CHAPTER 2
PRELIMINARIES

In this chapter, we give some definitions , notations and basic concepts which will be used in later chapters.

2.1 Euler Equation

Consider the differential equation of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)y(x) = b(x) \qquad (2.1.1)$$

where $a_i(x), i = 0, 1, 2, ..., n$ and $b(x)$ are defined in some interval I . The coefficients $a_i(x)$ are continuous in an interval I and $a_0(x) \neq 0$ for any x . Equation of this form are called *linear* differential equations .

If $b(x) = 0$ in I ,then

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)y(x) = 0 \qquad (2.1.2)$$

and (2.1.2) is said to be a *homogeneous equation* .

In (2.1.2) , let $a_i(x) = a_i x^{n-i}, (0 \leq i \leq n)$, where the a_i are constants . Then (2.1.2) becomes

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + ... + a_ny(x) = 0 \qquad (2.1.3)$$

which is known as the $n-$order Euler equation. We assume that the interval I does not contain the point $x = 0$.

Example 2.1.1

1. $m_0t^2y''(t) + m_1ty' + m_2y(t) = 0$ is the general form of the second order Euler equation.

2. $m_0t^3y''' + m_1t^2y''(t) + m_2ty' + m_3y(t) = 0$ is the general form of the third order Euler equation.

3. $m_0t^4y^{(4)} + m_1t^3y'''(t) + m_2t^2y'' + m_3ty' + m_4y(t) = 0$ is the general form of the fourth order Euler equation.

2.2 Distribution

Fundamental spaces of test functions

For $p = 0, 1, 2, ...$and a compact set $K \subset R$, we use the following standard notation:

$C^p \equiv C^p(R)$: The space of all complex-valued functions on R with continuous derivatives at least up to order p.

$C_0^p \equiv C_0^p(R)$: The subspace of C^p comprising all functions with compact support.

$C_K^p \equiv C_K^p(R)$:The subspace of C_0^p comprising all functions with compact support contained in the same fixed compact K.

For $p = \infty$, we define

$C^\infty \equiv C^\infty(R)$: The space of all complex-valued functions on R which have continuous derivatives of all order.

$C_0^\infty \equiv C_0^\infty(R)$: The subspace of C^∞ comprising all infinitely differentiable functions with compact support.

$C_K^\infty \equiv C_K^\infty(R)$: The subspace of C_0^∞ comprising all infinitely differentiable functions with compact support contained in the same fixes compact K.

Definition 2.2.1 The space of testing function , which is denoted by $\mathcal{D}$ consists of all complex-valued function $\varphi(t)$ with continuous derivatives of all orders and with bounded support, which means that function vanishes outside of some bounded region.

Function $\varphi(t)$ is called the testing function . The testing functions can be added and multiplied by real numbers to yield new testing functions , so that $\mathcal{D}$ is a linear space .

Definition 2.2.2 A functional on linear(vector) space is a mapping $\mu : \mathcal{D} \to C$,where C is the set of complex numbers. For all $\varphi \in \mathcal{D}$, the value of μ acting on φ is denoted by

$$\mu(\varphi) \text{ or } < \mu, \varphi > \ \in C$$

We are particularly interested in functionals which are

1) linearity : μ is said to be a linear funtional on $\mathcal{D}$ if and only if given any two testing function φ_1, φ_2 in $\mathcal{D}$ and any scalar α_1, α_2 in C , we have

$$< \mu, \alpha_1\varphi_1 + \alpha_2\varphi_2 >= \alpha_1 < \mu, \varphi_1 > +\alpha_2 < \mu, \varphi_2 >$$

2) continuity: μ is said to be continuous functional on $\mathcal{D}$ if and only if whenever a sequence $(\varphi_n)_{n\in N}$ converges to zero in $\mathcal{D}$ (in the agreed sense) the corresponding sequence of complex numbers $(< \mu, \varphi_n >)_{n\in N}$ converges to zero in the usual sense.

A continuous linear functional on a linear space $\mathcal{D}$ is a distribution . And the space of all such distribution is denoted by $\mathcal{D}'$.

