CHAPTER 3
MAIN RESULTS

Theorem

The types of solutions of the fourth order Euler equation of the form

$$t^4y^{(4)}(t) + t^3y^{'''}(t) + m_2t^2y^{''}(t) + m_1ty^{'}(t) + m_0y(t) = 0 \qquad (3.1)$$

where m_0, m_1 and m_2 are some integers , relating on the lattice plane which can be classified by the following cases.

Case 1 If the lattice plane is $m_0 = km_1 - (k^2+k)m_2 - (k^4+5k^3+8k^2+4k)$ where $k = 1, 2, \ldots$,then (3.1) has the weak solution $y(t) = \delta^{(k-1)}(t)$.

Case 2 If the lattice plane is $m_0 = -km_1 - (k^2-k)m_2 - (k^4-5k^3+8k^2-4k)$ where $k = 0, 1, 2, \ldots$, then (3.1) has the strong solution $y(t) = \dfrac{H(t)t^{k+1}}{(k+1)!}$, where $H(t)$ is a Heaviside function .

Proof :

By taking the Laplace transform to equation (3.1) , we obtain

$$\mathcal{L}\{t^4y^{(4)}(t) + t^3y^{'''}(t) + m_2t^2y^{''}(t) + m_1ty^{'}(t) + m_0y(t)\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{t^4y^{(4)}(t)\} + \mathcal{L}\{t^3y^{'''}(t)\} + \mathcal{L}\{m_2t^2y^{''}(t)\} + \mathcal{L}\{m_1ty^{'}(t)\} + \mathcal{L}\{m_0y(t)\} = 0$$

.

By using Example 2.3.2 (3) and (4) , we obtain

$$(-1)^4\frac{d^4}{ds^4}\mathcal{L}\{y^{(4)}(t)\} + (-1)^3\frac{d^3}{ds^3}\mathcal{L}\{y^{'''}(t)\} + m_2(-1)^2\frac{d^2}{ds^2}\mathcal{L}\{y^{''}(t)\}$$
$$+m_1(-1)\frac{d}{ds}\mathcal{L}\{y^{'}(t)\} + m_0\mathcal{L}\{y(t)\} = 0$$

or

$$\frac{d^4}{ds^4}s^4Y(s) - \frac{d^3}{ds^3}s^3Y(s) + m_2\frac{d^2}{ds^2}s^2Y(s) - m_1\frac{d}{ds}sY(s) + m_0Y(s) = 0. \quad (3.2)$$

Consider

$$\frac{d}{ds}sY(s) = sY'(s) + Y(s) \quad (3.3)$$

and

$$\frac{d}{ds}s^2Y(s) = s^2Y'(s) + 2sY(s)$$
$$\frac{d^2}{ds^2}s^2Y(s) = s^2Y''(s) + 4sY'(s) + 2Y(s) \quad (3.4)$$

and

$$\frac{d}{ds}s^3Y(s) = s^3Y'(s) + 3s^2Y(s)$$
$$\frac{d^2}{ds^2}s^3Y(s) = s^3Y''(s) + 3s^2Y'(s) + 3s^2Y'(s) + 6sY(s)$$
$$= s^3Y''(s) + 6s^2Y'(s) + 6sY(s)$$
$$\frac{d^3}{ds^3}s^3Y(s) = s^3Y'''(s) + 9s^2Y''(s) + 18sY'(s) + 6Y(s) \quad (3.5)$$

and

$$\frac{d}{ds}s^4Y(s) = s^4Y'(s) + 4s^3Y(s)$$
$$\frac{d^2}{ds^2}s^4Y(s) = s^4Y''(s) + 8s^3Y'(s) + 12s^2Y(s)$$
$$\frac{d^3}{ds^3}s^4Y(s) = s^4Y'''(s) + 12s^3Y''(s) + 36s^2Y'(s) + 24sY(s)$$
$$\frac{d^4}{ds^4}s^4Y(s) = s^4Y^{(4)}(s) + 16s^3Y'''(s) + 72s^2Y''(s) + 96sY'(s) + 24Y(s) \quad (3.6)$$

