Chapter 3
Main Results

Theorem *Given the equation*

$$ty^{(n)}(t) + kty^{(n-1)}(t) = f(t) \tag{3.1}$$

where $t \in R$, $f(t)$ and $y(t)$ are functions in space $\mathcal{D}'$ of distribution and k is any real number. Then we obtain the solution $y(t)$ of (3.1) as the following cases
(1) If $f(t)$ is a locally integrable, then

$$y(t) = \frac{-(k+C)}{(n-2)!}\Big[\int_0^\infty \frac{r^{n-2}}{k+C} f(t-r)dr + \int_0^\infty r^{n-2} f(t-r) \sum_{i=1}^{\infty} \frac{k^{i-1}(-r)^i}{(n+i-2)(n+i-3)\cdots(n-1)} dr\Big].$$

(2) If $f(t)$ is a singular distribution of the form $f(t) = \sum_{i=1}^{m} a_i \delta^{(i)}$ where $\delta^{(i)}$ is a Dirac-delta with i-derivatives

(i) for $m = 0$ and $n \geq 2$, then

$$y(t) = \frac{-a_0(k+C)t^{n-2}H(t)}{(n-2)!}\left[\frac{1}{k+C} + \sum_{i=1}^{\infty} \frac{k^{i-1}(-t)^i}{(n+i-2)(n+i-3)\cdots(n-1)}\right]$$

(ii) for $0 < m \leq n-2$, then

$$y(t) = \frac{-1}{(n-2)!}\left[\sum_{i=0}^{m} a_i (t^{n-2})^{(i)} H(t)\right] + (k+C)\left[\sum_{r=0}^{\infty}\sum_{i=1}^{m} \frac{(-k)^r}{(n+r-1)!} a_i (t^{n+r-1})^{(i)} H(t)\right]$$

(iii) for $m > n-2$, $m = n-1+l$ $(l = 0, 1, 2, \ldots)$, then

$$\begin{aligned} y(t) = \frac{-1}{(n-2)!}&\left[\sum_{i=0}^{n-2} a_i (t^{n-2})^{(i)} H(t)\right] \\ &+ (k+C)\left[\sum_{r=0}^{\infty} \frac{(-k)^r}{(n+r-1)!}\left(\sum_{i=0}^{n-2} a_i (t^{n+r-1})^{(i)} H(t)\right)\right] \\ &- \left[\sum_{i=0}^{l} a_{n+i-1}\delta^{(i)}(t)\right] \\ &+ (k+C)\left[\sum_{r=0}^{\infty} \frac{(-k)^r}{r!}\left(\sum_{i=0}^{l} a_{n+i-1}(t^r H(t))^{(i)}\right)\right]. \end{aligned}$$

Proof: Now we consider the *Green's function* for (3.1), that is consider the equation

$$ty^{(n)}(t) + kty^{(n-1)}(t) = \delta(t). \tag{3.2}$$

Take the Laplace Transform to (3.2), we obtain

$$-\frac{d}{ds}s^nY(s) - k\frac{d}{ds}s^{n-1}Y(s) = 1$$

or

$$s(s+k)Y'(s) + ((s+k)n-k)Y(s) = \frac{-1}{s^{n-2}}.$$

By separating variables,

$$\frac{d}{ds}[Y(s)s^{n-1}(s+k)] = -1.$$

Thus $\quad Y(s)s^{n-1}(s+k) = -s + C$ where C is any constant or

$$Y(s) = \frac{-s+C}{s^{n-1}(s+k)}.$$

By applying the binomial expansion

$$Y(s) = \frac{-1}{s^{n-1}} + \frac{k+C}{s^n} - \frac{k^2+Ck}{s^{n+1}} + \frac{k^3+Ck^2}{s^{n+1}} - \cdots .$$

Take the inverse Laplace Transform directly to $Y(s)$ and use formula of example 2.3.1, we obtain

$$\begin{aligned} y(t) &= -\frac{t^{n-2}H(t)}{(n-2)!} + \frac{(k+C)t^{n-1}H(t)}{(n-1)!} - \frac{k(k+C)t^nH(t)}{n!} \\ &\quad + \frac{k^2(k+C)t^{n+1}H(t)}{(n+1)!} - \cdots \\ &= \frac{-(k+C)t^{n-2}H(t)}{(n-2)!}\Big[\frac{1}{k+C} + \\ &\quad \sum_{i=1}^{\infty}\frac{k^{i-1}(-t)^i}{(n+i-2)(n+i-3)\cdots(n-1)}\Big]. \end{aligned}$$

