR = R$. Put $h = f$, we have $hi = f$. This means that S is P-R-injective. Because $R/0 \cong R$, by 3.1.2(2), we get that S is P-$R/0$-injective. If $I = R$, we have $R/R \cong 0$. Since S is P-0-injective, S is P-R/R-injective by 3.1.2(2). Hence R is a C-ring. □

(3). *Z_{4n} is not a C-ring where $n \geq 1$.*

Proof. First, we will show that $\bar{2}Z_{4n}$ is maximal. Let A be a right ideal of Z_{4n} and $\bar{2}Z_{4n} \subset A \subseteq Z_{4n}$. Thus there exists $\bar{a} \in A \setminus \bar{2}Z_{4n}$, so $\bar{a} \neq \bar{2}\,\bar{x}$ for all $\bar{x} \in Z_{4n}$. That is $a \neq 2x$ for all $x \in Z$. This implies that $(a, 2) = 1$. Thus $1 = as + 2r$ for some $s, r \in Z$. We get that $\bar{1} = \overline{as} + \overline{2r} \in A$, so $A = Z_{4n}$. Hence $\bar{2}Z_{4n}$ is maximal. This means that $Z_{4n}/2Z_{4n}$ is simple. We want to show that Z_{4n} is not a C-ring. Suppose that Z_{4n} is a C-ring. Define

$$f : \bar{2}Z_{4n} \rightarrow Z_{4n}/2Z_{4n} \quad \text{by } f(\bar{2}\,\bar{m}) = \bar{m} + \bar{2}Z_{4n} \quad \text{for all } \bar{m} \in Z_{4n}$$

Next, we will show that f is well-defined. Let $\bar{2}\,\bar{m} = \bar{0}$, then $4n \mid 2m$. That is $2m = 4nl$ for some $l \in Z$. We have $m = 2nl$. So $\bar{m} \in \bar{2}Z_{4n}$. Thus f is well-defined and it is clear that f is a Z_{4n}-homomorphism. Because Z_{4n} is a C-ring, there exists $h : Z_{4n} \rightarrow Z_{4n}/2Z_{4n}$ a Z_{4n}-homomorphism such that $hi = f$ where $i : \bar{2}Z_{4n} \rightarrow Z_{4n}$

is an inclusion map. Since $\bar{1} \in Z_{4n}$, $h(\bar{1}) \in {Z_{4n}}/{2Z_{4n}}$. Put $h(\bar{1}) = \overline{w} + \bar{2}Z_{4n}$ for some $\overline{w} \in Z_{4n}$. Consider $h(\bar{1}+\bar{1}) = h(\bar{1}) + h(\bar{1}) = (\overline{w} + \bar{2}Z_{4n}) + (\overline{w} + \bar{2}Z_{4n}) = \overline{2w} + \bar{2}Z_{4n}$ and $h(\bar{2}) = hi(\bar{2}) = f(\bar{2}) = f(\bar{2}\cdot\bar{1}) = \bar{1} + \bar{2}Z_{4n}$. So $\overline{2w} - \bar{1} \in \bar{2}Z_{4n}$ which is a contradiction. Hence Z_{4n} is not a C-ring. □

3.2.2 A characterization of *C*-rings.

The following conditions are equivalent :

(1) R is a C-ring.

(2) For each right ideal I of R, each principal right ideal P of R, each maximal subideal K containing I of P+I, there exists a maximal right ideal M containing I of R such that $K = M \cap (P+I)$.

Proof. (1)⟹(2). Assume that R is a C-ring. Let I be a right ideal of R, P be a principal right ideal of R and K be a maximal subideal containing I of $P+I$. We get that R/I is cyclic, $(P+I)/I$ is a cyclic submodule of R/I and $(P+I)/K$ is simple. Since $\dfrac{(P+I)/I}{K/I} \cong (P+I)/K$, $\dfrac{(P+I)/I}{K/I}$ is simple. Let $\pi : (P+I)/I \rightarrow \dfrac{(P+I)/I}{K/I}$ be a natural homomorphism and $i : (P+I)/I \rightarrow R/I$ be an inclusion map. Because R is a C-ring, there exists $h : R/I \rightarrow \dfrac{(P+I)/I}{K/I}$ an R-homomorphism such that $hi = \pi$. We have $Im\, h = 0$ or $Im\, h = \dfrac{(P+I)/I}{K/I}$. If $Im\, h = 0$, then $h = 0$. Thus $\pi = 0$. It follows that $K/I = Ker\, \pi = (P+I)/I$ which is a contradiction. We have $\dfrac{R/I}{Ker h} \cong Im\, h = \dfrac{(P+I)/I}{K/I}$. Therefore $Ker\, h$ is a maximal submodule of R/I. Put $Ker\, h = M/I$. Because $\dfrac{R/I}{M/I}$ is simple and $\dfrac{R/I}{M/I} \cong R/M$, we have M is a maximal right ideal containing I of R. Next, we will show that $K = M \cap (P+I)$.

Let $k \in K$, we have $0 = \pi(k+I) = hi\,(k+I) = h\,(k+I)$. That is $k+I \in Ker\, h = M/I$. Thus $k \in M$. It follows that $K \subseteq M \cap (P+I)$. Let $y \in M \cap (P+I)$, thus $y+I \in M/I$. We get that $0 = h(y+I) = hi\,(y+I) = \pi(y+I)$. That is $y+I \in Ker\, \pi = K/I$. Thus $y \in K$. It follows that $M \cap (P+I) \subseteq K$. Hence $K = M \cap (P+I)$.

(2) $\Rightarrow$ (1). Assume that (2) holds. We want to show that R is a C-ring. Let S be a simple right R-module, I be a right ideal of R. We will show that S is P-R/I-injective. Let $a \in R$, $f : (a+I)R \longrightarrow S$ be a nonzero homomorphism and $\iota : (a+I)R \longrightarrow R/I$ be an inclusion map. We have $(a+I)R = (aR+I)/I$. Let $Ker\, f = K/I$, we get that $(aR+I)/K \cong \dfrac{(aR+I)/I}{K/I} \cong Im\, f = S$. Since S is simple, K is a maximal subideal containing I of $aR+I$. By (2), there exists a maximal right ideal M containing I of R such that $K = M \cap (aR+I)$. If $aR+I \subseteq M$, then $aR+I = K$ which is a contradiction. Thus $aR+I \not\subseteq M$. We have $M \subset M + aR+I \subseteq R$. Because M is a maximal ideal of R, $R = M+aR+I$. Let $z+I \in R/I$, thus $z+I = m+ar+I$ for some $m \in M$ and for some $r \in R$ .Define

$$h : R/I \longrightarrow S \quad \text{by} \quad h(z+I) = f(ar+I)$$

Next, we will show that h is well-defined. Let $z_1+I, z_2+I \in R/I$ and $z_1+I = z_2+I$, thus $(m_1+ar_1+I) = (m_2+ar_2+I)$. We get that $(ar_1 - ar_2)+I = (m_2 - m_1)+I \in M/I$. That is $ar_1 - ar_2 \in M \cap (aR+I) = K$. So $(ar_1 - ar_2)+I \in K/I = Ker\, f$. This implies that $f((ar_1 - ar_2)+I) = 0$. That is $f(ar_1+I) = f(ar_2+I)$. Therefore h is well-defined and it is clear that h is an R-homomorphism.We will show that $h\iota = f$, let $ar+I = (ar+i)+I \in (aR+I)/I$. We get that $h\iota\,(ar+I) = h(ar+I) = h(0+ar+I) = f(ar+I)$. Therefore $h\iota = f$. Hence S is P-R/I-injective. It follows that R is a C-ring. □

3.2.3 Theorem. *Let M be a right R-module, N be a cyclic right R-module ($N = tR$) and ${}_N M = \{m \in M \,/\, r_R(t) \subseteq r_R(m)\}$. Then the following conditions are equivalent:*

(1) M is P-N-injective.

(2) For each $n = ta \in N$ and each $f \in Hom\,(nR, M)$, $f(n) \in {}_N Ma$.

(3) For each $n = ta \in N$, $l_M r_R(n) = {}_N Ma$.

(4) For each $n = ta \in N$ and each $m \in M$, $r_R(n) \subseteq r_R(m)$ implies $Sm \subseteq {}_N Ma$, where $S = End(M)$.