Substitute (3.3) , (3.4) , (3.5) and (3.6) into (3.2) , we obtain

$$[s^4Y^{(4)}(s) + 16s^3Y'''(s) + 72s^2Y''(s) + 96sY'(s) + 24Y(s)]$$
$$-[s^3Y'''(s) + 9s^2Y''(s) + 18sY'(s) + 6Y(s)] + m_2[s^2Y''(s) + 4sY'(s) + 2Y(s)]$$
$$-m_1[sY'(s) + Y(s)] + m_0Y(s) = 0$$

$$s^4Y^{(4)} + 15s^3Y'''(s) + (63 + m_2)s^2Y''(s) + (78 + 4m_2 - m_1)sY'(s) + (18 + 2m_2 - m_1 + m_0)Y(s) = 0 \qquad (3.7)$$

Let a solution of equation (3.7) be $Y(s) = s^r$,where r is any real constant. So

$Y'(s) = rs^{r-1}$

$Y''(s) = r(r-1)s^{r-2}$

$Y'''(s) = r(r-1)(r-2)s^{r-3}$

$Y^{(4)}(s) = r(r-1)(r-2)(r-3)s^{r-4}$

Substitute $Y(s), Y'(s), Y''(s), Y'''(s)$ and $Y^{(4)}(s)$ into (3.7),then we obtain

$$s^4r(r-1)(r-2)(r-3)s^{r-4} + 15s^3r(r-1)(r-2)s^{r-3} + (63 + m_2)s^2r(r-1)s^{r-2} + (78 + 4m_2 - m_1)srs^{r-1} + (18 + 2m_2 - m_1 + m_0)s^r = 0$$

Since $s^r \neq 0$, then

$$r(r-1)(r-2)(r-3) + 15r(r-1)(r-2) + (63 + m_2)r(r-1) + (78 + 4m_2 - m_1)r + (18 + 2m_2 - m_1 + m_0) = 0$$

or

$$r^4 + 9r^3 + (29 + m_2)r^2 + (39 + 3m_2 - m_1)r + (18 + 2m_2 - m_1 + m_0) = 0. \qquad (3.8)$$

Consider the value of r is the following 2 cases.

Case 1 If $r = 0, 1, 2, ...$, then by (3.8) , we obtain

If $r = 0$, then $m_0 = m_1 - 2m_2 - 18$

If $r = 1$, then $m_0 = 2m_1 - 6m_2 - 96$

If $r = 2$, then $m_0 = 3m_1 - 12m_2 - 300$

If $r = 3$, then $m_0 = 4m_1 - 20m_2 - 720$

If $r = 4$, then $m_0 = 5m_1 - 30m_2 - 1470$

If $r = 5$, then $m_0 = 6m_1 - 42m_2 - 2688$

By induction , we obtain If $r = k - 1$,then

$$m_0 = km_1 - (k^2 + k)m_2 - (k^4 + 5k^3 + 8k^2 + 4k) \qquad (3.9)$$

Since $Y(s) = s^r$, the solution of (3.7) are

$$Y(s) = s^r = s^{k-1} \text{ where } k = 1, 2, 3, ...$$

or
$$Y(s) = 1, s, s^2, s^3, ... \text{ respectively.}$$

By taking the inverse Laplace transform to $Y(s)$ and by Example 2.3.3 (2), we obtain the solution of (3.1) which are the singular distributions

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \delta^{(k-1)}(t) \text{ where } k = 1, 2, 3, ..$$
$$y(t) = \delta, \delta^{(1)}, \delta^{(2)}, ..., \delta^{(k-1)}$$

where $\delta^{(k-1)}$ is defined as Example 2.2.2

Then we obtain the singular distribution solution

δ corresponding to $m_0 = m_1 - 2m_2 - 18$

$\delta^{(1)}$ corresponding to $m_0 = 2m_1 - 6m_2 - 96$

$\delta^{(2)}$ corresponding to $m_0 = 3m_1 - 12m_2 - 300$

$\delta^{(3)}$ corresponding to $m_0 = 4m_1 - 20m_2 - 720$

$\delta^{(k-1)}$ corresponding to $m_0 = km_1 - (k^2 + k)m_2 - (k^4 + 5k^3 + 8k^2 + 4k)$