Since $L(y(t)) = \delta$, where $L = t\frac{d^n}{dt^n} + kt\frac{d^{n-1}}{dt^{(n-1)}}$, y(t) is called *Green's function* and we replace it by $G(t)$ then

$$L(G(t)) = \delta(t). \tag{3.3}$$

From (3.1) we write again by

$$L(y(t)) = f(t). \tag{3.4}$$

Convolving both sides of (3.4) by $G(t)$, we obtain

$$G(t) * L(y(t)) = G(t) * f(t). \tag{3.5}$$

By operating L of a convolution, we obtain

$$L(G(t)) * y(t) = G(t) * f(t).$$

From (3.3), we get $\delta(t) * y(t) = G(t) * f(t)$. Thus

$$y(t) = G(t) * f(t).$$

Now, for the case (1), if $f(t)$ is a locally integrable function, then

$$\begin{aligned}
G(t) * f(t) &= \int_{-\infty}^{\infty} G(r) f(t-r)dr \\
&= \int_{-\infty}^{\infty} \frac{-(k+C)r^{n-2}H(r)}{(n-2)!}\Big[\frac{1}{k+C} \\
&\quad + \sum_{i=1}^{\infty} \frac{k^{i-1}(-r)^i}{(n+i-2)(n+i-3)\cdots(n-1)}\Big] f(t-r)dr \\
&= -\frac{(k+C)}{(n-2)!}\Big[\int_0^{\infty} \frac{r^{n-2}}{k+C} f(t-r)dr + \\
&\quad \int_0^{\infty} r^{n-2} f(t-r) \sum_{i=1}^{\infty} \frac{k^{i-1}(-r)^i}{(n+i-2)(n+i-3)\cdots(n-1)} dr\Big].
\end{aligned}$$

Since $f(t)$ is locally integrable and G(t) is an integrable function, so the integral of right hand-side converges.

Hence $y(t) = (G * f)(t)$ exists and unique.

Case (2), if $f(t)$ is a singular distribution.

Consider

$$f(t) = \sum_{r=0}^{m} a_r \delta^{(r)}(t)$$

where a_r is a constant for all $r = 0, 1, 2, ..., m$

Then

$$G(t)*f(t) = -\frac{(k+C)t^{n-2}H(t)}{(n-2)!}\Big[\frac{1}{k+C}+\sum_{i=1}^{\infty}\frac{k^{i-1}(-t)^i}{(n+i-2)(n+i-3)\cdots(n-1)}\Big]$$

$$*\sum_{r=0}^{m}a_r\delta^{(r)}(t)$$

$$= \Big[-\frac{t^{n-2}H(t)}{(n-2)!}+\frac{(k+C)t^{n-1}H(t)}{(n-1)!}-\frac{k(k+C)t^nH(t)}{n!}+$$

$$\frac{k^2(k+C)t^{n+1}H(t)}{(n+1)!}-\cdots\Big]*\sum_{r=0}^{m}a_r\delta^{(r)}(t)$$

$$= -\frac{t^{n-2}H(t)}{(n-2)!}*\big[a_0\delta+a_1\delta^{(1)}+\cdots+a_m\delta^{(m)}\big]$$

$$+\frac{(k+C)t^{n-1}H(t)}{(n-1)!}*\big[a_0\delta+a_1\delta^{(1)}+\cdots+a_m\delta^{(m)}\big]$$

$$-\frac{k(k+C)t^nH(t)}{n!}*\big[a_0\delta+a_1\delta^{(1)}+\cdots+a_m\delta^{(m)}\big]$$

$$+\frac{k^2(k+C)t^{n+1}H(t)}{(n+1)!}*\big[a_0\delta+a_1\delta^{(1)}+\cdots+a_m\delta^{(m)}\big]-\cdots$$

$$= -\frac{1}{(n-2)!}\big[a_0(t^{n-2}H(t)*\delta)+a_1(t^{n-2}H(t)*\delta^{(1)})$$

$$+\cdots+a_m(t^{n-2}H(t)*\delta^{(m)})\big]$$

$$+\frac{k+C}{(n-1)!}\big[a_0(t^{n-1}H(t)*\delta)+a_1(t^{n-1}H(t)*\delta^{(1)})$$

$$+\cdots+a_m(t^{n-1}H(t)*\delta^{(m)})\big]$$

$$-\frac{k(k+C)}{n!}\big[a_0(t^nH(t)*\delta)+a_1(t^nH(t)*\delta^{(1)})$$

$$+\cdots+a_m(t^nH(t)*\delta^{(m)})\big]$$

$$+\frac{k^2(k+C)}{(n+1)!}\big[a_0(t^{n+1}H(t)*\delta)+a_1(t^{n+1}H(t)*\delta^{(1)})$$

