Chapter 2

Free Vibration of Laminated Transversely Isotropic Cylinder

2.1 Transversely Isotropic Material

A material is considered as transversely isotropic if its elastic properties are invariant with respect to an arbitrary rotation about a given axis. The mechanical properties are the same in all directions on a plane perpendicular to this axis. Thus, there are five independent constants that characterize transverse isotropy. (Reddy, 1997)

One of the examples of transversely isotropic material is fiber-reinforced material, it is often assumed that the material behavior in the 2 direction is identical to the material behavior in the 3 direction, where the 1 direction is the fiber direction while the 2 and 3 directions are the transverse directions (matrix directions). Thus, the material is said to be isotropic in the 2-3 plane, or transversely isotropic in the 2-3 plane. (Hyer, 1998) Figure 2.1 illustrates the principal coordinate system of a fiber-reinforced material.

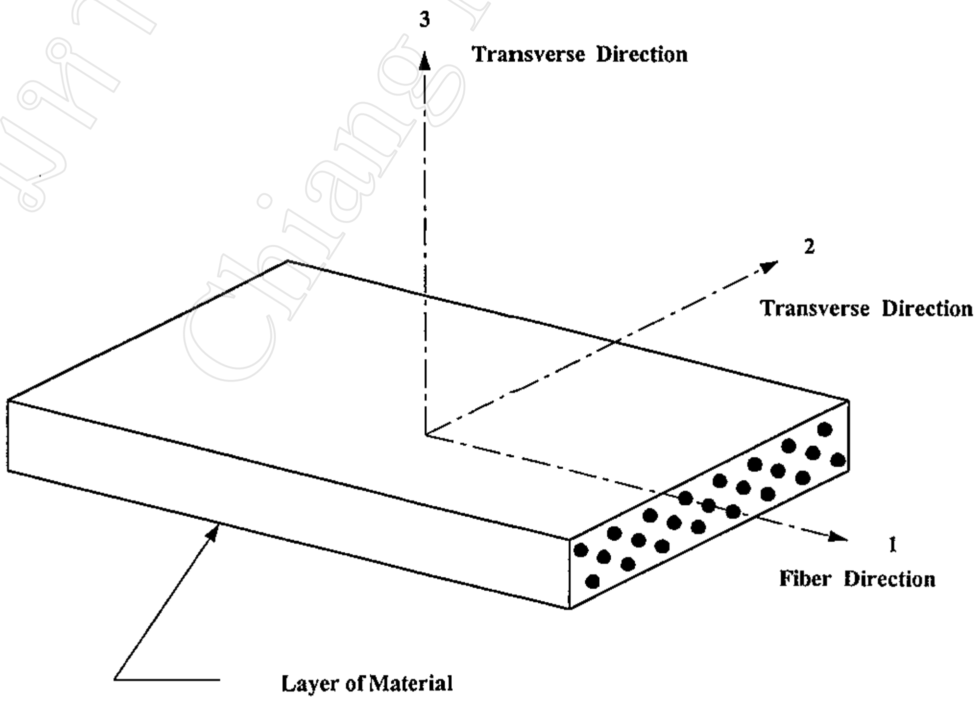

Figure 2.1 The principal coordinate system of a fiber-reinforced material

2.2 Governing Equations

The focus of this study is a hollow cylinder composed of layers of transversely isotropic materials as shown in Figure 2.2. The analytical formulation of this problem will be developed in such a way that the number and the properties of layers can be varied arbitrarily without changing the solution procedure.

When the k^{th} layer of a cylinder bounded by $r = r_k$ and $r = r_{k+1}$ surfaces shown in Figure 2.3 is considered, the equations of motion based on the three-dimensional elasticity theory of each layer in the cylindrical coordinates (r, θ, z) are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2},$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = \rho \frac{\partial^2 v}{\partial t^2},$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho \frac{\partial^2 w}{\partial t^2}, \qquad (1)$$

where σ_{ij} (i, j = r, θ, z) are the stress components. u, v, and w the displacements in r, θ, and z directions, respectively, and ρ the mass density of the material.

The stress-strain relations of a transversely isotropic material can be expressed in the cylindrical coordinates as:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{Bmatrix} \qquad (2)$$

and the strain-displacement relations are:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \quad ; \quad \varepsilon_{\theta\theta} = \frac{u}{r} + \frac{\partial v}{r\partial\theta} \quad ; \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \, ,$$

$$\gamma_{\theta z} = \frac{\partial w}{r\partial\theta} + \frac{\partial v}{\partial z} \quad ; \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \quad ; \quad \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{\partial u}{r\partial\theta} - \frac{v}{r} , \qquad (3)$$

where C_{ij} (i, j = 1, 2,..., 6) are the elastic coefficients. ε_{ii} and γ_{ij} (i, j = r, θ, z) are the normal strains and the shear strains, respectively.