Case 2 If $r = -1, -2, -3, ...,$ then from (3.8) , we obtain

If $r = -1$, then $m_0 = 0$

If $r = -2$, then $m_0 = -m_1$

If $r = -3$, then $m_0 = -2m_1 - 2m_2$

If $r = -4$, then $m_0 = -3m_1 - 6m_2 - 6$

If $r = -5$, then $m_0 = -4m_1 - 12m_2 - 48$

By induction , we obtain If $r = -(k+1)$, then $m_0 = -km_1 - (k^2 - k)m_2 - (k^4 - 5k^3 + 8k^2 - 4k)$ (3.10)

Since $Y(s) = s^r$, the solution of (3.7) are

$$Y(s) = s^r = s^{-(k+1)} \text{ where } k = 0, 1, 2, 3, ...$$

or
$$Y(s) = s^{-1}, s^{-2}, s^{-3}, ... = \frac{1}{s}, \frac{1}{s^2}, \frac{1}{s^3}, ... \text{ respectively}$$

Similarly , take the inverse Laplace transform to $Y(s)$ and by Example 2.3.3 (1) , we obtain the solution of (3.1)

$$y(t) = L^{-1}\{Y(s)\} = \frac{H(t)t^{k+1}}{(k+1)!}$$

for $k = 0, 1, 2, ...$

We obtain the classical solution of (3.1):

$H(t)t$ corresponding to $m_0 = 0$

$H(t)\frac{t^2}{2!}$ corresponding to $m_0 = -m_1$

$H(t)\frac{t^3}{3!}$ corresponding to $m_0 = -2m_1 - 2m_2$

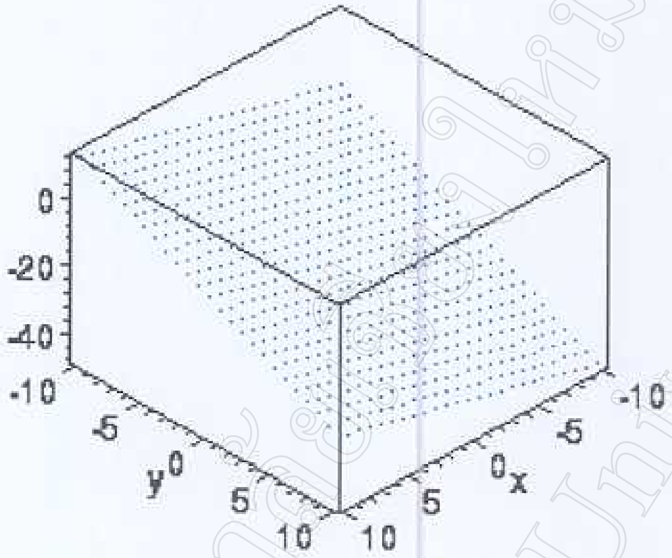
$H(t)\frac{t^4}{4!}$ corresponding to $m_0 = -3m_1 - 6m_2 - 6$

$H(t)\frac{t^{k+1}}{(k+1)!}$ corresponding to $m_0 = -km_1 - (k^2 - k)m_2 - (k^4 - 5k^3 + 8k^2 - 4k)$

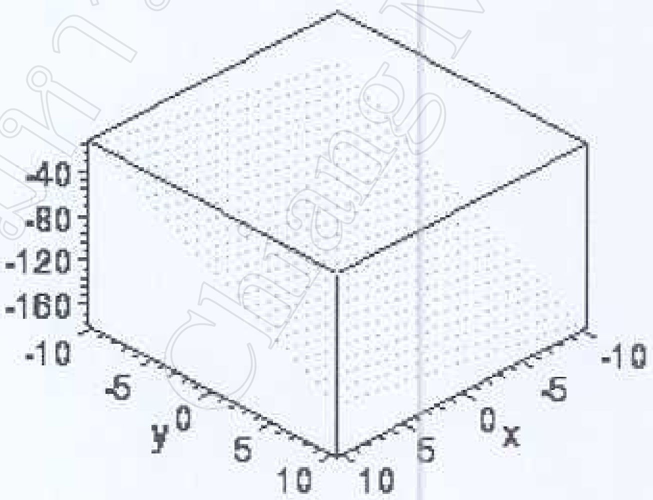
That completes the proof of this Theorem $\triangle$

The equation (3.9) and (3.10) are called Lattice Plane of the fourth order Euler equation (3.1) and can be shown by the following graphic .