$$+\cdots+a_m(t^{n+1}H(t)*\delta^{(m)})\big]$$

$$-\cdots$$

$$
\begin{aligned}
G(t)*f(t) = &-\frac{1}{(n-2)!}\big[a_0(t^{n-2}H(t)) + a_1(t^{n-2}H(t))^{(1)} \\
&+\cdots + a_m(t^{n-2}H(t))^{(m)}\big] \\
&+\frac{k+C}{(n-1)!}\big[a_0(t^{n-1}H(t)) + a_1(t^{n-1}H(t))^{(1)} \\
&+\cdots + a_m(t^{n-1}H(t))^{(m)}\big] \\
&-\frac{k(k+C)}{n!}\big[a_0(t^{n}H(t)) + a_1(t^{n}H(t))^{(1)} \\
&+\cdots + a_m(t^{n}H(t))^{(m)}\big] \\
&+\frac{k^2(k+C)}{(n+1)!}\big[a_0(t^{n+1}H(t)) + a_1(t^{n+1}H(t))^{(1)} \\
&+\cdots + a_m(t^{n+1}H(t))^{(m)}\big] \\
&-\cdots \\
= &-\frac{1}{(n-2)!}\left[\sum_{i=0}^{m} a_i(t^{n-2}H(t))^{(i)}\right] + \frac{k+C}{(n-1)!}\left[\sum_{i=0}^{m} a_i(t^{n-1}H(t))^{(i)}\right] \\
&-\frac{k(k+C)}{n!}\left[\sum_{i=0}^{m} a_i(t^{n}H(t))^{(i)}\right] + \frac{k^2(k+C)}{(n+1)!}\left[\sum_{i=0}^{m} a_i(t^{n+1}H(t))^{(i)}\right] \\
&-\cdots \\
= &-\frac{1}{(n-2)!}\left[\sum_{i=0}^{m} a_i(t^{n-2}H(t))^{(i)}\right] \\
&+(k+C)\left[\sum_{r=0}^{\infty}\sum_{i=0}^{m}\frac{(-k)^r}{(n+r-1)}a_i(t^{n+r-1}H(t))^{(i)}\right]
\end{aligned}
$$

(i) if $m = 0$ for $n \geq 2$, then

$$
\begin{aligned}
y(t) &= G(t)*f(t) \\
&= G(t)*a_0\delta(t) \\
&= a_0G(t).
\end{aligned}
$$

(ii) If $0 < m \leq n-2$, then

$$
\begin{aligned}
G(t) * f(t) = & -\frac{1}{(n-2)!}\big[a_0(t^{n-2}H(t)) + a_1(t^{n-2}H(t))^{(1)} \\
& + \cdots + a_m(t^{n-2}H(t))^{(m)}\big] \\
& + \frac{k+C}{(n-1)!}\big[a_0(t^{n-1}H(t)) + a_1(t^{n-1}H(t))^{(1)} \\
& + \cdots + a_m(t^{n-1}H(t))^{(m)}\big] \\
& - \frac{k(k+C)}{n!}\big[a_0(t^{n}H(t)) + a_1(t^{n}H(t))^{(1)} \\
& + \cdots + a_m(t^{n}H(t))^{(m)}\big] \\
& + \cdots \\
= & -\frac{1}{(n-2)!}\big[a_0(t^{n-2}H(t)) + a_1((n-2)t^{n-3}H(t)) + \cdots \\
& + a_m((n-2)(n-3)\cdots(n-m+1)t^{n-(m+2)}H(t))\big] \\
& + \frac{k+C}{(n-1)!}\big[a_0(t^{n-1}H(t)) + a_1((n-1)t^{n-2}H(t)) + \cdots \\
& + a_m((n-1)(n-2)\cdots(n-m)t^{n-(m+1)}H(t))\big] \\
& - \frac{k(k+C)}{n!}\big[a_0(t^{n}H(t)) + a_1(nt^{n-1}H(t)) + \cdots \\
& + a_m(n(n-1)(n-2)t^{n-m}H(t))\big] \\
& + \cdots \\
= & -\frac{1}{(n-2)!}\left[\sum_{i=0}^{m} a_i(t^{n-2})^{(i)}H(t)\right] \\
& + (k+C)\left[\sum_{r=0}^{\infty}\frac{(-k)^r}{(n+r-1)!}\sum_{i=o}^{m} a_i(t^{n+r-1})^{(i)}H(t)\right].
\end{aligned}
$$

(iii) For $m > n-2$, let $m = n+l-1 \quad (l = 0, 1, 2, \ldots)$. By computing directly the same as before we obtain

$$
\begin{aligned}
G(t) * f(t) = & \frac{-1}{(n-2)!}\left[\sum_{i=0}^{n-2} a_i(t^{n-2})^{(i)}H(t)\right] \\
& + (k+C)\left[\sum_{r=0}^{\infty}\frac{(-k)^r}{(n+r-1)!}\left(\sum_{i=0}^{n-2} a_i(t^{n+r-1})^{(i)}H(t)\right)\right] \\
& - \left[\sum_{i=0}^{l} a_{n+i-1}\delta^{(i)}(t)\right] \\
& + (k+C)\left[\sum_{r=0}^{\infty}\frac{(-k)^r}{r!}\left(\sum_{i=0}^{l} a_{n+i-1}(t^r H(t))^{(i)}\right)\right].
\end{aligned}
$$