Then, the functions belonging to $\mathcal{D}$ will generally be called testing function, while the functionals belonging to $\mathcal{D}'$ will be called generalized functions or distribution.

Example 2.2.1

(1) The locally integrable function f is a distribution , that is, generated by the locally integrable function f . Then we define $< \mu_f, \varphi >= \int_K f(t)\varphi(t)dt$, where K is a support of φ and $\varphi \in \mathcal{D}$.

(2) The Dirac delta functional is a distribution defined by $< \delta, \varphi >= \varphi(0)$ for $\varphi \in \mathcal{D}$ and the support of δ is $\{0\}$.

Definition 2.2.3 Let $\mu(t)$ be a locally integrable function (i.e., a function that is inergrable in the Lebesgue sense over every finite interval) corresponding to $\mu(t)$, we can define a distribution μ through the convergent integral

$$< \mu, \varphi >=< \mu(t), \varphi(t) >= \int_{-\infty}^{\infty} \mu(t)\varphi(t)dt \tag{2.2.1}$$

,where $\mu \in \mathcal{D}'$.

Definition 2.2.4 A distribution μ that is generated by a locally integrable function is called a *regular distribution* . A distribution that is not generated by a locally integrable function is called a *singular distribution.*

An example of a distribution that is a singular distribution is the so-called Delta function or Dirac delta function that denoted by δ , and is defined by the equation

$$< \delta, \varphi > = \varphi(0) \tag{2.2.2}$$

Clearly , (2.2.2) is a continuous linear functional on $\mathcal{D}$. However, this distribution cannot be obtained from a locally integrable function through the use of (2.2.1). Indeed, if there were such a function $\delta(t)$, then we would have

$$\varphi(0) = \int_{-\infty}^{\infty} \delta(t)\varphi(t)dt$$

for all $\varphi(t) \in \mathcal{D}$.

If

$$\varphi(t) = \begin{cases} exp(\dfrac{-a^2}{a^2 - t^2}) & ; -a < t < a \\ 0 & ; otherwise \end{cases}$$

,we have

$$\varphi(0) = \frac{1}{e} = \int_{-a}^{a} \delta(t) exp(\frac{-a^2}{a^2 - t^2})dt \tag{2.2.3}$$

If $\delta(t)$ were a locally integrable function, the right-hand side of (2.2.3) converges to zero as $a \to 0$. This would be contradiction. Hence , δ is not a regular distribution. It is singular distribution .

Definition 2.2.5 The sequence of distribution $\{\mu_k\}_{k=1}^{\infty}$ is said to converge in $\mathcal{D}'$ if , for every φ in $\mathcal{D}$, the sequence of number $\{< \mu_k, \varphi >\}_{k=1}^{\infty}$ converge in the ordinary sense of the convergence of numbers. The limit of $\{< \mu_k, \varphi >\}_{k=1}^{\infty}$ which we shall denote by $< \mu, \varphi >$ defines a functional μ acting on the space $\mathcal{D}$. In this case we shall also say that μ is the limit in $\mathcal{D}'$ of $\{\mu_k\}_{k=1}^{\infty}$ and we write $\lim_{k \to \infty} \mu_k = \mu$.

Theorem 2.2.1 If a sequence of distributions $\{\mu_k\}_{k=1}^{\infty}$ converges in $\mathcal{D}'$ to the functional μ , then μ is also a distribution .In other words , the space $\mathcal{D}'$ is closed under convergence .

Proof : See [8] page 37.

Definition 2.2.6 (The Differentiation of Distributions)

Let μ be any distribution , then its derivatives is the distribution $\mu^{'}$ defined by

$$< \mu^{'}(t), \varphi(t) >=< \mu, -\varphi^{'}(t)$$

for all $\varphi \in \mathcal{D}$,where $\varphi^{'}$ denotes the ordinary classical derivative of the function φ .