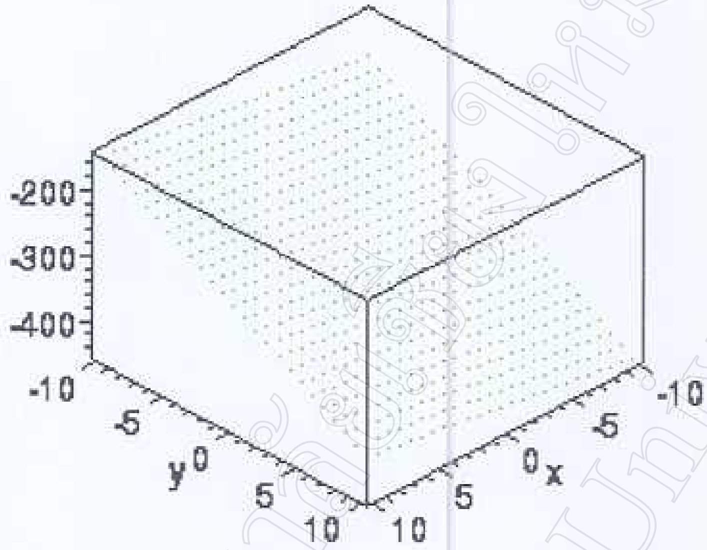
m_0 = m_1 - 2m_2 - 18 corresponding to r=0



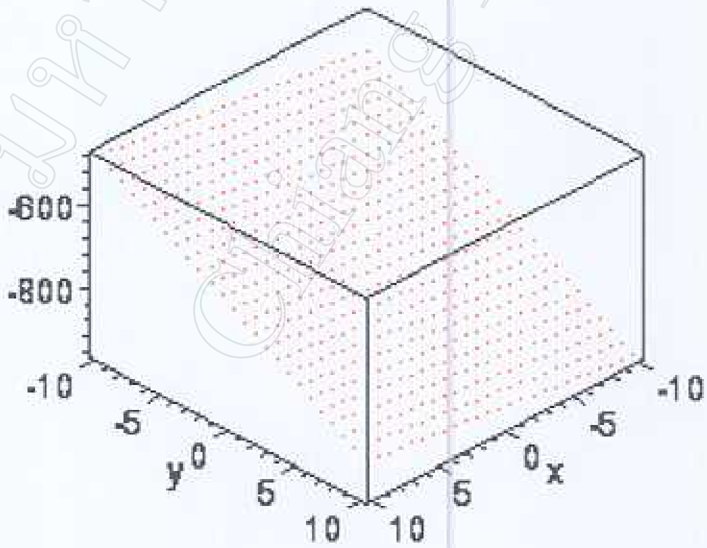
m_0 = 2m_1 - 6m_2 - 96 corresponding to r=1



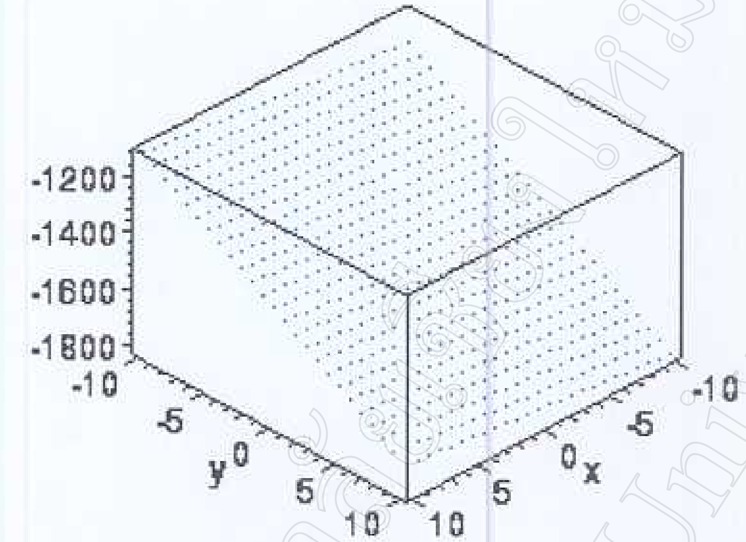
m_0 = 3m_1 - 12m_2 - 300 corresponding to r=2



