Chapter 2
Preliminaries

Consider the equation

$$ty^{(n)}(t) + kty^{(n-1)}(t) = f(t) \tag{2.1}$$

where $f(t)$ and $y(t)$ are functions in the space $\mathcal{D}'$ of distribution, k is any real number and $n \geq 2$, $t \in (-\infty, \infty)$.

We are finding the solutions of (2.1). Before finding such solutions, the following definition and basic concepts are needed.

2.1 Distribution

Fundamental space of test functions

For $p = 0, 1, 2, ...$ and a compact set $K \subset R$, we use the following standard notation

$C^p \equiv C^p(R)$: The space of all complex valued functions on R with continuous derivatives at least up to order p.

$C_0^p \equiv C_0^p(R)$: The subspace of C^p comprising of all functions with compact supports.

$C_K^p \equiv C_K^p(R)$: The subspace of C_0^p comprising of all functions with supports contained in the same fixed compact subset K of R.

For $p = \infty$ we define

$C^\infty \equiv C^\infty(R)$: The space of all complex valued functions on R which have continuous derivative of all order.

$C_0^\infty \equiv C_0^\infty(R)$: The subspace of C^∞ with compact supports.

$C_K^\infty \equiv C_K^\infty(R)$:The subspace of C_0^∞ comprising of all functions with supports contained in the same fixed compact subset K of R.

Definition 2.1.1 We define the space $\mathcal{D}_K \equiv \mathcal{D}_K(R)$ to be space C_K^∞ and $\mathcal{D} \equiv \mathcal{D}(R)$ to be space C_0^∞.

Definition 2.1.2 A functional on vector space $\mathcal{D}$ is a mapping $\mu : \mathcal{D} \to C$ where C is a complex number. For all $\varphi \in \mathcal{D}$, the value of μ acting on φ is denoted by

$$\mu(\varphi)\text{or}\langle \mu, \varphi \rangle \in C.$$

We are interested in functionals which are
(a) linearity: that is

$$\langle \mu, \alpha_1\varphi_1 + \alpha_2\varphi_2 \rangle = \alpha_1 \langle \mu, \varphi_1 \rangle + \alpha_2 \langle \mu, \varphi_2 \rangle$$

for any scalar α_1 and α_2, and
(b) continuous in the some sense: μ is said to be a continuous fucntional on $\mathcal{D}$ if and only if whenever a sequence $(\varphi_n)_{n\in N}$ converges to zero in $\mathcal{D}$ (in the agreed sense) the corresponding sequence of complex number $(\langle \mu, \varphi \rangle)_{n\in N}$ converges to zero in the usual sense.

The set of all continuous linear functionals on the linear space $\mathcal{D}$ is denoted by $\mathcal{D}'$. It forms a linear space in its own right under the natural componentwise definitions on vector addition and multiplication by scalars:

$$\langle \mu + \nu, \varphi \rangle = \langle \mu, \varphi \rangle + \langle \nu, \varphi \rangle$$
$$\langle \alpha\mu, \varphi \rangle = \alpha \langle \mu, \varphi \rangle$$

where $\mu, \nu \in \mathcal{D}'$ and α is any scalar. $\mathcal{D}'$ is called the dual space of $\mathcal{D}$. The function φ is called a testing function and the function μ is called the generalized function or distribution.

Definition 2.1.3 Let $\mu(t)$ be a locally integrable function (i.e., a function that is integrable in the Lebesgue sense over every finite interval) corresponding to $\mu(t)$, we can define a distribution μ through the convergent integral

$$\langle \mu, \varphi \rangle = \langle \mu(t), \varphi(t) \rangle = \int_{-\infty}^{\infty} \mu(t)\varphi(t)dt \qquad (2.1.1)$$

$\mu \in \mathcal{D}'$ (Zemanian [2], page 7).

Definition 2.1.4 The distribution that can be generated through the equation (2.1.1) from locally integrable functions is called the regular distribution. A distribution that is not generated by locally integrable function is called a singular distribution.

An example of a singular distribution is the Dirac-delta δ which is defined by the equation

$$\langle \delta, \varphi \rangle = \varphi(0).$$

Definition 2.1.5 The sequence of distribution $\{\mu_k\}_{k=1}^{\infty}$ is said to converge in $\mathcal{D}'$ if, for every φ in $\mathcal{D}$, the sequence of number $\{\langle \mu_k, \varphi \rangle\}_{k=1}^{\infty}$ converge in the ordinary sense of the convergence of numbers. The limit of $\{\langle \mu_k, \varphi \rangle\}_{k=1}^{\infty}$ which we shall denote by $\langle \mu, \varphi \rangle$ defines a functional μ acting on the space $\mathcal{D}$. In this case we shall also say that μ is the limit in $\mathcal{D}'$ of $\{\mu_k\}_{k=1}^{\infty}$ and we write $\lim_{k \to \infty} \mu_k = \mu$.

Theorem 2.1.1 If a sequence of distributions $\{\mu_k\}_{k=1}^{\infty}$ converges in $\mathcal{D}'$ to the functional μ, then μ is also a distribution. In other words, the space $\mathcal{D}'$ is closed under convergence.