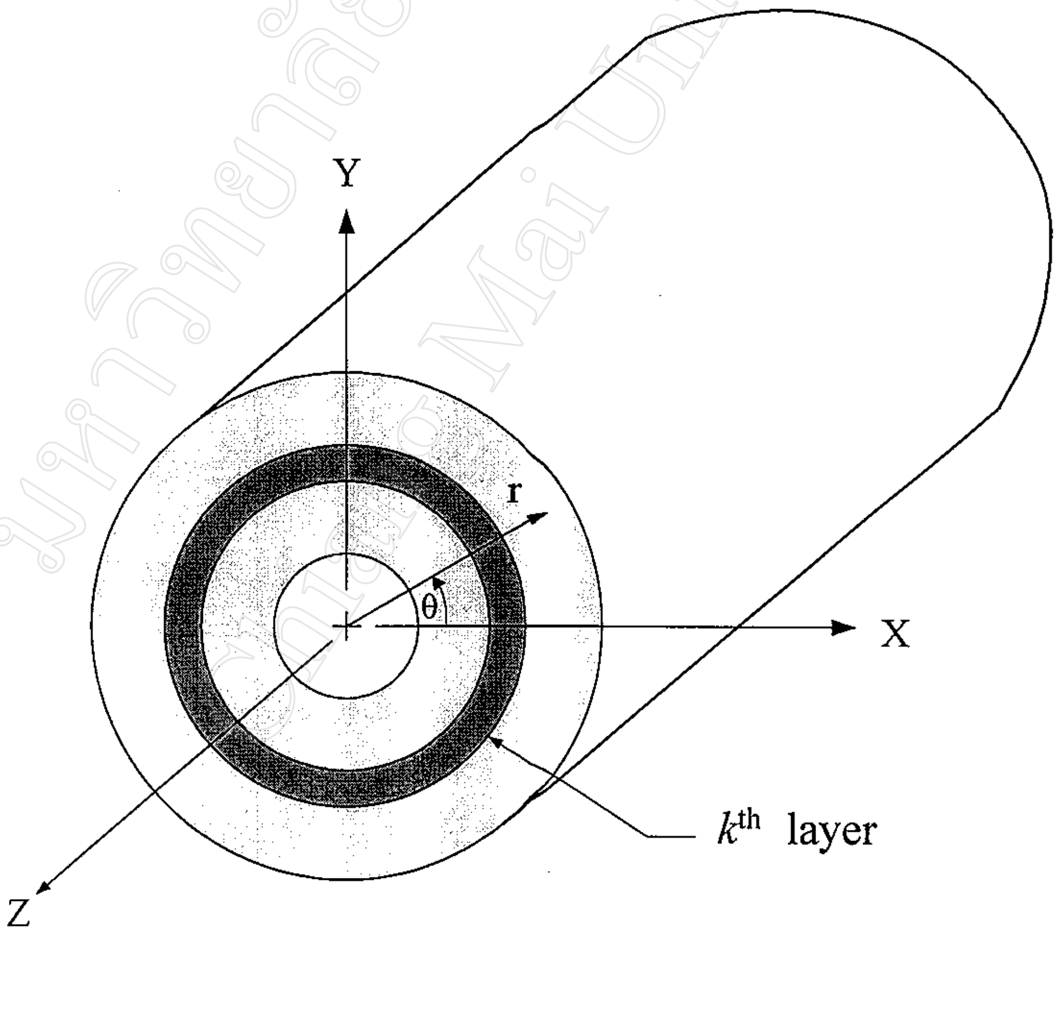

Figure 2.2 Laminated hollow cylinder

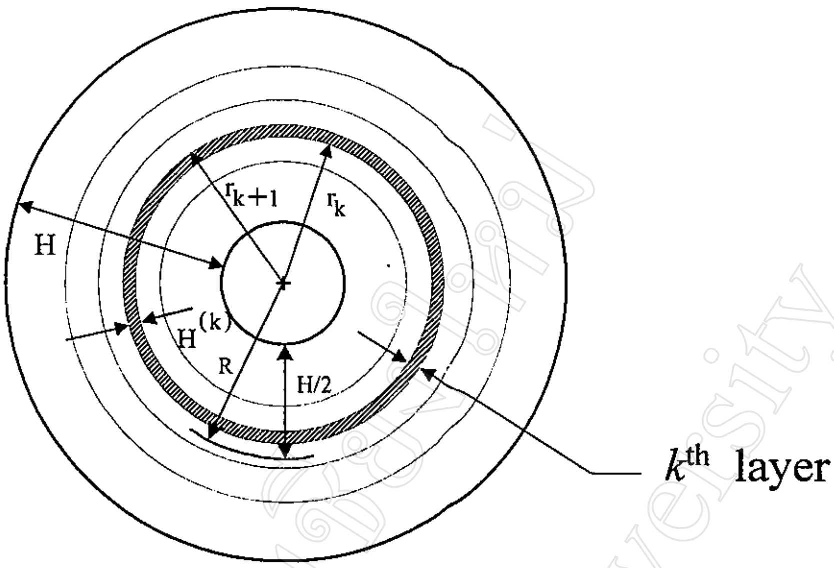

Figure 2.3 Cross section of laminated hollow cylinder

Substitution of Eqs. (2) and (3) into Eqs. (1) yields:

$$C_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}-\frac{1}{r^2}\frac{\partial v}{\partial \theta}\right)+C_{12}\frac{1}{r}\frac{\partial v^2}{\partial r\partial \theta}+C_{13}\frac{\partial^2 w}{\partial r\partial z}$$
$$+\ C_{44}\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial r\partial z}\right)+C_{66}\left(\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta}-\frac{1}{r^2}\frac{\partial v}{\partial \theta}\right)=\rho\frac{\partial^2 u}{\partial t^2},$$

$$C_{11}\left(\frac{1}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right)+C_{12}\frac{1}{r}\frac{\partial^2 u}{\partial r\partial \theta}+C_{13}\frac{1}{r}\frac{\partial^2 w}{\partial \theta\partial z}+C_{44}\left(\frac{\partial^2 v}{\partial z^2}+\frac{1}{r}\frac{\partial^2 w}{\partial \theta\partial z}\right)$$
$$+\ C_{66}\left(\frac{1}{r}\frac{\partial^2 u}{\partial r\partial \theta}+\frac{1}{r^2}\frac{\partial u}{\partial \theta}+\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)=\rho\frac{\partial^2 v}{\partial t^2},$$

$$C_{13}\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{i}{r}\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\partial z}\right)+C_{33}\frac{\partial^2 w}{\partial z^2}$$
$$+\ C_{44}\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\partial z}+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)=\rho\frac{\partial^2 w}{\partial t^2}, \qquad (4)$$

where $C_{66} = (C_{11} - C_{12})/2$. Eqs. (4) can be rewritten as:

$$C_{11}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)+C_{44}\frac{\partial^2 u}{\partial z^2}+C_{66}\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta}$$