In general , the kth-order of derivatives of a disribution μ is denoted by $\mu^{(k)}$ and defined by

$$< \mu^{(k)}, \varphi >=< \mu, (-1)^k \varphi^{(k)} > \quad \text{for } \varphi \in \mathcal{D}.$$

Example 2.2.2

(1) The first derivative of the Dirac delta functional ($\delta^{(1)}$) defined by

$$\begin{aligned} < \delta^{(1)}, \varphi > &=< \delta, (-1)\varphi^{'} > \\ &= (-1) < \delta, \varphi^{'} > \\ &= -\varphi^{'}(0) \end{aligned}$$

for $\varphi \in \mathcal{D}$.

(2) The $kth-$order of derivatives of the Dirac delta functional is denoted by $\delta^{(k)}$ and defined by

$$\begin{aligned} < \delta^{(k)}, \varphi > &=< \delta, (-1)^{(k)} \varphi^{(k)} > \\ &= (-1)^{(k)} \varphi^{(k)}(0) \end{aligned}$$

for $\varphi \in \mathcal{D}$

(3) $H(t)$ is called the Heaviside function that defined by

$$H(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

The first derivative of the Heaviside function denoted by $H'(t)$ and defined by

$$\begin{aligned}
< H'(t), \varphi > &= < H(t), -\varphi' > \\
&= -\int_{-\infty}^{\infty} H(t)\varphi'(t)dt \\
&= -\int_{0}^{\infty} \varphi'(t)dt \\
&= -(\varphi(t))_0^{\infty} \\
&= -\varphi(\infty) + \varphi(0) \\
&= \varphi(0) \\
&= < \delta, \varphi > .
\end{aligned}$$

Thus $H' = \delta$.

Definition 2.2.7 (The Multiplication of a Distribution by Infinitely Differentiable Function)

Let $\alpha(t)$ be the infinitely differentiable function and define the product of $\alpha(t)$ with any distribution μ in $\mathcal{D}'$ by

$$< \alpha\mu, \varphi > = < \mu, \alpha\varphi > \text{ for all } \varphi \in \mathcal{D}.$$

Example 2.2.3

(1) $$\begin{aligned}
< \alpha(t)\delta, \varphi(t) > &= < \delta, \alpha(t)\varphi(t) > \text{ by definition 2.2.7} \\
&= \alpha(0)\varphi(0) \\
&= \mu(0) < \delta, \varphi(t) > \\
&= < \alpha(0)\delta, \varphi(t) >
\end{aligned}$$

Thus $\alpha(t)\delta = \alpha(0)\delta$ for all $\varphi \in \mathcal{D}$

(2) $$\begin{aligned}
< t\delta, \varphi(t) > &= < \delta, t\varphi(t) > \\
&= 0\varphi(0) \\
&= 0 \\
&= < 0, \varphi(t) >
\end{aligned}$$

Thus $t\delta = 0$

(3) $< t\delta^{(1)}, \varphi(t) > = < \delta^{(1)}, t\varphi(t) >$

$$=< \delta, (-1)(t\varphi(t))^{(1)} >$$
$$=< \delta, -\varphi > - < \delta, t\varphi^{(1)}(t) >$$
$$=-\varphi(0)$$
$$=< -\delta, \varphi >$$

Thus $t\delta^{(1)} = -\delta$

(4) In general , $t\delta^{(m)} = -m\delta^{(m-1)}, m = 1, 2, 3, \ldots$

$$< t\delta^{(m)}(t), \varphi(t) > =< \delta^{(m)}(t), t\varphi(t) >$$
$$=< \delta(t), (-1)^m(t\varphi(t))^{(m)} >$$
$$= (-1)^m < \delta(t), m\varphi^{(m-1)}(t) >$$
$$= (-1)^m m < \delta(t), \varphi^{(m-1)}(t) >$$
$$= (-1)^m m < (-1)^{(m-1)}\delta^{(m-1)}(t), \varphi(t) >$$
$$=< -m\delta^{(m-1)}(t), \varphi(t) >$$

Thus $\qquad t\delta^{(m)}(t) = -m\delta^{(m-1)}(t)$

Definition 2.2.8 (The space $\mathcal{S}$ of testing functions of rapid descent)

Let $\mathcal{S}$ is the collection of all complex valued functions φ on R which are infinitely differentiable and $\varphi(t)$ in $\mathcal{S}$ satisfies the infinite set of inequalities

$$|t^m\varphi^{(k)}(t)| \leq C_{mk} \ , \ -\infty < t < \infty$$

where m and k run through all nonnegative integers . Here the C_{mk} are constants (with respect to t) which depend upon m and k .