Proof: See [2] page 37. □

Definition 2.1.6 (Differentiation of Distributions)

The first derivative $\mu'(t)$ of any distribution $\mu(t)$, where t is one-dimensional, is the functional on $\mathcal{D}$ given by

$$\langle \mu'(t), \varphi(t) \rangle = \langle \mu(t), -\varphi'(t) \rangle \quad \varphi \in \mathcal{D}.$$

Example

(1) The first derivative of the Dirac-delta functional $\delta^{(1)}$ denoted by the equation

$$\langle \delta', \varphi \rangle = \langle \delta, -\varphi' \rangle = -\varphi'(0).$$

(2) We define

$$H(t) = \begin{cases} 1 & \text{for } 0 < t \\ 0 & \text{for } 0 > t \end{cases}$$

$H(t)$ is called the Heaviside function and H is a locally integrable function

$$\begin{aligned}\langle H'(t),\varphi\rangle &= \langle H,-\varphi'\rangle \\ &= -\int_{-\infty}^{\infty} H(t)\varphi'(t)dt \\ &= -\int_{0}^{\infty} \varphi'(t)dt \\ &= -(\varphi(t))_0^{\infty} \\ &= -\varphi(\infty)+\varphi(0) = \varphi(0) = \langle \delta,\varphi\rangle.\end{aligned}$$

Thus $H' = \delta$.

Definition 2.1.7 (The multiplication of distribution by the infinitely differentiable functions)

Let $\mu(t)$ be an infinitely differentiable function, define

$$\langle \mu(t)T,\varphi(t)\rangle = \langle T,\mu(t)\varphi(t)\rangle \tag{2.1.2}$$

for all $\varphi \in \mathcal{D}$.

Theorem 2.1.2.

1. $\mu(t)\delta = \mu(0)\delta, \mu$ is infinitely differentiable
2. $t\delta(t) = 0$

Proof:

1. $\langle \mu(t)\delta,\varphi(t)\rangle = \langle \delta,\mu(t)\varphi(t)\rangle$ by (2.1.2)
$$\begin{aligned} &= \mu(0)\varphi(0) \\ &= \mu(0)\langle \delta,\varphi(t)\rangle \\ &= \langle \mu(0)\delta,\varphi(t)\rangle.\end{aligned}$$
Thus $\mu(t)\delta = \mu(0)\delta \quad$ for all $\varphi \in \mathcal{D}$.
2. $\langle t\delta,\varphi(t)\rangle = \langle \delta,t\varphi(t)\rangle, \quad$ for all $\varphi \in \mathcal{D}$
$$= 0\varphi(0) = 0 = \langle 0,\varphi(t)\rangle.$$
Thus $t\delta = 0$.

Theorem 2.1.3 A necessary and sufficient condition for a distribution $\mu(t)$ defined over R to satisfy the equation

$$t^m \mu(t) = 0 \tag{2.1.3}$$

where m is a positive integer, is that $\mu(t)$ be a linear combination of the Dirac-delta functional and its derivatives of order not greater than $m-1$. That is

$$\mu = \sum_{k=0}^{m-1} c_k \delta^{(k)} \tag{2.1.4}$$

where the c_r are arbitary constants.
Proof: That (2.1.4) is necessary for (2.1.3) to hold is obvious. To prove the sufficiency of (2.1.3), we need the following Lemmas.

Lemma 2.1.1 For a testing function $\chi(t)$ in $\mathcal{D}$ to have the form

$$\chi(t) = t^m \varphi(t) \quad \varphi(t) \in \mathcal{D} \tag{2.1.5}$$

it is necessary and sufficient that

$$\chi^{(k)} = 0 \qquad k = 0, 1, ..., m-1 \tag{2.1.6}$$

Proof: See [2] page 81. □
Lemma 2.1.2 Let $\lambda(t)$ be a fixed testing function in $\mathcal{D}$ such that $\lambda(0) = 1$ and $\lambda^{(k)}(0) = 0$ $(k = 1, 2, ..., m-1)$. Then any testing function $\psi(t)$ in $\mathcal{D}$ can be uniquely decomposed according to

$$\psi(t) = \lambda \sum_{k=0}^{m-1} \frac{1}{k!} \psi^{(k)}(0) t^k + \chi(t)$$

where $\chi(t)$ is in $\mathcal{D}$ and satisfies (2.1.6).
Proof: See [2] page 82. □
Proof of Theorem 2.1.3 Let $t^m \mu(t) = 0$, then for any $\lambda(t)$ of the form given in Lemma 2.1.1,

$$\langle \mu, \chi \rangle = \langle \mu(t), t^m \varphi(t) \rangle = \langle t^m \mu(t), \varphi(t) \rangle = 0. \tag{2.1.7}$$