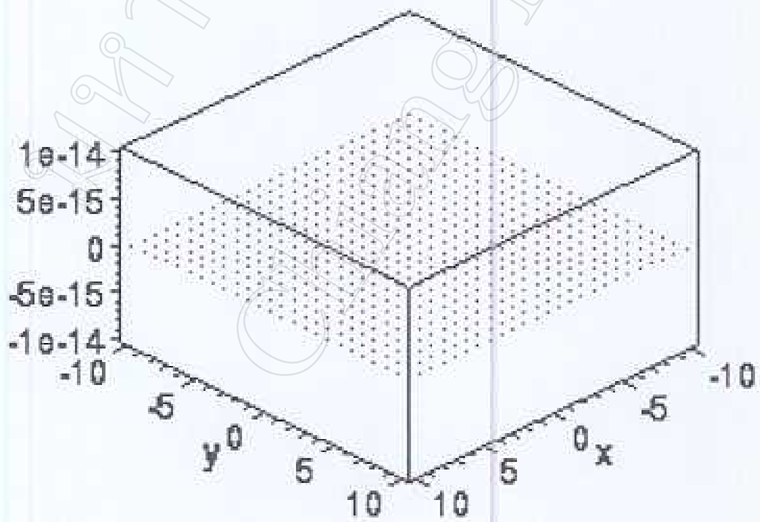
m_0 = 4m_1 - 20m_2 - 720 corresponding to r=3



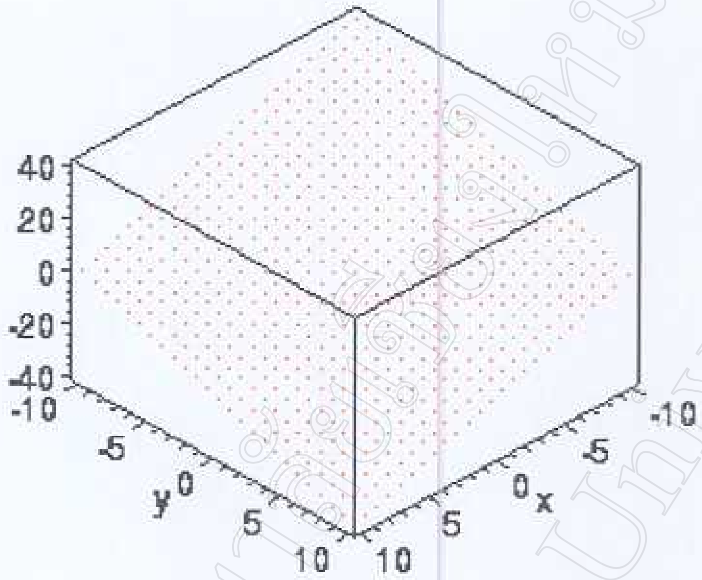
m_0 = 5m_1 -30 m_2 - 1470 corresponding to r=4



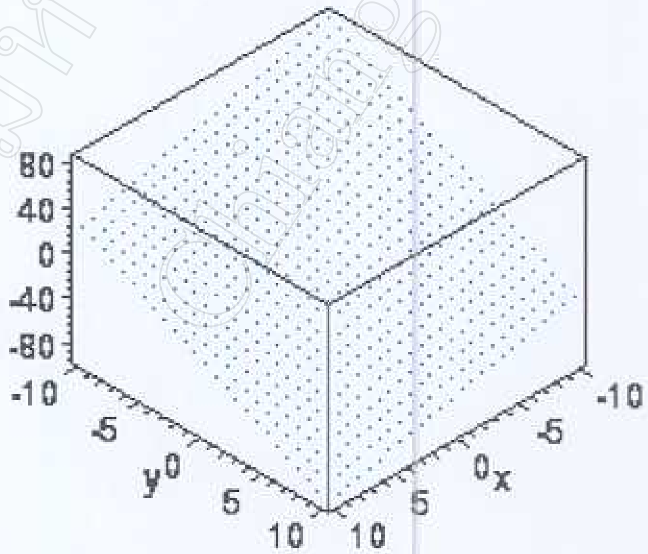
m_0 = 0m_1 - 0m_2 -0 corresponding to r=-1



m_0 = -2m_1 - 2m_2 -0 corresponding to r=-3



m_0 = -3m_1 - 6m_2 - 6 corresponding to r=-4



m_0 = -4m_1 - 12m_2 - 48 corresponding to r=-5