Note : $\mathcal{D} \subset \mathcal{S}$.

Definition 2.2.9 (Convergence in $\mathcal{S}$)

A sequence of functions $\{\varphi_k(t)\}_{k=1}^{\infty}$ is said to converges in $\mathcal{S}$ if every functon $\varphi_k(t)$ is in $\mathcal{S}$ and if , for each nonnegative m and k , the sequence $\{t^m\varphi_k(t)\}_{k=1}^{\infty}$ converge uniformly in R .

Theorem 2.2.2 The space $\mathcal{D}$ is dense in the space $\mathcal{S}$ in the sense that for each φ in $\mathcal{S}$ there exists a sequence $\{\varphi_k(t)\}_{k=1}^{\infty}$ with every $\varphi_k(t)$ in $\mathcal{D}$ which converges in $\mathcal{S}$ to $\varphi(t)$.

Proof : See [8] page 101 .

Definition 2.2.10 (The space $\mathcal{S}'$ of distribution of slow growth)

A distribution μ is said to be of slow growth if it is a continuous linear functional on the space $\mathcal{S}$ of testing functions of rapid descent (such distributions are also called tempered distributions) . That is , a distribution μ of slow growth is a rule that assigns a number $<\mu, \varphi>$ to each φ in $\mathcal{S}$ in such a way that the following condition are fulfilled .

Linearity : If φ_1 and φ_2 are in $\mathcal{S}$ and if any numbers α and β , then

$$<\mu, \alpha\varphi_1 + \beta\varphi_2> = \alpha <\mu, \varphi_1> + \beta <\mu, \varphi_2>$$

Continuity : If $\{\varphi_k\}_{k=1}^{\infty}$ is any sequence that converges in $\mathcal{S}$ to zero , then

$$\lim_{k\to 0} <\mu, \varphi_k> = 0$$

The space of all distribution of slow growth is denoted by $\mathcal{S}'$. $\mathcal{S}'$ is also called the dual space of $\mathcal{S}$.

Note : $\mathcal{S}'$ is a proper subspace of $\mathcal{D}'$.

2.3 Laplace transform of distribution

Definition 2.3.1 Let $f(t)$ be a locally integrable function that satisfies the following condition.

1. $f(t) = 0$ for $-\infty < t < T$ where T is a real constant.
2. There exists a real number c such that $f(t)e^{-ct}$ is absolutely integrable over $-\infty < x < \infty$.Then the Laplace transform of $f(t)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_T^{\infty} f(t)e^{-st}dt \qquad (2.3.1)$$

where s is a complex variable. It can be shown that $F(s)$ is an analytic function for the half-plane $Re(s) > \sigma_a$, where σ_a is an abscissa of absolute convergence for $\mathcal{L}\{f(t)\}$. Now (2.3.1) can be replaced by the notation

$$F(s) = \mathcal{L}\{f(t)\} = < f(t), e^{-st} > \tag{2.3.2}$$

.

Suppose $f(t)$ is a distribution , that is , $f \in \mathcal{D}'$ .We try to define the Laplace transform of right-side distributions .

Suppose $f(t)$ is a distribution whose support is bounded on the left and there exists a real number c for which $e^{-ct}f(t)$ is in the space $\mathcal{S}'$ of a tempered distributions . Define

$$F(s) = \mathcal{L}\{f(t)\} = < e^{-st}f(t), X(t)e^{-(s-c)t} > \tag{2.3.3}$$

where $X(t)$ is any infinitely differentiable function which bounded support on the left and equals to 1 over the neighborhood of support of $f(t)$.