Thus, for every ψ in $\mathcal{D}$, we have by Lemma 2.1.2 and by (2.1.7) that

$$
\begin{aligned}
\langle \mu, \psi \rangle &= \langle \mu(t), \lambda(t) \sum_{k=0}^{m-1} \frac{1}{k!} \psi^{(k)}(0) t^k + \chi(t) \rangle \\
&= \langle \mu(t), \lambda(t) \sum_{k=0}^{m-1} \frac{1}{k!} \psi^{(k)}(0) t^k \rangle + \langle \mu(t), \chi(t) \rangle \\
&= \sum_{k=0}^{m-1} \frac{1}{k!} \psi^{(k)}(0) \langle \mu(t), \lambda(t) t^k \rangle.
\end{aligned}
$$

Define the constant c_k by

$$c_k = \frac{1}{k!}(-1)^k \langle \mu(t), t^k \lambda(t) \rangle.$$

Since $\psi^{(k)}(0) = (-1)^k \langle \delta^{(k)}, \psi(t) \rangle$, we have

$$\langle \mu, \psi \rangle = \langle \sum_{k=0}^{m-1} c_k \delta^{(k)}, \psi(t) \rangle.$$

Hence $\mu(t) = \sum_{k=o}^{m-1} c_k \delta^{(k)}$. □

Theorem 2.1.4 Let μ be a distribution that is defined over some neighborhood of a fixed finite closed interval I in R. There exists a nonnegative integer r and a finite positive constant C such that for every φ in $\mathcal{D}_I$

$$|\langle \mu, \varphi \rangle| \leq C \sup_t |\varphi^{(r)}(t)|. \tag{2.1.8}$$

Both C and r depend in general upon μ and I.

Proof: Suppose that for a given μ no relatation such as (2.1.8) can hold. Therefore for any positive integer k there exists a testing function φ_k in $\mathcal{D}_I$ such that

$$
\begin{aligned}
|\langle \mu, \varphi_k \rangle| &> k(b-a)^k \sup_{t \in I} |\varphi_k^{(k)}(t)| \\
&= k \gamma_k(\varphi_k).
\end{aligned} \tag{2.1.9}
$$

where $\gamma_k(\varphi_k) = (b-a)^k \sup_{t \in I} |\varphi_k^{(k)}(t)|$.

Let $\theta_k = \dfrac{\varphi_k}{k \gamma_k(\varphi_k)}$, then $\theta_k \in \mathcal{D}_I$.

Let m be some nonnegative integer. For $k \geq m$,

$$\gamma_m(\theta_k) \leq \gamma_k(\theta_k) = \frac{\gamma_k(\varphi_k)}{k \gamma_k(\varphi_k)} = \frac{1}{k}.$$

Thus $\gamma_k(\theta_k) \to 0$ as $k \to \infty$, consequently $\theta_k \to 0$ in $\mathcal{D}_I$.

Since μ is a continuous functional on $\mathcal{D}_I$, we have $\langle \mu, \theta_k \rangle \to 0$ as $k \to \infty$. But (2.1.9) implies that

$$|\langle \mu, \theta_k \rangle| = \frac{|\langle \mu, \varphi_k \rangle|}{k\gamma_k(\varphi_k)} > \frac{k\gamma_k(\varphi_k)}{k\gamma_k(\varphi_k)} = 1.$$

Thus contradiction. □

Definition 2.1.8 (The space $\mathcal{S}$ of testing functions of rapid descent)

Let $\mathcal{S}$ is the collection of all complex valued function φ on R which are infinitely differentiable and $\varphi(t)$ in $\mathcal{S}$ satisfies the infinite set of inequalities

$$|t^m \varphi^{(k)}(t)| \leq C_{mk} \qquad -\infty < t < \infty$$

where m and k run through all nonnegative integers. Here the C_{mk} are constants (with respect to t) which depend upon m and k.

Note $\mathcal{D}$ is a proper subspace of $\mathcal{S}$.

Definition 2.1.9 (Convergence in $\mathcal{S}$)

A sequence of functions $\{\varphi_k(t)\}_{k=1}^{\infty}$ is said to converge in $\mathcal{S}$ if every function $\varphi_k(t)$ is in $\mathcal{S}$ and if, for each nonnegative m and k, the sequence $\{t^m \varphi_k(t)\}_{k=1}^{\infty}$ converge uniformly in R.

Theorem 2.1.5 The space $\mathcal{D}$ is dense in the space $\mathcal{S}$ in the sense that for each φ in $\mathcal{S}$ there exists a sequence $\{\varphi_k(t)\}_{k=1}^{\infty}$ with every $\varphi_k(t)$ in $\mathcal{D}$ which converges in $\mathcal{S}$ to $\varphi(t)$.

Proof: See [2] page 101. □

Definition 2.1.10 (The space $\mathcal{S}'$ of distribution of slow growth)

A distribution μ is said to be of slow growth if it is a continuous linear functional on the space $\mathcal{S}$ of testing functions of rapid descent(such distributions are also called tempered distributions). That is, a distribution μ of slow growth is a rule that assigns a number $\langle \mu, \varphi \rangle$ to each φ in $\mathcal{S}$ in such a way that the following condition are fulfilled.