(5) For each $n = ta \in N$ and each $b \in R$, $l_M[bR \cap r_R(n)] = l_M(b) + {}_N Ma$.

Proof. (1)$\Rightarrow$(2). Assume that (1) holds. Let $n = ta \in N$ and $f \in Hom\,(nR, M)$. Since M is P-N-injective, there exists $h : N \longrightarrow M$ an R-homomorphism such that $hi = f$ where $i : nR \longrightarrow N$ is an inclusion map. Thus $f(n) = hi(n) = h(n) = h(ta) = h(t)a$ and we have $h(t) \in M$. We will show that $h(t) \in {}_N M$, i.e., $r_R(t) \subseteq r_R(h(t))$. Let $y \in r_R(t)$, thus $ty = 0$. We get that $h(t)y = h(ty) = h(0) = 0$. That is $y \in r_R(h(t))$. Hence $h(t) \in {}_N M$. It follows that $f(n) \in {}_N Ma$.

(2)$\Rightarrow$(3). Assume that (2) holds. Let $n = ta \in N$ and $y \in {}_N Ma$. Then $y = ma$ for some $m \in {}_N M$. Thus $r_R(t) \subseteq r_R(m)$. We want to prove that $y \in l_M r_R(n)$, i.e., $yz = 0$ for all $z \in r_R(n)$. Let $z \in r_R(n)$, thus $0 = nz = taz$. That is $az \in r_R(t) \subseteq r_R(m)$. We get that $maz = 0$. Consider $yz = maz = 0$, thus $y \in l_M r_R(n)$. Therefore ${}_N Ma \subseteq l_M r_R(n)$. Conversely, let $x \in l_M\, r_R(n)$. Define

$$f : nR \longrightarrow xR \quad \text{by} \quad f(nr) = xr \quad \text{for all } r \in R$$

Thus f is well-defined and an R-homomorphism. Let $i : xR \longrightarrow M$ be an inclusion map and $h = if \in Hom\,(nR, M)$. By (2), we get that $h(n) \in {}_N Ma$. Consider $n \in nR$, we have $h(n) = if(n) = f(n) = f(n1) = x1 = x$. So $x \in {}_N Ma$. Therefore $l_M r_R(n) \subseteq {}_N Ma$.

(3)$\Rightarrow$ (4). Assume that (3) holds. Let $n = ta \in N$, $m \in M$ and $r_R(n) \subseteq r_R(m)$. Then $\{m\} \subseteq l_M r_R(m) \subseteq l_M r_R(n) = {}_N Ma$. So $m = m'a$ for some $m' \in {}_N M$, thus $r_R(t) \subseteq r_R(m')$. Let $x \in Sm$, then $x = f(m)$ for some $f \in S$. We have $f(m) = f(m'a) = f(m')a$. Next, we will show that $f(m') \in {}_N M$, i.e., $r_R(t) \subseteq r_R(f(m'))$. Let $y \in r_R(t)$, thus $y \in r_R(m')$. That is $m'y = 0$, we have $0 = f(m'y) = f(m')y$. This means that $y \in r_R(f(m'))$. Therefore $f(m') \in {}_N M$. It follows that $x = f(m) \in {}_N Ma$. Hence $Sm \subseteq {}_N Ma$.

(4)$\Rightarrow$(1). Assume that (4) holds. Let $n = ta \in N$, $f \in Hom(nR, M)$ and $i : nR \rightarrow N$ be an inclusion map. For each $x \in r_R(n)$, we get that $nx = 0$. So $f(n)x = f(nx) = f(0) = 0$. Hence $x \in r_R(f(n))$. That is $r_R(n) \subseteq r_R(f(n))$. By assumption, we get that $S(f(n)) \subseteq {}_N Ma$. Thus $f(n) = ma$ for some $m \in {}_N M$. Thus $r_R(t) \subseteq r_R(m)$. Define

$$h : N \rightarrow M \text{ by } h(tr) = mr \text{ for all } r \in R$$

Then h is well-defined and an R-homomorphism. Let $nr \in nR$ for some $r \in R$. We get that $hi(nr) = h(nr) = h(tar) = mar = f(n)r = f(nr)$, so $hi = f$. Hence M is P-N-injective

(4)$\Rightarrow$(5). Assume that (4) holds. Let $n = ta \in N$ and $b \in R$. We will show that $l_M[bR \cap r_R(n)] = l_M(b) + {}_N Ma$. Let $x \in l_M[bR \cap r_R(n)]$, then $xy = 0$ for all $y \in bR \cap r_R(n)$. For each $r \in r_R(nb)$, $nbr = 0$. So $br \in bR \cap r_R(n)$ and thus $xbr = 0$, i.e., $r \in r_R(xb)$. Therefore $r_R(nb) \subseteq r_R(xb)$. This implies that $xb \in {}_N Mab$, by (4). Thus $xb = mab$ for some $m \in {}_N M$. We have $(x - ma)b = 0$, thus $x - ma \in l_M(b)$. So $x - ma = z$ for some $z \in l_M(b)$.Therefore $x = z + ma \in l_M(b) + {}_N Ma$. This means that $l_M[bR \cap r_R(n)] \subseteq l_M(b) + {}_N Ma$. Conversely, let $x \in l_M(b) + {}_N Ma$. Then $x = y + ma$ for some $y \in l_M(b)$, $m \in {}_N M$. Thus $r_R(t) \subseteq r_R(m)$. We get that $xb = yb + mab = mab$. We want to show that $x \in l_M[bR \cap r_R(n)]$. Let $y \in bR \cap r_R(n)$, thus $y = br$ for some $r \in R$ and $ny = 0$. Therefore $0 = ny = nbr$

$= tabr$. We get that $abr \in r_R(t) \subseteq r_R(m)$. That is $mabr = 0$. Consider $xy = xbr = mabr = 0$. This implies that $x \in l_M[bR \cap r_R(n)]$. Thus $l_M(b) + {}_NMa \subseteq l_M[bR \cap r_R(n)]$

(5)$\Rightarrow$(3). Assume that

$l_M[bR \cap r_R(n)] = l_M(b) + {}_NMa$ for all $n{=}ta \in N$, for all $b \in R$. ------------(*)

Put $b = 1$ in (*), we get that $l_M[R \cap r_R(n)] = l_M(1) + {}_NMa$. Therefore $l_M r_R(n) = {}_NMa$. □

By putting $N = R$ in Theorem 3.2.3, we then get ${}_NM = M$. So we have :

3.2.4 Corollary. *Let M be a right R-module and $S = End(M)$. Then the following conditions are equivalent:*

(1) M is P-injective.

(2) For each $a \in R$ and each $f \in Hom(aR, M)$, $f(a) \in Ma$.

(3) For each $a \in R$, $l_M r_R(a) = Ma$.

(4) For each $a \in R$ and each $m \in M$, $r_R(a) \subseteq r_R(m)$ implies $Sm \subseteq Ma$

(5) For each $a, b \in R$, $l_M[bR \cap r_R(a)] = l_M(b) + Ma$.

Since for each simple right R-module S, we have $S \cong R/r_R(s)$ for all $0 \neq s \in S$. Thus $r_R(s)$ is a maximal right ideal of R. Therefore $\mathcal{M} = \{R/M \,/\, M$ is a maximal right ideal of $R\}$ is a class of representatives of simple right R-modules. For convenience if given a right ideal I of R and a maximal right ideal M of R, we will denote that ${}_{IM}R = \{x \in R \,/\, xI \subseteq M\}$ and ${}_{IM}\overline{R} = \{x + M \in R/M \,/\, xI \subseteq M\}$. Equivalently, ${}_{IM}R = \{x \in R \,/\, r_R(1+I) \subseteq r_R(x)\}$ and ${}_{IM}\overline{R} = \{x + M \in R/M \,/\, r_R(1+I) \subseteq r_R(x+M)\}$

3.2.5 Other characterizations of *C*-rings.

Let R be a ring . Then the following conditions are equivalent :

(1) R is a C-ring.

(2) For each right ideal I of R, each maximal right ideal M of R, each $a \in R$ *and each* $f \in Hom\,((a+I)R, R/M)$, $f(a+I) \in {}_{IM}\overline{R}\,a$.