For $Re(s) > c$, $X(t)e^{-(s-c)t}$ is a testing function in the space S of a testing function of rapid descent. Equation (2.3.3) can be deduced to the definition

$$F(s) = \mathcal{L}\{f(t)\} = < f(t), e^{-st} > \tag{2.3.4}$$

Then equation (2.3.4) possesses a sense , that is, the same notation as (2.3.2).

Now , $F(s)$ is a function of s defined over the right-side half-plane $Re(s) > c$, and Zemanian [8] has proved that $F(s)$ is an analytic function in the region of convergence $Re(s) > \sigma_1$, where σ_1 is the abscissa of convergence for which $e^{-ct}f(t) \in S'$.

Consider

$$\begin{aligned}
\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_T^{\infty} e^{-st}\{c_1 f_1(t) + c_2 f_2(t)\}dt \\
&= \int_T^{\infty} e^{-st} c_1 f_1(t)dt + \int_T^{\infty} e^{-st} c_2 f_2(t)dt \\
&= c_1 \int_T^{\infty} e^{-st} f_1(t)dt + c_2 \int_T^{\infty} e^{-st} f_2(t)dt \\
&= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.
\end{aligned}$$

Hence

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}. \tag{2.3.5}$$

Theorem 2.3.1 Let $f(t)$ be a continuous function that satisfies condition of definition 2.3.1 and let σ_a be the abcissa of absolute convergence for $\mathcal{L}\{f(t)\} = F(s)$. Then , $F(s)$ is an analytic function for $Re(s) > \sigma_a$ and

$$F^{(k)}(s) = \int_T^\infty (-t)^k f(t)e^{-st}dt \quad ; Re(s) > \sigma_a \tag{2.3.6}$$

Proof : See [8] page 215 .

Example 2.3.1

1. $\mathcal{L}\{\frac{t^k H(t)}{k!}\} = \frac{1}{s^{k+1}}$, $Re(s) > 0, k = 0, 1, 2, \dots$, where

$$H(t) = \begin{cases} 1 & \text{for } t \in [0, \infty) \\ 0 & \text{for } t \in (-\infty, 0) \end{cases}$$

is a Heaviside function

Proof : By definition 2.3.1 , we obtain

$$\begin{aligned} \mathcal{L}\{\frac{t^k H(t)}{k!}\} &= \int_0^\infty \frac{e^{-st}t^k}{k!}dt \\ &= \frac{1}{k!}\int_0^\infty e^{-st}t^k dt \end{aligned}$$

Let $x = st$, then $dx = sdt$. So

$$\begin{aligned} \mathcal{L}\{\frac{t^k H(t)}{k!}\} &= \frac{1}{sk!}\int_0^\infty (\frac{x}{s})^k e^{-x}dx \\ &= \frac{1}{k!s^{k+1}}\int_0^\infty e^{-x}x^k dx \\ &= \frac{1}{k!s^{k+1}}\Gamma(k+1) \quad ; Re(s) > 0. \end{aligned}$$

Since $\Gamma(k+1) = k!$,

$$\begin{aligned} \mathcal{L}\{\frac{t^k H(t)}{k!}\} &= \frac{k!}{k!s^{k+1}} \\ &= \frac{1}{s^{k+1}} \quad ; \; Re(s) > 0 \end{aligned}$$

Theorem 2.3.2 Suppose that f is continuous and that f' is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constant K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \le M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and moreover

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \tag{2.3.7}$$

Proof : See [1] page 296.

Theorem 2.3.3 Suppose that the function $f, f', ..., f^{(n-1)}$ are continuous , and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constant k, a and M such that

$$|f(t)| \le ke^{at}, |f'(t)| \le ke^{at}, ..., |f^{(n-1)}(t)| \le ke^{at} \text{ for } t \ge M.$$

Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{2.3.8}$$

Proof : By mathematical induction ,
we let $P(n) = \mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$ for $n = 1$ and by theorem 2.3.2 , we canl see that $P(1)$ is true .