Linearity If φ_1 and φ_2 are in $\mathcal{S}$ and if any numbers α and β, then

$$\langle \mu, \alpha\varphi_1 + \beta\varphi_2 \rangle = \alpha\langle \mu, \varphi_1 \rangle + \beta\langle \mu, \varphi_2 \rangle.$$

Continuity If $\{\varphi_k\}_{k=1}^{\infty}$ is any sequence that converges in $\mathcal{S}$ to zero, then

$$\lim_{k\to 0}\langle \mu, \varphi_k\rangle = 0.$$

The space of all distribution of slow growth is denoted by $\mathcal{S}'$. $\mathcal{S}'$ is also called the dual space of $\mathcal{S}$.

Note $\mathcal{S}'$ is a proper subspace of $\mathcal{D}'$.

Theorem 2.1.6 If $\mu(\tau)$ is in $\mathcal{S}'_\tau$ and $\varphi(t,\tau)$ is in $\mathcal{S}_{t,\tau}$ then

$$\psi(t) = \langle \mu(\tau), \varphi(t,\tau)\rangle \quad \text{is in } \mathcal{S}_t \text{ and}$$
$$\psi^{(k)}(t) = \langle \mu(\tau), \varphi^{(k)}(t,\tau)\rangle$$

where $\mathcal{S}_t, \mathcal{S}_\tau$ and $\mathcal{S}_{t,\tau}$ is denoted the spaces of testing functions of rapid descent defined over the space of variable t, τ and (t,τ) respectively.

Proof: See [2] page 112. □

2.2 Convolution

Definition 2.2.1 (Direct product)

Let $\mu(t)$ be a distribution in $\mathcal{D}'_t$ and $\nu(\tau)$ be a distribution in $\mathcal{D}'_\tau$. If $\varphi(t,\tau)$ is an element of $\mathcal{D}_{t,\tau}$, we define the direct product

$$\langle \mu(t)\times\nu(\tau), \varphi(t,\tau)\rangle = \langle \mu(t), \langle \nu(\tau), \varphi(t,\tau)\rangle\rangle.$$

Theorem 2.2.1 The direct product $\mu(t)\times\nu(\tau)$ of two distributions $\mu(t)$ and $\nu(\tau)$ is a distribution in $\mathcal{D}'_{t,\tau}$.

Proof: See [2] page 115. □

Theorem 2.2.2 The direct product of two distributions of slow growth is another distribution of slow growth.

That is if $\mu(t)$ is in $\mathcal{S}'_t$ and $\nu(\tau)$ is in $\mathcal{S}'_\tau$ then $\mu(t)\times\nu_\tau$ is in $\mathcal{S}'_{t,\tau}$.

Proof: See [2] page 116. □

Example 2.2.1 The direct product of the delta function over R with it self yields the delta functional over R^2. That is

$$\delta(t)\times\delta(\tau) = \delta(t,\tau),$$

because with φ in $\mathcal{D}_{t,\tau}$

$$\begin{aligned}\langle \delta(t) \times \delta(\tau), \varphi(t,\tau)\rangle &= \langle \delta(t), \langle \delta(\tau), \varphi(t,\tau)\rangle\rangle \\ &= \langle \delta(t), \varphi(t,0)\rangle = \varphi(0,0) \\ &= \langle \delta(t,\tau), \varphi(t,\tau)\rangle.\end{aligned}$$

Theorem 2.2.3 The support of the direct product of two distribution is the cartesian product of their supports. That is, if Ω_μ is the support of the distribution $\mu(t)$ and Ω_ν is the support of the distribution $\nu(\tau)$, the support of $\mu(t) \times \nu(\tau)$ is the set $\Omega_\mu \times \Omega_\nu$.

Proof: See [2] page 118. □

Lemma 2.2.1 The space of all testing functions of the form

$$\varphi(t,\tau) = \sum_k \psi_k(t)\theta_k(\tau)$$

where the $\psi_k(t)$ are in $\mathcal{D}_t$, the $\theta_k(\tau)$ are in $\mathcal{D}_\tau$ and the summation has a finite number of terms, is dense in $\mathcal{D}_{t,\tau}$.

Proof: See [2] page 119. □

Theorem 2.2.4 The direct product of two distribution in commutative

$$\mu(t) \times \nu(\tau) = \nu(\tau) \times \mu(t).$$

That is, for every testing function $\varphi(t,\tau)$ in $\mathcal{D}_{t,\tau}$, we have

$$\langle \mu(t), \langle \nu(\tau), \varphi(t,\tau)\rangle\rangle = \langle \nu(\tau), \langle \mu(t), \varphi(t,\tau)\rangle\rangle. \tag{2.2.1}$$

Proof: See [2] page 120. □

Corollary 2.2.1 The equation (2.2.1) still holds when $\mu(t)$ is in $\mathcal{S}'_t$, $\nu(\tau)$ is in $\mathcal{S}'_\tau$, and $\varphi(t,\tau)$ is in $\mathcal{S}_{t,\tau}$.