(3) For each right ideal I of R, each maximal right ideal M of R and each $a \in R$, $l_{R/M}\, r_R(a+I) = {}_{IM}\overline{R}\,a$.

(4) For each right ideal I of R, each maximal right ideal M of R and each $a, b \in R$, $r_R(a+I) \subseteq r_R(b+M)$ *implies* $S(b+M) \subseteq {}_{IM}\overline{R}\,a$, *where* $S = End(R/M)$

(5) For each right ideal I of R, each maximal right ideal M of R and each $a, b \in R$, $l_{R/M}\,[bR \cap r_R(a+I)] = l_{R/M}\,(b) + {}_{IM}\overline{R}\,a$.

Proof. (1)$\Rightarrow$(2). Assume that R is a C-ring. Let I be a right ideal of R, M be a maximal right ideal of R, $a \in R$ and $f \in Hom\,((a+I)R, R/M)$. Since R/M is simple, by assumption we get R/M is P-R/I-injective. Then there exists $h : R/I \to R/M$ an R-homomorphism such that $hi = f$ where $i : (a+I)R \to R/I$ is an inclusion map. Thus $f(a+I) = hi(a+I) = h(a+I) = h(1+I)\,a$. Because $h(1+I) \in R/M$, let $h(1+I) = z+M$ for some $z \in R$. We will show that $z+M \in {}_{IM}\overline{R}$, i.e., $zI \subseteq M$. For each $x \in I$, we have $x+I = 0$. Thus $h(1+I)x = h(x+I) = h(0) = 0$, we get that $x \in r_R(h(1+I)) = r_R(z+M)$. Therefore $I \subseteq r_R(z+M)$. Let $y \in zI$, thus $y = zw$ for some $w \in I \subseteq r_R(z+M)$. That is $y \in M$, so $zI \subseteq M$. Hence $z+M \in {}_{IM}\overline{R}$. It follows that $f(a+I) = h(1+I)\,a \in {}_{IM}\overline{R}\,a$.

(2)$\Rightarrow$(3). Assume that (2) holds. Let I be a right ideal of R, M be a maximal right ideal of R, $a \in R$ and $x \in {}_{IM}\overline{R}\,a$. Thus $x = (z+M)a$ for some $z+M \in {}_{IM}\overline{R}$. We get that $zI \subseteq M$. We will show that $x \in l_{R/M}\, r_R(a+I)$, i.e., $xy = 0$ for all $y \in r_R(a+I)$. Let $y \in r_R(a+I)$, thus $ay \in I$. We get that $zay \in zI \subseteq M$. Consider

$xy = (z+M)ay = zay + M = M$. This implies that $x \in l_{R/M}\, r_R(a+I)$. Therefore ${}_{IM}\overline{R}\,a \subseteq l_{R/M}\, r_R(a+I)$. Conversely, let $x \in l_{R/M}\, r_R(a+I)$ .Thus $xy = 0$ for all $y \in r_R(a+I)$. Define

$$f : (a+I)R \longrightarrow xR \quad \text{by} \quad f(ar+I) = xr \ \text{ for all } \ r \in R$$

Clearly f is well-defined and an R-homomorphism.Let $i : xR \longrightarrow R/M$ is an inclusion map and $h = if \in Hom\,((a+I)R,\ R/M)$. By (2), we get that $h(a+I) \in {}_{IM}\overline{R}\,a$. Consider $h(a+I) = if(a+I) = f(a+I) = x$, thus $x \in {}_{IM}\overline{R}\,a$. Therefore $l_{R/M}\, r_R(a+I)$. $\subseteq {}_{IM}\overline{R}\,a$.

(3)$\Rightarrow$(4). Assume that (3) holds. Let I be a right ideal of R, M be a maximal right ideal of R, $a, b \in R$ and $r_R(a+I) \subseteq r_R(b+M)$. Then $\{b+M\} \subseteq l_{R/M}\, r_R(b+M) \subseteq l_{R/M}\, r_R(a+I) = {}_{IM}\overline{R}\,a$. Hence $b+M = (r+M)a$ for some $r+M \in {}_{IM}\overline{R}$, thus $rI \subseteq M$. Let $x \in S(b+M)$, then $x = f(b+M)$ for some $f \in S$. We have $f(b+M) = f((r+M)a) = f(r+M)a$. Since $f(r+M) \in R/M$, let $f(r+M) = z+M$ for some $z \in R$. Next, we will show that $z+M \in {}_{IM}\overline{R}$, i.e., $zI \subseteq M$. For each $x \in I$, we have $rx \in rI \subseteq M$. Thus $rx+M = 0$. Consider $f(r+M)x = f(rx+M) = f(0) = 0$, we get that $x \in r_R(f(r+M)) = r_R(z+M)$. Therefore $I \subseteq r_R(z+M)$. Let $y \in zI$, thus $y = zw$ for some $w \in I \subseteq r_R(z+M)$. That is $y \in M$, so $zI \subseteq M$. Hence $z+M \in {}_{IM}\overline{R}$. It follows that $f(b+M) = f(r+M)\,a \in {}_{IM}\overline{R}\,a$. Hence $S(b+M) \subseteq {}_{IM}\overline{R}\,a$.

(4)$\Rightarrow$(1). Assume that (4) holds. We will show that R is a C-ring. Let S be a simple right R-module and N be a cyclic right R-module. Thus $S \cong R/M$ for some maximal right ideal M of R and $N \cong R/I$ for some right ideal I of R. We show that R/M is P-R/I-injective. Let $a \in R$, $f : (a+I)R \longrightarrow R/M$ be an R-homomorphism and $i : (a+I)R \longrightarrow R/I$ be an inclusion map. For each $x \in r_R(a+I)$, we get that $ax+I = 0$. So $f(a+I)x = f(ax+I) = f(0) = 0$. Hence $x \in r_R(f(a+I))$. That is $r_R(a+I) \subseteq r_R(f(a+I))$. By assumption, we get that $S(f(a+I)) \subseteq {}_{IM}\overline{R}\,a$. Thus $f(a+I) = (z+M)a$ for some $z+M \in {}_{IM}\overline{R}$. We have $zI \subseteq M$. Define

$$h : R/I \to R/M \quad \text{by } h(r+I) = zr+M \quad \text{for all } r \in R$$

Then h is well-defined and an R-homomorphism. Let $ar+I \in (a+I)R$ for some $r \in R$. Consider $hi(ar+I) = h(ar+I) = zar+M = (z+M)ar = f(a+I)r = f(ar+I)$, we get that $hi = f$. Hence R/M is P-R/I-injective. It follows that R is a C-ring.

(4)⟹(5). Assume that (4) holds. Let I be a right ideal of R, M be a maximal right ideal of R, and $a, b \in R$. We will show that $l_{R/M}\,[bR \cap r_R(a+I)] = l_{R/M}\,(b) + {}_{IM}\overline{R}\,a$. Let $x \in l_{R/M}\,[bR \cap r_R(a+I)]$, then $xy = 0$ for all $y \in bR \cap r_R(a+I)$. For each $r \in r_R(ab+I)$, $abr+I = 0$. So $br \in bR \cap r_R(a+I)$ and thus $xbr = 0$, i.e., $r \in r_R(xb)$. That is $r_R(ab+I) \subseteq r_R(xb)$. This implies that $xb \in {}_{IM}\overline{R}\,ab$, by (4). Thus $xb = zab$ for some $z \in {}_{IM}\overline{R}$. We have $(x-za)\,b = 0$, thus $x-za \in l_{R/M}\,(b)$. So $x-za = w$ for some $w \in l_{R/M}\,(b)$. Therefore $x = w + za \in l_{R/M}\,(b) + {}_{IM}\overline{R}\,a$. This means that $l_{R/M}\,[bR \cap r_R(a+I)] \subseteq l_{R/M}\,(b) + {}_{IM}\overline{R}\,a$. Conversely, let $x \in l_{R/M}\,(b) + {}_{IM}\overline{R}\,a$. Then $x = y + \bar{z}\,a$ for some $y \in l_{R/M}\,(b)$, $\bar{z} \in {}_{IM}\overline{R}$. Let $\bar{z} = z+M$, thus $zI \subseteq M$. We get that $xb = yb + \bar{z}\,ab = \bar{z}\,ab$. We want to show that $x \in l_{R/M}\,[bR \cap r_R(a+I)]$, i.e., $xy = 0$ for all $y \in bR \cap r_R(a+I)$. Let $y \in bR \cap r_R(a+I)$, thus $y = br$ for some $r \in R$ and $ay+I = I$. That is $I = abr + I$, we have $abr \in I$. We get that $zabr \in zI \subseteq M$. Consider $xy = xbr = \bar{z}\,abr = zabr+M = M$. This implies that $x \in l_{R/M}\,[bR \cap r_R(a+I)]$. Thus $l_{R/M}\,(b) + {}_{IM}\overline{R}\,a \subseteq l_{R/M}\,[bR \cap r_R(a+I)]$.