Suppose $P(k)$ is true. Next we want to show that $P(k+1)$ is also true . Consider

$$\begin{aligned}\mathcal{L}\{f^{(k+1)}f(t)\} &= \mathcal{L}\{\frac{d}{dt}f^{(k)}(t)\} \\ &= s\mathcal{L}\{f^{(k)}(t)\} - f^{(k)}(0) \quad \textit{by theorem } 2.3.2 \\ &= s[s^k\mathcal{L}\{f(t)\} - s^{k-1} - ... - sf^{(k-2)}(0) - f^{(k-1)}(0)] - f^{(k)}(0) \\ &= s^{k+1}\mathcal{L}\{f(t)\} - s^kf(0) - ... - s^2f^{(k-2)}(0) - sf^{(n-1)}(0) - f^{(k)}(0)\end{aligned}$$

Hence $P(k+1)$ is true .
Then we obtain

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Corollary 2.3.1 Suppose that the function $f, f', ..., f^{(n-1)}$ are continuous , and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constant k, a and M such that

$$|f(t)| \le ke^{at}, |f'(t)| \le ke^{at}, ..., |f^{(n-1)}(t)| \le ke^{at} \text{ for } t \ge M.$$

Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and $f(0) = f'(0) = ... = f^{(n-1)} = 0$ and is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} \tag{2.3.9}$$

Example 2.3.2 The following formulas are the consequences of definition 2.3.1

1. $\mathcal{L}\{\delta\} = 1$ for $-\infty < Re(s) < \infty$ and δ is the Dirac delta functional whost support concentrated on a single point.

2. $\mathcal{L}\{\delta^{(k)}\} = s^k, -\infty < Re(s) < \infty, k$ is a positive integer.

3. $\mathcal{L}\{t^k f(t)\} = (-1)^k F^{(k)}, Re(s) > \sigma_1$, where $f(t)$ is a distribution in the space $\mathcal{D}'_R$ of distribution whose supports are bounded on the left.

4. $\mathcal{L}\{f^{(k)}(t)\} = s^k F(s), Re(s) > \sigma_1$, where $f(t) \in \mathcal{D}'_R$.

5. $\mathcal{L}\{\frac{t^k e^{\alpha t}}{k!} H(t)\} = \frac{1}{(s-\alpha)^{k+1}}, Re(s) > Re(\alpha)$ and $H(t)$ is the Heaviside function.

Definition 2.3.2 (The inverse Laplace transform)

Let $f(t)$ be a locally integrable function that satisfies the conditions of definition 2.3.1 , the inverse Laplace transform of $F(s)$, denote by $\mathcal{L}^{-1}\{F(s)\}$ and defined by

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{y \to \infty} \int_{c-iy}^{c+iy} F(s) e^{st} ds \tag{2.3.10}$$

where $Re(s) \ge c > \sigma_a$

Example 2.3.3

1. $\mathcal{L}^{-1}\{\frac{1}{s^{k+1}}\} = \frac{H(t)t^k}{k!}$; $Re(s) > 0$

2. $\mathcal{L}^{-1}\{s^k\} = \delta^{(k)}$ $\quad -\infty < Re(s) < \infty$

Lemma 2.3.1 Assume that over the half-plane $Re(s) \geq a$, $F(s)$ is analytic and satisfies the inequality

$$|F(s)| \leq \frac{c}{|s|^2} \tag{2.3.11}$$

where c is a constant .

If the integral $\mathcal{L}^{-1}\{F(s)\}$ is taken over some vertical line in the half-plane $Re(s) \geq a$, $\mathcal{L}^{-1}\{F(s)\} = f(t)$ exists and is a continuous function for all t and $f(t) = 0$ for $t < 0$.

Proof : See [8] page 217.

Lemma 2.3.2 Let $F(s)$ be a function that is analytic over a half-plane $Re(s) \geq a$ and is bounded according to $|F(s)| \leq P(|s|)$, $Re(s) \geq c$ where $P(|s|)$ is some polynomial in $|s|$. Then , $F(s)$ is the Laplace Transform of a distribution $f(t)$ whose support is bounded on the left at $t = 0$.

Proof : See [8] page 236 .