Defintion 2.2.2 Let $\mu(t)$ and $\nu(t)$ be two continuous functions with bounded support. A convolution between $\mu(t)$ and $\nu(t)$ is denoted by $\mu(t) * \nu(t)$ and defined by

$$\mu(t) * \nu(t) = \int_{-\infty}^{\infty} \mu(t)\nu(t-\tau)d\tau.$$

The above equation cannot be used when μ and ν are arbitary distribution because, for one reason, two distribution cannot be multiplies in general. Let φ be in $\mathcal{D}$, we may write

$$\langle \mu * \nu, \varphi \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(\tau)\nu(t-\tau)\varphi(t)d\tau dt.$$

By applying the change of variable $\tau = x$ and $t = x + y$

$$\langle \mu * \nu, \varphi \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x)\nu(y)\varphi(x+y)dxdy.$$

The last expression has a form that is similar to that of the direct product of two regular distribution. Thus

$$\begin{aligned}\langle \mu * \nu, \varphi \rangle &= \langle \mu(t) \times \nu(\tau), \varphi(t+\tau) \rangle \\ &= \langle \mu(t), \langle \nu(\tau), \varphi(t+\tau) \rangle \rangle. \end{aligned} \tag{2.2.2}$$

However, a problem arises in this case. Even though the function $\varphi(t+\tau)$ is infinitely smooth, it is not a testing function, since its support is not bounded in the (t, τ) plane. If the support of $\mu(t) \times \nu(\tau)$ intersects the support of $\varphi(t+\tau)$ in a bounded set, say Ω, we can replace the right hand side of (2.2.2) by

$$\langle \mu(t) \times \nu(\tau), \lambda(t,\tau)\varphi(t+\tau) \rangle \tag{2.2.3}$$

where $\lambda(t,\tau)$ is some testing function in $\mathcal{D}_{t,\tau}$ that is equal to one over some neighborhood of Ω. Thus $\lambda(t,\tau)\varphi(t+\tau) \in \mathcal{D}_{t,\tau}$.

We have yet to determine under what conditions the intersection of the support of $\mu(t) \times \nu(\tau)$ and $\varphi(t+\tau)$ is always bounded for all φ in $\mathcal{D}$ and whether $\mu * \nu$ is a distribution.

Theorem 2.2.5 Let μ and ν be two distribution over R and let their convolution $\mu * \nu$ be defined by (2.2.2), when the right-hand side of (2.2.2) is understood to be (2.2.3). Then, $\mu * \nu$ will exist as a distribution over R under any one of the following conditions:

a. Either μ or ν has a bounded support.

b. Both μ and ν have supports bounded on the left
[i.e.,there exists some constant T_1 such that $f(t) = g(t) = 0$ for $t < T_1$].

c. Both μ and ν have supports bounded on the right
[i.e.,there exists some constant T_2 such that $f(t) = g(t) = 0$ for $t > T_2$].

Proof: See [2] page 124. □

Corallary 2.2.1 The convolution of two distribution is commutative $\mu * \nu = \nu * \mu$. That is, for every φ in $\mathcal{D}$,

$$\langle \mu(t), \langle \nu(\tau), \varphi(t+\tau)\rangle\rangle = \langle \nu(\tau), \langle \mu(t), \varphi(t+\tau)\rangle\rangle.$$

Theorem 2.2.6 Let Ω_μ and Ω_ν be the respective supports of the distributions μ and ν, which are defined over R, and let $\Omega_\mu + \Omega_\nu$ be the set in R each of whose points can be written as the sum of a point in Ω_μ and a point in Ω_ν. Then, the support of $\mu * \nu$ is contained in $\Omega_\mu + \Omega_\nu$.

Proof: See[2] page 125. □

Corollary 2.2.2 If both of the supports of the distribution μ and ν are either (a) bounded or (b) bounded on the left or (c) bounded on the right, then the support of $\mu * \nu$ is respectively either (a) bounded or (b) bounded on the left or (c) bounded on the right.

Example 2.2.2 Given $\mu \in \mathcal{D}'$ be any distribution, then

1. $\delta * \mu = \mu$
2. $\delta^{(m)} * \mu = \mu^{(m)}$ where $\delta^{(m)}$ and $\mu^{(m)}$ are the distributions with m-derivatives.

Proof 1. $\langle (\delta * \mu)(t), \varphi(t)\rangle = \langle \mu(t), \langle \delta(r), \varphi(t+r)\rangle\rangle$

$$= \langle \mu(t), \varphi(t)\rangle.$$

Thus $\delta * \mu = \mu$.

2.
$$\begin{aligned}\langle (\delta^{(m)} * \mu)(t), \varphi(t)\rangle &= \langle \mu(t), \langle \delta^{(m)}(r), \varphi(t+r)\rangle\rangle \\ &= \langle \mu(t), \langle \delta(r), (-1)^m \varphi^{(m)}(t+r)\rangle\rangle \\ &= \langle \mu(t), (-1)^m \varphi^{(m)}(t)\rangle \\ &= \langle \mu^{(m)}(t), \varphi(t)\rangle.\end{aligned}$$

Thus $\delta^{(m)} * \mu = \mu^{(m)}$.

Definition 2.2.3 (Convolution equation)

Consider a convolution equation in the form

$$f * u = g \tag{2.2.4}$$

when f and g are known distributions and u is an unknown distribution. We may seek all possible solutions u in $\mathcal{D}'$. We want (2.2.4) has a unique solution u, thus we restrict the convolution to some convolution algebra.