(5)⟹(3). Assume that

$$l_{R/M}\,[bR \cap r_R(a+I)] = l_{R/M}\,(b) + {}_{IM}\overline{R}\,a. \quad \text{for all } a, b \in R. \quad \text{------------}(*)$$

Put $b = 1$ in (*), we get that $l_{R/M}\,[R \cap r_R(a+I)] = l_{R/M}\,(I) + {}_{IM}\overline{R}\,a$. Therefore $l_{R/M}\; r_R(a+I) = {}_{IM}\overline{R}\,a$ □

3.2.6 Example. Z_6 *is a C-ring.*

Proof. Since $0, \langle \bar{2} \rangle, \langle \bar{3} \rangle$ and Z_6 are the only right ideals of Z_6, $\langle \bar{2} \rangle$ and $\langle \bar{3} \rangle$ are maximal. We will show that $l_{Z_6/M}\; r_{Z_6}(\bar{a} + I) = {}_{IM}\overline{Z_6}(\bar{a})$ for all right ideal I of R, for all maximal right ideal M of R and for all $\bar{a} \in Z_6$.

Case $M = \langle \bar{2} \rangle$.

(1). If $I = \langle \bar{3} \rangle$, we have ${}_{\langle \bar{3} \rangle \langle \bar{2} \rangle}\overline{Z_6} = \{\, \bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \;/\; \bar{x}\langle \bar{3} \rangle \subseteq \langle \bar{2} \rangle \}$ $= \{\langle \bar{2} \rangle\}$. Thus ${}_{\langle \bar{3} \rangle \langle \bar{2} \rangle}\overline{Z_6}(\bar{a}) = \{\langle \bar{2} \rangle\}$ for all $\bar{a} \in Z_6$.

Consider $\bar{a} \in \{\bar{0}, \bar{3}\}$, we have $r_{Z_6}(\bar{a} + \langle \bar{3} \rangle) = \{\bar{x} \in Z_6 \;/\; \overline{ax} \in \langle \bar{3} \rangle\}$ $= Z_6$. Thus $l_{Z_6/\langle \bar{2} \rangle}\; r_{Z_6}(\bar{a} + \langle \bar{3} \rangle) = \{\, \bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \;/\; \overline{xw} \in \langle \bar{2} \rangle$ for all $\bar{w} \in Z_6\} = \{\langle \bar{2} \rangle\}$.

Consider $\bar{a} \in \{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}$, we have $r_{Z_6}(\bar{a} + \langle \bar{3} \rangle) = \{\, \bar{x} \in Z_6 \;/\; \overline{ax} \in \langle \bar{3} \rangle \} = \langle \bar{3} \rangle$. Thus $l_{Z_6/\langle \bar{2} \rangle}\; r_{Z_6}(\bar{a} + \langle \bar{3} \rangle) = \{\, \bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \;/\; \overline{xw} \in \langle \bar{2} \rangle$ for all $\bar{w} \in \langle \bar{3} \rangle\} = \{\langle \bar{2} \rangle\}$.

Therefore $l_{Z_6/\langle \bar{2} \rangle}\; r_{Z_6}(\bar{a} + \langle \bar{3} \rangle) = {}_{\langle \bar{3} \rangle \langle \bar{2} \rangle}\overline{Z_6}(\bar{a})$ for all $\bar{a} \in Z_6$.

(2). If $I = \langle \bar{2} \rangle$, we have ${}_{\langle \bar{2} \rangle \langle \bar{2} \rangle}\overline{Z_6} = \{\, \bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \;/\; \bar{x}\langle \bar{2} \rangle \subseteq \langle \bar{2} \rangle \}$ $= Z_6/\langle \bar{2} \rangle$. Thus ${}_{\langle \bar{2} \rangle \langle \bar{2} \rangle}\overline{Z_6}(\bar{a}) = \{\langle \bar{2} \rangle\}$ when $\bar{a} \in \{\bar{0}, \bar{2}, \bar{4}\}$ and ${}_{\langle \bar{2} \rangle \langle \bar{2} \rangle}\overline{Z_6}(\bar{a}) = Z_6/\langle \bar{2} \rangle$ when $\bar{a} \in \{\bar{1}, \bar{3}, \bar{5}\}$.

Consider $\bar{a} \in \{\bar{0}, \bar{2}, \bar{4}\}$, we have $r_{Z_6}(\bar{a} + \langle \bar{2} \rangle) = \{\bar{x} \in Z_6 \,/\, \overline{ax} \in \langle \bar{2} \rangle\}$ $= Z_6$. Thus $l_{Z_6/\langle \bar{2} \rangle}\, r_{Z_6}(\bar{a} + \langle \bar{2} \rangle) = \{\bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \,/\, \overline{xw} \in \langle \bar{2} \rangle$ for all $\bar{w} \in Z_6\} = \{\langle \bar{2} \rangle\}$.

Consider $\bar{a} \in \{\bar{1}, \bar{3}, \bar{5}\}$, we have $r_{Z_6}(\bar{a} + \langle \bar{2} \rangle) = \{\bar{x} \in Z_6 \,/\, \overline{ax} \in \langle \bar{2} \rangle\}$ $= \langle \bar{2} \rangle$. Thus $l_{Z_6/\langle \bar{2} \rangle}\, r_{Z_6}(\bar{a} + \langle \bar{2} \rangle) = \{\bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \,/\, \overline{xw} \in \langle \bar{2} \rangle$ for all $\bar{w} \in \langle \bar{2} \rangle\} = Z_6/\langle \bar{2} \rangle$.

Therefore $l_{Z_6/\langle \bar{2} \rangle}\, r_{Z_6}(\bar{a} + \langle \bar{2} \rangle) = {}_{\langle \bar{2} \rangle \langle \bar{2} \rangle}\overline{Z_6}(\bar{a})$ for all $\bar{a} \in Z_6$.

(3). If $I = Z_6$, we have ${}_{Z_6 \langle \bar{2} \rangle}\overline{Z_6} = \{\bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \,/\, \bar{x} Z_6 \subseteq \langle \bar{2} \rangle\} = \{\langle \bar{2} \rangle\}$. Thus ${}_{Z_6 \langle \bar{2} \rangle}\overline{Z_6}(\bar{a}) = \{\langle \bar{2} \rangle\}$ for all $\bar{a} \in Z_6$. For each $\bar{a} \in Z_6$, $r_{Z_6}(\bar{a} + Z_6) = Z_6$. So $l_{Z_6/\langle \bar{2} \rangle}\, r_{Z_6}(\bar{a} + Z_6) = \{\bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \,/\, \overline{xw} \in \langle \bar{2} \rangle$ for all $\bar{w} \in Z_6\}$ $= \{\langle \bar{2} \rangle\}$. Therefore $l_{Z_6/\langle \bar{2} \rangle}\, r_{Z_6}(\bar{a} + Z_6) = {}_{Z_6 \langle \bar{2} \rangle}\overline{Z_6}(\bar{a})$ for all $\bar{a} \in Z_6$.

(4). If $I = 0$, we have ${}_{0 \langle \bar{2} \rangle}\overline{Z_6} = \{\bar{x} + \langle \bar{2} \rangle \in Z_6/\langle \bar{2} \rangle \,/\, \bar{x}\, 0 \subseteq \langle \bar{2} \rangle\} = Z_6/\langle \bar{2} \rangle$. Thus ${}_{0 \langle \bar{2} \rangle}\overline{Z_6}(\bar{a}) = \{\langle \bar{2} \rangle\}$ when $\bar{a} \in \{\bar{0}, \bar{2}, \bar{4}\}$ and ${}_{0 \langle \bar{2} \rangle}\overline{Z_6}(\bar{a}) = Z_6/\langle \bar{2} \rangle$ when $\bar{a} \in \{\bar{1}, \bar{3}, \bar{5}\}$.

Consider $\bar{a} = \bar{0}$, we have $r_{Z_6}(\bar{a}) = Z_6$. So $l_{Z_6/\langle \bar{2} \rangle}\, r_{Z_6}(\bar{a}) = \{\bar{x} + \langle \bar{2} \rangle$ $\in Z_6/\langle \bar{2} \rangle \,/\, \overline{xw} \in \langle \bar{2} \rangle$ for all $\bar{w} \in Z_6\} = \{\langle \bar{2} \rangle\}$.

Consider $\bar{a} = \bar{3}$, we have $r_{Z_6}(\bar{a}) = \{\bar{x} \in Z_6 \,/\, \overline{3x} = 0\} = \langle \bar{2} \rangle$.

Thus $l_{Z_6/\langle \bar{2}\rangle}\, r_{Z_6}(\bar{a}) = \{\bar{x} + \langle \bar{2}\rangle \in Z_6/\langle \bar{2}\rangle \ / \ \overline{xw} \in \langle \bar{2}\rangle \text{ for all } \bar{w} \in \langle \bar{2}\rangle\}$
$= Z_6/\langle \bar{2}\rangle$.

Consider $\bar{a} \in \{\bar{2}, \bar{4}\}$, we have $r_{Z_6}(\bar{a}) = \{\bar{x} \in Z_6 \ / \ \overline{ax} = 0\} = \langle \bar{3}\rangle$.

Thus $l_{Z_6/\langle \bar{2}\rangle}\, r_{Z_6}(\bar{a}) = \{\bar{x} + \langle \bar{2}\rangle \in Z_6/\langle \bar{2}\rangle \ / \ \overline{xw} \in \langle \bar{2}\rangle \text{ for all } \bar{w} \in \langle \bar{3}\rangle\}$
$= \{\langle \bar{2}\rangle\}$.

Consider $\bar{a} \in \{\bar{1}, \bar{5}\}$, we have $r_{Z_6}(\bar{a}) = \{\bar{x} \in Z_6 \ / \ \overline{ax} = 0\} = 0$. Thus
$l_{Z_6/\langle \bar{2}\rangle}\, r_{Z_6}(\bar{a}) = Z_6/\langle \bar{2}\rangle$.

Therefore $l_{Z_6/\langle \bar{2}\rangle}\, r_{Z_6}(\bar{a}) = {}_{0\langle\bar{2}\rangle}\overline{Z_6}(\bar{a})$ for all $\bar{a} \in Z_6$.

For $M = \langle \bar{3}\rangle$, we can prove in the same way. Hence Z_6 is a *C*-ring. □

3.2.7 Lemma. *Let E be a right R-module and E be P-N-injective for all cyclic right R-modules N. Then the following conditions are equivalent :*

(1) E cogenerates every cyclic right R-module.

(2) Hom (T, E) $\neq 0$ for all simple right R-modules T.

(3) E cogenerates every simple right R-module.

Proof. (1) ⟹ (3). Let *S* be a simple right *R*-module , thus *S* is cyclic. By assumption , we have *E* cogenerates *S.*

(3)⟹ (2). Assume that *E* cogenerates every simple right *R*-module. Let *T* be a simple right *R*-module. By assumption, *E* cogenerates *T*. Thus $Rej_T(E) = 0$. We want to show that $Hom(T, E) \neq 0$. Suppose that $Hom(T, E) = 0$. For each $h \in Hom(T, E)$, we have *h* is a zero homomorphism. Therefore $Ker\, h = T$. That is $0 = Rej_T(E) = \bigcap\{Ker\, h \,/\, h \in Hom\,(T, E)\} = T$ which is a contradiction. Hence $Hom\,(T, E) \neq 0$.

(2) $\Rightarrow$(1). Assume that (2) holds.We will show that E cogenerates every cyclic right R-module. Let N be a cyclic right R-module and $0 \neq n \in N$. We have nR has a maximal submodule. Let L be a maximal submodule of nR. Thus nR/L is simple. By assumption, there exists $0 \neq f \in Hom\,(nR/L, E)$. Let $\eta : nR \longrightarrow nR/L$ be a natural homomorphism. We get that $f\eta$ is a nonzero homomorphism. Since E is P-N-injective, there exists $h \in Hom\,(N, E)$ such that $hi = f\eta$ where $i : nR \longrightarrow N$ is an inclusion map. Consider $h(n) = hi(n) = f\eta(n) \neq 0$. This implies that h is one-to-one. Thus $Ker\, h = 0$. We get that $Rej_N(E) = \bigcap\{Ker\, h \,/\, h \in Hom\,(N, E)\} = 0$. Therefore E cogenerates every cyclic right R-module. □

3.2.8 Theorem. *The following conditions are equivalent for a ring R :*

(1) Each simple right R-module is injective.

(2) Each simple right R-module is P-M-injective for all right R-modules M.

(3) Each simple right R-module is P-N-injective for all cyclic right R-modules N.

(4) The radical of N, Rad N = 0 for all cyclic right R-modules N.

(5) Each right ideal is an intersection of maximal right ideals.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let S be a simple right R-module and M be a right R-module. We will show that S is P-M-injective. Let $m \in M$, $f : mR \longrightarrow S$ be an R-homomorphism and $i : mR \longrightarrow M$ be an inclusion map. Since S is injective, thus there exists $h : M \longrightarrow S$ an R-homomorphism such that $hi = f$. Hence S is P-M-injective.

(2) $\Rightarrow$ (3) Let S be a simple right R-module and N be a cyclic right R-module. By assumption , we have S is P-N-injective.

(3) $\Rightarrow$ (4) Assume that (3) holds. Let N be a cyclic right R-module. We want to show that $\text{Rad}\, N = 0$. Let $\{ S_i \ / \ i \in I \}$ be a set of representatives of distinct isomorphism classes of simple right R-modules. Let S be a simple right R-module. We have $S \cong S_j$ for some $j \in I$. Let $f: S \rightarrow S_j$ be an R-isomorphism and $\eta_j : S_j \rightarrow \prod_{i \in I} S_i$ be an injection map. We get that $\eta_j f : S \rightarrow \prod_{i \in I} S_i$ is one-to-one. Therefore $\prod_{i \in I} S_i$ cogenerates every simple right R-module. Since S_i is simple for all $i \in I$ and by assumption, S_i is P-N-injective for all $i \in I$. We have $\prod_{i \in I} S_i$ is P-N-injective. We get that $\prod_{i \in I} S_i$ cogenerates every cyclic right R-module, by Lemma 3.2.7. Thus $\prod_{i \in I} S_i$ cogenerates N. Then there exists $0 \rightarrow N \rightarrow (\prod_{i \in I} S_i)^{\Lambda}$ a monomorphism for some index set Λ. This implies that N is cogenerated by the class of simple R-modules. Hence $\text{Rad}\, N = 0$.

(4) $\Rightarrow$ (5) Assume that (4) holds. Let I be a right ideal of R, thus R/I is cyclic. By assumption, we have $\{I\} = \text{Rad}(R/I) = \bigcap \{ M/I \subset_{>} R/I \ / \ M/I$ is maximal in $R/I \}$. We want to prove that $\bigcap \{ M \ / \ I \subseteq M$ is maximal in $R \} = \{I\}$. Let $x \in \bigcap \{ M \ / \ I \subseteq M$ is maximal in $R \}$, then $x \in M$ for all maximal right ideals M containing I of R. Thus for each maximal right ideal M containing I of R, $x + I \in M/I$. Therefore $x + I \in \bigcap \{ M/I \subset_{>} R/I \ / \ M/I$ is maximal in $R/I \} = \{I\}$. We have $x \in I$. Hence $\bigcap \{ M \ / \ I \subseteq M$ is maximal in $R \} \subseteq \{I\}$. It follows that $\bigcap \{ M \ / \ I \subseteq M$ is maximal in $R \} = \{I\}$.