A space $\mathcal{A}'$ of distribution is said to be a convolution algebra if it possesses the following properties:

1. $\mathcal{A}'$ is a linear space.
2. $\mathcal{A}'$ is closed under convolution.
3. Convolution is associative for any three distributions in $\mathcal{A}'$.

Theorem 2.2.7 Let an equation $f * u = g$ where $f, g \in \mathcal{D}'_R$(the space of distribution whose support bounded from the left or sometimes call the right side distribution). A necessary and sufficient condition for the equation to have at least one solution in $\mathcal{D}'_R$ for every g in $\mathcal{D}'_R$ is that f possess an inverse f^{*-1} in $\mathcal{D}'_R$. When f does possess an inverse in $\mathcal{D}'_R$, this inverse is unique and the equation possesses a unique solution in $\mathcal{D}'_R$, given by

$$u = \delta * u = f^{*-1} * f * u = f^{*-1} * g. \tag{2.2.5}$$

Proof: If the equation $f * u = g$ has at least one solution in $\mathcal{D}'_R$, then one of the solution of the equation with $g = \delta$ will be one of the inverse of f. Conversely, if f has an inverse in $\mathcal{D}'_R$, the equation (2.2.5) will be one of the solution of the equation.

Finally, we will show the equation possesses a unique solution in $\mathcal{D}'_R$. For if u and v are both in $\mathcal{D}'_R$ and are solutions, then

$$f * u = g \quad \text{and} \quad f * v = g.$$

On the convolving all terms by this particular inverse f^{*-1}, we get

$$u = f^{*-1} * g \quad \text{and} \quad v = f^{*-1} * g.$$

Since convolution is a single-valued operation, $u = v$.
This also implies that f^{*-1} is unique in $\mathcal{D}'_R$. □

Theorem 2.2.8 If h and j are in $\mathcal{D}'_R$ and possess inverses in $\mathcal{D}'_R$, then

$$(h * j)^{*-1} = h^{*-1} * j^{*-1}. \tag{2.2.6}$$

Proof: Since convolution in $\mathcal{D}'_R$ is associative and commutative,

$$(h * j) * (h^{*-1} * j^{*-1}) = (h * h^{*-1}) * (j * j^{*-1}) = \delta * \delta = \delta.$$

This establishes (2.2.6). □

Definition 2.2.4 Let L denote the general differential operator of the form

$$L = a_n \frac{d^n}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{d}{dt} + a_0 \tag{2.2.7}$$

when the $a_i (i = 0, 1, 2, ..., n)$ are constants, $a_n \neq 0$, and $n \geq 1$. We wish to solve the equation

$$Lu = g \tag{2.2.8}$$

where g is a known distribution in $\mathcal{D}'_R$ and u is unknown but also required to be in $\mathcal{D}'_R$. The equation (2.2.8) may be written as a convolution equation:

$$(L\delta) * u = g \tag{2.2.9}$$

$$L\delta = a_n \delta^{(n)} + a_{n-1} \delta^{(n-1)} + \cdots + a_1 \delta^{(1)} + a_0 \delta. \tag{2.2.10}$$

Thus, the technique developed for the convolution algebra $\mathcal{D}'_R$ may be applied here and, as we have shown the problem becomes simply that of finding in $\mathcal{D}'_R$ an inverse for $L\delta$.

Theorem 2.2.9 The distribution $L\delta$, given by the equation (2.2.10) with the $a_i (i = 1, 2, ..., n; n \geq 1)$ being constants and $a_n \neq 0$, has an inverse in $\mathcal{D}'_R$. This inverse is $H(t)\phi(t)$, where $\phi(t)$ is that classical solution of the homogeneous equation $Lu = 0$ which satisfies the initial conditions

$$\phi(0) = \phi^{(1)}(0) = \cdots = \phi^{(n-2)}(0) = 0 \tag{2.2.11}$$

$\phi^{(n-1)}(0) = \frac{1}{a_n}$.

$H(t)\phi(t)$ is called the Green's function for L.

Proof: We have

$$\begin{aligned}
\langle (H(x)\phi(x))', \varphi(x) \rangle &= \langle (H(x)\phi(x)), -\varphi'(x) \rangle \\
&= \langle H(x), -\phi(x)\varphi'(x) \rangle \\
&= \int_0^\infty (-\phi(x))\varphi'(x)dx \\
&= [-\phi(x)\varphi(x)]_0^\infty + \int_0^\infty \varphi(x)\phi'(x)dx \\
&= \phi(0)\varphi(0) + \langle H(x)\phi'(x), \varphi(x) \rangle \\
&= \langle \phi(0)\delta, \varphi(x) \rangle + \langle H(x)\phi'(x), \varphi(x) \rangle \\
&= \langle \phi(0)\delta + H(x)\phi'(x), \varphi(x) \rangle.
\end{aligned}$$