(5) $\Rightarrow$ (1) Assume that (5) holds. Let S be a simple right R-module. We will show that S is injective. Let I be a right ideal of R, $f: I \rightarrow S$ be a nonzero R-homomorphism and $\iota : I \rightarrow R$ be an inclusion map. Thus $I/Ker f \cong Im f = S$. Then there exists $x \in I \setminus Ker f$ such that $Ker f \subset Ker f + xR$. Since $Ker f \subset Ker f + xR \subseteq I$ and $Ker f$ is maximal, we have $Ker f + xR = I$. Because $Ker f$ is a right

ideal of R and by assumption, we have $Ker f = \bigcap_{i \in \Lambda} M_i$ where M_i is maximal for all $i \in \Lambda$. If $I \subseteq M_i$ for all $i \in \Lambda$, then $I \subseteq \bigcap_{i \in \Lambda} M_i = Ker f$ which is a contradiction. Thus there exists $j \in \Lambda$ such that $I \not\subset M_j$. Because $M_j \subset M_j + I \subseteq R$ and M_j is maximal, $M_j + I = R$. Consider $M_j \cap I = M_j \cap (Ker f + xR) = Ker f + (M_j \cap xR)$. Since $Ker f + (M_j \cap xR) \subset_{>} I$, $(Ker f + (M_j \cap xR))/Ker f \subset_{>} I/Ker f$. Because $I/Ker f$ is simple, we get that $Ker f + (M_j \cap xR) = Ker f$ or $Ker f + (M_j \cap xR) = I$. If $Ker f + (M_j \cap xR) = I$, then $M_j \cap I = Ker f + (M_j \cap xR) = I$. This implies that $I \subseteq M_j$ which is a contradiction. Thus $Ker f + (M_j \cap xR) = Ker f$. We get that $M_j \cap I = Ker f$. Define

$$h : M_j + I \longrightarrow S \text{ by } h(m_j + i) = f(i) \quad \text{for all } m_j + i \in M_j + I$$

Next, we will show that h is well-defined. Let $m_j + i$, $m'_j + i' \in M_j + I$ and $m_j + i = m'_j + i'$, thus $i' - i = m_j - m'_j \in M_j$. We have $i' - i \in M_j \cap I = Ker f$. That is $0 = f(i' - i) = f(i') - f(i)$. Therefore h is well-defined and an R-homomorphism. Let $x \in I$, thus $hi(x) = h(x) = h(0 + x) = f(x)$. Hence $hi = f$. It follows that S is injective. □

Definition. *Let R be a ring. R is a right V-ring if every simple right R-module is injective.*

By the definition of V-rings and Theorem 3.2.8, we can conclude that V-ring and C-ring are the same ring. Thus we have :

3.2.9 Theorem. *The following conditions are equivalent for a ring R :*

(1) R is a C-ring.

(2) For each right ideal I of R, each maximal right ideal M of R, each $a \in R$ *and each* $f \in Hom((a+I)R, R/M)$, $f(a+I) \in {}_{IM}\overline{R}a$.

(3) For each right ideal I of R, each maximal right ideal M of R and each $a \in R$, $l_{R/M}\, r_R(a+I) = {}_{IM}\overline{R}a$.

(4) For each right ideal I of R, each maximal right ideal M of R and each $a, b \in R$, $r_R(a+I) \subseteq r_R(b+M)$ *implies* $S(b+M) \subseteq {}_{IM}\overline{R}a$, *where* $S = End(R/M)$

(5) For each right ideal I of R, each maximal right ideal M of R and each $a, b \in R$, $l_{R/M}\,[bR \cap r_R(a+I)] = l_{R/M}(b) + {}_{IM}\overline{R}a$.

(6) R is a V-ring.

(7) Each simple right R-module is injective.

(8) Each simple right R-module is P-M-injective for all right R-modules M.

(9) Each simple right R-module is P-N-injective for all cyclic right R-modules N.

(10) The radical of N, Rad N = 0 for all cyclic right R-modules N.

(11) Each right ideal is an intersection of maximal right ideals.

Proof. (1) - (5) are equivalent by Theorem 3.2.5 , (6) - (11) are equivalent by Theorem 3.2.8 and it is clear that (1) $\Leftrightarrow$ (6) . □

3. Cyclically Injective Rings , *P-V*-Rings and Regular Rings

Definition. *Let R be a ring. R is a right P-V-ring if every simple right R-module is P-injective .*

3.3.1 Example. [[5] , 2.3.6 Example]

The endomorphism ring of a countable infinite dimentional left vector space is a P-V-ring but not a C-ring.

Proof. (For this example, we will write $(a)f$ instead of $f(a)$ where $f \in Hom_F(A, B)$ and $a \in A$.) Before we prove this example, we will give the following remark. For two vector spaces V, W and a linear map $f: V \to W$ we get from the basis extension theorem

(1). If f is a monomorphism, then there is a homomorphism $h: W \to V$ with $fh = 1_V$.

(2). If f is an epimorphism, then there is a homomorphism $k: W \to V$ with $kf = 1_W$.

Let V be a countable infinite dimentional left vector space over a field F with a basis $\{v_n\}_{n \in N}$ and $S = End({}_FV)$. We will show that S is a regular ring. Let $f \in S$, $\overline{V} = V/_{Kerf}$ and $\eta: V \to \overline{V}$ be a natural homomorphism. Since $Ker\,\eta = Ker f$, by Factor Theorem, there exists $g: \overline{V} \to V$ a monomorphism such that $\eta g = f$. Since η is an epimorphism and both $\overline{V}$ and V are vector spaces, there exists $\overline{\eta}: \overline{V} \to V$ an R-homomorphism such that $\overline{\eta}\eta = 1_{\overline{V}}$. Since g is a monomorphism and both $\overline{V}$ and V are vector spaces, there exists $\overline{g}: V \to \overline{V}$ such that $g\overline{g} = 1_{\overline{V}}$. Consider $f = \eta g = \eta 1_{\overline{V}} g = \eta\overline{\eta}\eta g = \eta\overline{\eta}f = \eta 1_{\overline{V}}\overline{\eta}f = \eta g\overline{g}\,\overline{\eta}f = f\overline{g}\,\overline{\eta}f$ with $\overline{g}\,\overline{\eta} \in S$. Therefore S is a regular ring. Hence S is a P-V-ring. Moreover, we have V is a right S-module. Next, we will show that V_S is simple, i.e., $u_1S = V$ for all $0 \neq u_1 \in V$. Let $0 \neq u_1 \in V$, thus $u_1 = \sum_{k \in N} r_k v_k$ for some $r_k \in F$. We get that $u_1S \subseteq V$. We will show that $\{u_1\}$ is linearly independent. Let $r \in F$ and $ru_1 = 0$, thus $\sum_{k \in N}(rr_k)v_k = r(\sum_{k \in N} r_k v_k) = 0$. We have $rr_k = 0$ for all $k \in N$. If $r_k = 0$ for all $k \in N$, then $u_1 = \sum_{k \in N} r_k v_k = 0$ which is a contradiction. Thus $r = 0$. This implies that $\{u_1\}$ is linearly independent. Then we have $\{u_1\}$ can be extended to a basis of ${}_FV$. Let $\{u_1, u_2, u_3, \ldots.\}$ be a basis of ${}_FV$. We want to show that $V \subseteq u_1S$. Let $v \in V$. Define

$h : V \longrightarrow V$ by $(u_1)h = v$, $(u_i)h = 0$ for all $i \in N\backslash\{1\}$ and $(\sum_{k\in N} r_k u_k)h = \sum_{k\in N} r_k((u_k)h)$

Let $\sum_{k\in N} r_k u_k \in V$ and $\sum_{k\in N} r_k u_k = 0$, thus $r_k = 0$ for all $k \in N$. We get that $(\sum_{k\in N} r_k u_k)h = \sum_{k\in N} r_k((u_k)h) = 0$. Therefore h is well-defined and an F-homomorphism. Moreover, we have $h \in S$ and $(u_1)h = v$. We get that $V \subseteq u_1S$. So $V = u_1S$. This implies that V_S is simple. Let $I = \{f \in S / (v_k)f \neq 0$ for only finitely many $k \in N\}$. We have I is an ideal of S. We want to show that S is not a C-ring. Suppose that S is a C-ring. Define

$$g : I \longrightarrow V_S \text{ by } g(f) = \sum_{k\in N} (v_k)f \quad \text{for all } f \in I$$

Therefore g is well-defined and an S-homomorphism. Since V is simple and S is a C-ring, there exists $g^* : S \longrightarrow V$ an S-homomorphism such that $i g^* = g$ where $i : I \longrightarrow S$ is an inclusion map. Since $1_V \in S$, let $g^* 1_V = \sum_{k\in N} r_k v_k$ for some $r_k \in F$. For each $f \in I$, we have $\sum_{k\in N} (v_k)f = g(f) = i g^*(f) = g^*(f) = g^*(1_V f) = (g^* 1_V)f = (\sum_{k\in N} r_k v_k)f = \sum_{k\in N} r_k((v_k)f)$. Let $k \in N$. Define

$f^* : V \longrightarrow V$ by $(v_k)f^* = v_k$, $(v_j)f^* = 0$ for all $j \in N\backslash k$ and $(\sum_{k\in N} r_k v_k)f^* = \sum_{k\in N} r_k((v_k)f^*)$

Therefore f^* is well-defined and an F-homomorphism. Moreover, we have $f^* \in I$. Thus $v_k = \sum_{k\in N} (v_k)f^* = \sum_{k\in N} r_k((v_k)f^*) = r_k v_k$. We get that $(r_k - 1)v_k = 0$, so $r_k = 1$. This implies that $r_k = 1$ for all $k \in N$ which is a contradiction. Hence S is not a C-ring. □

Definition. *Let R be a ring, R is a right duo ring if every right ideal of R is a left ideal of R.*

Recall that for a right ideal I of R and a maximal right ideal M of R, ${}_{IM}R = \{x / xI \subseteq M\}$

3.3.2 Proposition. *If R is a right duo ring, then ${}_{IM}R = M$ or ${}_{IM}R = R$.*

Proof. Let R be a right duo ring and ${}_{IM}R \neq M$. Then there exists $x \in {}_{IM}R \setminus M$. Therefore $xI \subseteq M$. We get that $xa \in M$ for all $a \in I$. Since M is maximal and $x \notin M$, $xR + M = R$. Thus $xr + m = 1$ for some $r \in R$, $m \in M$. We have $a = x(ra) + ma \in M$ for all $a \in I$, since R is right duo. This implies that $I \subseteq M$. Next, we will show that ${}_{IM}R = R$. Let $y \in R$, thus $ya \in I \subseteq M$ for all $a \in I$. Therefore $y \in {}_{IM}R$. Hence ${}_{IM}R = R$. □

From above Proposition, we have ${}_{IM}\overline{R} = 0$ or ${}_{IM}\overline{R} = R/M$.

3.3.3 Lemma. *The following conditions are equivalent for a ring R :*

(1) R is a P-V-ring.

(2) For each maximal right ideal M of R, each $a \in R$ and each $f \in Hom(aR, R/M)$, $f(a) \in (R/M)a$.

(3) For each maximal right ideal M of R and each $a \in R$, $l_{R/M}\, r_R(a) = (R/M)a$.

(4) For each maximal right ideal M of R and each $a, b \in R$, $r_R(a) \subseteq r_R(b+M)$ implies $S(b+M) \subseteq (R/M)a$, where $S = End(R/M)$.

(5) For each maximal right ideal M of R and each $a, b \in R$, $l_{R/M}\,[bR \cap r_R(a)] = l_{R/M}(b) + (R/M)a$.

Proof. See [4] page 29. □

3.3.4 Theorem. *The following conditions are equivalent for a right duo ring R :*

(1) R is a C-ring.

(2) For each maximal right ideal M of R, each $a \in R$ *and each* $f \in Hom(aR, R/M)$, $f(a) \in (R/M)a$.

(3) For each maximal right ideal M of R and each $a \in R$, $l_{R/M}\, r_R(a) = (R/M)a$.

(4) For each maximal right ideal M of R and each $a, b \in R$, $r_R(a) \subseteq r_R(b+M)$ *implies* $S(b+M) \subseteq (R/M)a$, *where* $S = End(R/M)$.

(5) For each maximal right ideal M of R and each $a, b \in R$, $l_{R/M}[bR \cap r_R(a)] = l_{R/M}(b) + (R/M)a$.

(6) R is a P-V-ring.

Proof. (2) - (6) are equivalent by Lemma 3.3.3. Since every *C*-ring is a *P-V*-ring, (1) implies (6). Thus we only prove (4) $\Rightarrow$ (1).

(4)$\Rightarrow$(1). Assume that (4) holds. We want to show that R is a *C*-ring. Let S be a simple right R-module and N be a cyclic right R-module. Thus $S \cong R/M$ for some maximal right ideal M of R and $N \cong R/I$ for some right ideal I of R. We will show that R/M is P-R/I-injective. Let $a \in R$, $f: (a+I)R \longrightarrow R/M$ be a nonzero R-homomorphism and $i: (a+I)R \longrightarrow R/I$ be an inclusion map. For each $x \in r_R(a)$, we get that $ax = 0$. That is $ax + I = I$. So $f(a+I)x = f(ax+I) = f(0) = 0$. Hence $x \in r_R(f(a+I))$. That is $r_R(a) \subseteq r_R(f(a+I))$. By assumption, we get that $S(f(a+I)) \subseteq (R/M)a$. Thus $f(a+I) = (z+M)a$ for some $z+M \in R/M$. Define

$$h : R/I \longrightarrow R/M \quad \text{by} \quad h(r+I) = zr+M \quad \text{for all } r \in R.$$

Next, we will show that h is well-defined. Let $r+I \in R/I$ and $r+I = I$, thus $r \in I$. We get that $zr \in I$. Consider $f(a+I) \in R/M$, we show that $f(a+I) \in {}_{IM}\overline{R}$, i.e.,

$r_R(1+I) \subseteq r_R(f(a+I))$. Let $y \in r_R(1+I)$, thus $y \in I$. Since R is a right duo ring, $ay \in I$. Thus $f(a+I)y = 0$. So $y \in r_R(f(a+I))$. Since $f(a+I)$ is nonzero, by Proposition 3.3.2 we have $f(a+I) \in {}_{IM}\overline{R} = R/M$. It follows that ${}_{IM}R = R$. We have $bI \subseteq M$ for all $b \in R$. Hence $zr \in M$. Then h is well-defined and an R-homomorphism. Let $ar + I \in (a+I)R$ for some $r \in R$. Consider $hi(ar+I) = h(ar+I) = zar+M = (z+M)ar = f(a+I)r = f(ar+I)$, we get that $hi = f$. Hence R/M is P-R/I-injective. It follows that R is a C-ring. □

In general not every P-V-ring is a V-ring, as example 3.3.1 shows. But for a right duo ring, R is a V-ring if and only if R is a P-V-ring.

Definition. *An element a of the ring R is called a (von Neumann) regular element if there is $b \in R$ with $aba = a$. A ring R is called (von Neumann) regular if every element in R is regular.*

For V-ring and regular ring we have :

3.3.5 Theorem. [[2], Theorem 4.8].

Let R be a right duo ring. Then R is a V-ring if and only if R is a regular ring.

By Theorem 3.2.9, 3.3.4 and 3.3.5 we get :

3.3.6 Theorem. *The following conditions are equivalent for a right duo ring R :*

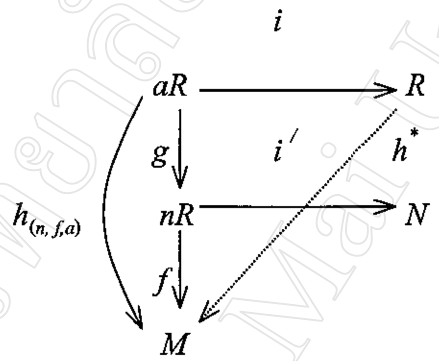
(1) R is a V-ring;

(2) R is a P-V-ring;

(3) R is a regular ring.

We now give another characterization of a regular ring when it is right duo. At first we need the following lemma.

Let M be a right R-module and $N = tR$ be a cyclic right R-module. Assume that M is P-injective. Then for each $n = ta \in N$ and for each $f: nR \longrightarrow M$ an R-homomorphism there exists $g : aR \longrightarrow nR$ defined by $ar \mapsto nr$ an R-homo morphism. Let $h_{(n,f,a)} = fg$. Since M is P-injective, there exists $h^{*}: R \longrightarrow M$ be an R-homomorphism such that $h^{*}i = h_{(n,f,a)} = fg$.