Thus $(H(x)\phi(x))' = H(x)\phi'(x) + \phi(0)\delta$. Similarly

$$\begin{aligned}(H(x)\phi(x))'' &= H(x)\phi''(x) + \phi'(0)\delta + \phi(0)\delta' \\ (H(x)\phi(x))^{(n-1)} &= H(x)\phi^{(n-1)}(x) + \phi^{(n-2)}(0)\delta + \cdots + \phi(0)\delta^{(n-2)} \\ (H(x)\phi(x))^{(n)} &= H(x)\phi^{(n)}(x) + \phi^{(n-1)}(0)\delta + \cdots + \phi(0)\delta^{(n-1)}.\end{aligned}$$

By the initial condition, we obtain

$$\begin{aligned}(H(x)\phi(x))^{(\nu)} &= H(x)\phi^{(\nu)}(x) \qquad \nu = 1, 2, ..., n-1 \\ (H(x)\phi(x))^{(n)} &= H(x)\phi^{(n)}(x) + \frac{1}{a_n}\delta.\end{aligned}$$

Thus $(L\delta) * (H(x)\phi(x)) = L(H(x)\phi(x)) = H(x)L\phi(x) + \delta$.
Since $L\phi(x) = 0$, hence $(L\delta) * (H(x)\phi(x)) = \delta$.
It follows that $(L\delta)^{*-1} = H(x)\phi(x)$. □

Example 2.2.3 Find the particular solution of the equation

$$y''(x) + 4y(x) = x, \quad 0 \le x \le \infty$$

Now consider $y''(x) + 4y(x) = 0$. The classical solution is

$$\phi(x) = A\cos 2x + B\sin 2x$$

under the condition $\phi(0) = 0$, $\phi'(0) = 1$.
Thus $A = 0$, $B = \frac{1}{2}$. $\phi(x) = \frac{1}{2}\sin 2x$.
Let $y(x) \in \mathcal{D}'_R$, then the particular solution is $y(x) = (H(x)\phi(x)) * x$. Thus

$$\begin{aligned}y(x) &= \int_{-\infty}^{\infty} H(r)\phi(r)(x-r)dr \\ &= \frac{1}{2}\int_{-\infty}^{\infty} H(r)\sin 2r(x-r)dr \\ &= \frac{1}{2}[x\int_0^x \sin 2rdr] - \frac{1}{2}\int_0^x r\sin 2rdr \\ &= -\frac{1}{4}x\cos 2x + \frac{1}{4}x + \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x \\ &= -\frac{1}{8}\sin 2x + \frac{1}{4}x.\end{aligned}$$

Theorem 2.2.10 Let $Lu = g$ be a linear differential equation with constant coefficients, where L is given by (2.2.5) and g is a continuous function for $t \ge t_0$. The solution to $Lu = g$ that satisfies

$$u(t_0) = u_0, u^{(1)}(t_0) = u_1, ..., u^{(n-1)}(t_0) = u_{n-1}$$

is given by

$$u(t) = \int_{t_0}^{t} \varphi(t-\tau)g(\tau)d\tau + \sum_{i=0}^{n-1} b_i \varphi^{(i)}(t-t_0)$$

where $\varphi(t)$ is the classical solution to the homogeneous equation $Lu = 0$ that satisfies the initial conditions (2.2.11) and

$$b_i = a_{i+1}u_0 + a_{i+1}u_1 + \cdots + a_n u_{n-i-1}.$$

Proof: See [2] page 162. □

2.3 Laplace Transform of Distribution

Definition 2.3.1 Let $f(t)$ be a locally integrable function that satisfies conditions A:

1. $f(t) = 0$ for $-\infty < t < T$.
2. There exists a real number c such that $f(t)e^{-ct}$ is absolutely integrable over $-\infty < t < \infty$.

Let s denote the complex variable $s = \sigma + i\omega$. The Laplace transformation is an operation $\mathcal{L}$ that assigns a function $F(s)$ of the complex variable s to each locally integrable function $f(t)$ that satisfies conditions A. $\mathcal{L}$ is defined by

$$\mathcal{L}f(t) = F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{2.3.1}$$

The function $F(s)$ is called the Laplace transform of $f(t)$.

The fact that $f(t) = 0$ for $-\infty < t < T$ allows us to write

$$\mathcal{L}f(t) = F(s) = \int_{T}^{\infty} f(t)e^{-st}dt \tag{2.3.2}$$

We shall refer to (2.3.2) as a right-side Laplace transform.

(2.3.2) may be broken up into the sum of two integrals

$$\mathcal{L}f(t) = \mathcal{L}_B f(t) + \mathcal{L}_+ f(t). \tag{2.3.3}$$

Here, $$\mathcal{L}_B f(t) = F_B(s) = \int_{T}^{0} f(t)e^{-st}dt. \tag{2.3.4}$$

$$\mathcal{L}_+ f(t) = F_+(s) = \int_{0}^{\infty} f(t)e^{-st}dt. \tag{2.3.5}$$

The second part of condition A implies that (2.3.4) converges absolutely for every (finite) value of s. Furthermore, since

$$|f(t)e^{-st}| \leq |f(t)e^{-ct}|$$

whenever $t > 0$ and Re $s \geq c$, (2.3.5) converges absolutely for all s in the half plane Re $s \geq c$. We can therefore conclude that (2.3.2) converges absolutely for all s in this half-plane.

The greatest lower bound σ_a on all possible values of c, for which the second of condition A holds, is called the abscissa of absolute convergence and the open half-plane Re$s > \sigma_a$ is called the half-plane(or region) of absolute convergence for the Laplace transform (2.3.2). We show refer to a half-plane that is bounded on the left but extends infinitely to the right as a right-sided half-plane. Note that the Laplace transform of a right sided function that satisfies conditions A has as its region of convergence either a right-sided half-plane or the entire s plane.

Theorem 2.3.1 Let $f(t)$ be a continuous function that satisfies condition A and let σ_a be the abscissa of absolute convergence for $\mathcal{L}f(t) = F(s)$. Then, $F(s)$ is an analytic function for Re $s > \sigma_a$ and

$$F^{(k)}(s) = \int_T^\infty (-t)^k f(t)e^{-st}dt \qquad \text{Re } s > \sigma_a \tag{2.3.6}$$

Proof: See [2] page 215. □

Theorem 2.3.2 Let the locally integrable function $f(t)$ satisfy conditions A, let its ordinary first derivative $f'(t)$ exists and be continuous throughout some open interval $a < t < b$, and let σ_a be the abscissa of absolute convergence for $\mathcal{L}f(t) = F(s)$. For each real constant c greater than σ_a and for $a < t < b$,

$$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i}\lim_{y\to\infty}\int_{c-iy}^{c+iy} F(s)e^{st}ds \tag{2.3.7}$$

where the path of integration is along the vertical line $s = c + iy$.
Proof: See [2] page 216. □

Example 2.3.1 Let us establish that

$$\mathcal{L}[\frac{t^k}{k!}H(t)] = \frac{1}{s^{k+1}}, \qquad \text{Re } s > 0 \ \ ; k = 0, 1, 2, ... \tag{2.3.8}$$

where

$$H(t) = \begin{cases} 1 & \text{if } o \le t < \infty \\ 0 & \text{if } -\infty < t < 0 \end{cases}$$

a heaviside function.

Proof: By Mathematical induction, for $k = 0$, $\mathcal{L}(1) = \frac{1}{s}$

Now, let k be a positive integer. Through an integration by parts, we may write

$$\begin{aligned} \int_0^\infty \frac{t^k}{k!} e^{-st} dt &= \frac{1}{s} \int_0^\infty \frac{t^{k-1}}{(k-1)!} e^{-st} dt \\ &= \frac{1}{s} \mathcal{L}[\frac{t^{k-1}}{(k-1)!} H(t)] \quad \text{Re } s > 0. \end{aligned}$$

According to the last expression, (2.3.8) will certainly hold so long as it holds when k is replace by $k-1$. This prove (2.3.8).

Definition 2.3.2 The Laplace transforms of right-side distributions, recall the Laplace transform $G(s)$ of a locally integrable function $g(t)$ that satisfies the conditions of definition 2.3.1

$$G(s) = \mathcal{L}g(t) = \int_T^\infty g(t) e^{-st} dt \quad , \text{Re } s > \sigma_a.$$

This relation may be written in the form

$$G(s) = \langle g(t), e^{-st} \rangle.$$

Now we try to define the Laplace transform of right-side distributions. Let $f(t)$ be a distribution whose support is bounded on the left and there exists a real number c for which $e^{-ct} f(t)$ is a Tempered Distribution. Define

$$F(s) = \mathcal{L}f(t) = \langle e^{-ct} f(t), \lambda(t) e^{-(s-c)t} \rangle \tag{2.3.9}$$

where $\lambda(t)$ is any infinitely differentiable function with bounded support on the left, which equal to one over the neighborhood of the support of $f(t)$.For Re $s > c$, $\lambda(t) e^{-(s-c)t}$ is a testing function in the space $\mathcal{S}$ of testing functions of rapid descent and that $e^{-ct} f(t)$ is in the space $\mathcal{S}'$ of Tempered Distributions.

(2.3.9) can be deduced to

$$F(s) = \mathcal{L}f(t) = \langle f(t), e^{-st} \rangle. \tag{2.3.10}$$

Now $F(s)$ is a function of s defined over the right-sided half-plane Re $s > c$.

Example 2.3.2 The following formulas are the consequence of definition 2.3.2

1. $\mathcal{L}\delta = 1$ for $-\infty < \text{Re } s < \infty$ where δ is Dirac-delta functional whose support concentrated on a single point.
2. $\mathcal{L}\delta(t-\tau) = e^{-s\tau}$ for $-\infty < \text{Re } s < \infty$ and τ is a real constant.
3. $\mathcal{L}\delta^{(k)} = s^k$ for $-\infty < \text{Re } s < \infty$ and k is a positive integer.
4. $\mathcal{L}\delta^{(k)}(t-\tau) = s^k e^{-s\tau}$ for $-\infty < \text{Re } s < \infty$.
5. $\mathcal{L}(t^k f(t)) = (-1)^k F^{(k)}(s)$ for Re $s > \sigma$, where $f(t)$ is a distribution in $\mathcal{D}'_R$.
6. $\mathcal{L}f^{(k)}(t) = s^k F(s)$ for Re $s > \sigma$, where $f(t)$ is a distribution in $\mathcal{D}'_R$.