Therefore we will denote

$Hom\,(R, M)_N = \{ h^{*} \in Hom\,(R, M) \;/\; h^{*}$ is an extension of $h_{(n,f,a)}$ for some $n \in N$, $f \in Hom\,(nR, M)$ and $a \in R \}$

Thus for each $h^{*} \in Hom\,(R, M)_N$, there is an $a \in R$ which corresponds to this h^{*} and we will denote it by a^{*}.

3.3.7 Lemma. *Let M be a right R-module. Then the following conditions are equivalent :*

(1) M is P-injective.

(2) M is P-N-injective for all cyclic right R-modules $N = tR$ *with* $h^{*}(1)a^{*} \in {}_{N}Ma^{*}$ *for all* $h^{*} \in Hom\,(R, M)_N$.

(3) M is P-N-injective for all cyclic right R-modules $N = tR$ *with* $r_R(t) \subseteq r_R(h^*(1))$ *for all* $h^* \in Hom(R, M)_N$

(4) M is P-N-injective for all cyclic right R-modules $N = tR$ *with* $r_R(t) \subseteq \cap P$ *where P is a nonzero principal right ideal of R.*

Proof. (1)⇒(2) Assume that (1) holds. Let $N = tR$ be a cyclic right R-module with $h^*(1)a^* \in {}_NMa^*$ for all $h^* \in Hom(R, M)_N$. We want to prove that M is P-N-injective. Let $n = ta \in N$, $f : nR \to M$ be an R-homomorphism and $i : nR \to N$ be an inclusion map. Define

$$g : aR \to nR \quad \text{by} \quad g(ar) = nr = tar \quad \text{for all } r \in R$$

clearly g is well-defined and an R-homomorphism. Putting $h_{(n,f,a)} = fg : aR \to M$ and let $i' : aR \to R$ be an inclusion map. Since M is P-injective, there exists $h^* : R \to M$ be an R-homomorphism such that $h^*i' = h_{(n,f,a)}$. Consider $h^*(a) = h^*(1)a \in {}_NMa$. Let $h^*(a) = ma$ for some $m \in {}_NM$, thus $r_R(t) \subseteq r_R(m)$. Define

$$f^* : N \to M \quad \text{by} \quad f^*(tr) = mr \quad \text{for all } r \in R$$

Let $x \in R$ and $tx = 0$. Then $x \in r_R(t) \subseteq r_R(m)$. We get that $mr = 0$. Hence f^* is well-defined and it is clear that f^* is an R-homomorphism. Next, we will show that $f^*i = f$. Let $nr \in nR$. Then $f^*i(nr) = f^*(nr) = f^*(tar) = mar = h^*(a)r = h^*i'(ar) = h_{(n,f,a)}(ar) = fg(ar) = f(nr)$. It follows that M is P-N-injective .

(2)⇒(3) Assume that (2) holds. Let $N = tR$ be a cyclic right R-module with $r_R(t) \subseteq r_R(h^*(1))$ for all $h^* \in Hom(R, M)_N$. Since $r_R(t) \subseteq r_R(h^*(1))$, $h^*(1) \in {}_NM$. Therefore $h^*(1)a^* \in {}_NMa^*$. By assumption, we have M is P-N-injective.

(3)⇒(4) Assume that (3) holds. Let $N = tR$ be a cyclic right R-module with $r_R(t) \subseteq \cap P$ where P is a nonzero principal right ideal of R. We want to show that $r_R(t) \subseteq r_R(h^*(1))$ for all $h^* \in Hom(R, M)_N$. Let $h^* \in Hom(R, M)_N$, thus h^* is an extension of $h_{(n,f,a)}$ for some $n \in N$, $f \in Hom(nR, M)$ and $a \in R$. We have

$h_{(n,f,a)}(r_R(t)) = fg(r_R(t)) = f(t(r_R(t))) = f(0) = 0$ where $g: aR \longrightarrow nR$ is an R-homomorphism defined by $g(ar) = nr$. That is $r_R(t) \subseteq Ker\, h_{(n,f,a)}$. Consider $x \in Ker\, h^*$ iff $h^*(x) = 0$ iff $x \in r_R(h^*(1))$. This implies that $Ker\, h^* = r_R(h^*(1))$. Since h^* is an extension of $h_{(n,f,a)}$, $Ker\, h_{(n,f,a)} \subseteq Ker\, h^*$. We get that $r_R(t) \subseteq r_R(h^*(1))$. By (3), we have M is P-N-injective.

(4)⇒(1) Assume that (4) holds. Putting $N = R$. Since $R = 1R$, $0 = r_R(1) \subseteq \bigcap P$ where P is a nonzero principal right ideal of R. By assumption, we have M is P-injective. □

3.3.8 Corollary. *Let R be a regular and right duo ring. Then every right R-module is P-N-injective for all cyclic right R-modules N.*

Proof. (⇒) Let M be a right R-module and $N = tR$ be a cyclic right R-module. We want to show that M is P-N-injective. Let $h^* \in Hom(R, M)_N$, thus h^* is an extension of $h_{(n,f,a)}$ for some $n \in N$, $f \in Hom(nR, M)$ and $a \in R$. We get that $h^*(1)a = h^*(a) = h^*i'(a) = fg(a) = f(n)$ where $g: aR \longrightarrow nR$ is an R-homomorphism defined by $g(ar) = nr$ and $i': aR \longrightarrow R$ is an inclusion map. Since R is a regular ring, $a = axa$ for some $x \in R$. Consider $h^*(1)a = h^*(1)axa = f(n)xa = f(nx)a$. Next, we will show that $r_R(t) \subseteq r_R(f(nx))$. Let $y \in r_R(t)$. Since R is a right duo ring, $axy \in r_R(t)$. Thus $nxy = taxy = 0$. We have $f(nx)y = 0$. Therefore $y \in r_R(f(nx))$. This implies that $f(nx) \in {}_N M$. It follows that $h^*(1)a = f(nx)a \in {}_N Ma$. By Lemma 3.3.7, we have M is P-N-injective.

Definition. *Let K and N be right R-modules. K is an N-cyclic submodule of N if K is a submodule of N and $K \cong N/L$ for some submodule L of N.*

Definition. *Let M and N be right R-modules. M is semi-N-injective if every R-homomorphism $f: K \longrightarrow M$, K an N-cyclic submodule of N, extends to N.*

3.3.9 Proposition. *Let M be a right R- module. If M is P-N-injective for all cyclic right R-modules N, then it is semi -N -injective for all cyclic right R-modules N.*

Proof. Assume that M is P-N-injective for all cyclic right R-modules N. Let A be a cyclic right R-module . By assumption, we have M is P-A-injective. We want to prove that M is semi-A-injective. Let K be an A-cyclic submodule of A, $f : K \longrightarrow M$ be an R-homomorphism and $i : K \longrightarrow A$ be an inclusion map. Therefore $K \subset_{>} A$ and $K \cong A/L$ for some $L \subset_{>} A$. We get that K is cyclic. Since M is P-A-injective, there exists $h : A \longrightarrow M$ an R-homomorphism such that $hi = f$. Hence M is semi-A-injective. □

3.3.10 Proposition. *The following conditions are equivalent for a right duo ring R :*

(1) R is regular;

(2) Every right R-module is P-N-injective for all cyclic right R-modules N;

(3) Every right R-module is semi-N-injective for all cyclic right R-modules N;

(4) Every right R-module is P-injective;

(5) R is a V-ring;

(6) R is a P-V-ring.

Proof. (1) ⇒(2) is Corollary 3.3.8 , (2) ⇒(3) is Proposition 3.3.9, (4) ⇒(1) see [4] page 18 and (1)⇔ (5) ⇔ (6) is Theorem 3.3.6. Thus we only prove (3) ⇒ (4).

(3)⇒(4). Let M be a right R-module. By (3) , we have M is semi-R-

injective. We want to show that M is P-injective. Let $0 \neq a \in R$, $f: aR \longrightarrow M$ be an R-homomorphism and $i: aR \longrightarrow R$ be an inclusion map. Since $aR \cong R/r_R(a)$, we get that aR is an R-cyclic submodule of R. Because M is semi-R-injective, there exists $h: R \longrightarrow M$ an R-homomorphism such that $hi = f$. Hence M is P-injective. □