$$-(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial v}{\partial \theta}+(C_{13}+C_{44})\frac{\partial^2 w}{\partial r\partial z} = \rho\frac{\partial^2 u}{\partial t^2},$$

$$C_{11}\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+C_{44}\frac{\partial^2 v}{\partial z^2}+C_{66}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right)+(C_{11}+C_{66})\frac{1}{r^2}\frac{\partial u}{\partial \theta}$$

$$+(C_{12}+C_{66})\frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}+(C_{13}+C_{44})\frac{1}{r}\frac{\partial^2 w}{\partial \theta\partial z} = \rho\frac{\partial^2 v}{\partial t^2},$$

$$C_{33}\frac{\partial^2 w}{\partial z^2}+C_{44}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)$$

$$+(C_{13}+C_{44})\left(\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta\partial z}\right) = \rho\frac{\partial^2 w}{\partial t^2}. \quad (5)$$

For the free vibration problem of an infinitely long circular cylinder, the general solution to the equations of motion, Eqs. (5), can be represented as (Chou and Achenbach, 1972):

$$\begin{aligned} u &= U(r)\cos n\theta \cos(\xi z-\omega t), \\ v &= V(r)\sin n\theta \cos(\xi z-\omega t), \\ w &= W(r)\cos n\theta \sin(\xi z-\omega t), \end{aligned} \quad (6)$$

where U(r), V(r), and W(r) are functions of radial coordinate only, ξ the axial wavenumber, n the circumferential wavenumber, and ω the circular frequency.

Substitution of Eqs. (6) into Eqs. (5) leads to a set of three coupled displacement equations of motion:

$$C_{11}U''\cos n\theta\cos(\xi z-\omega t)+C_{11}\frac{1}{r}U'\cos n\theta\cos(\xi z-\omega t)$$
$$-C_{11}\frac{1}{r^2}U\cos n\theta\cos(\xi z-\omega t)-C_{44}\xi^2U\cos n\theta\cos(\xi z-\omega t)$$
$$-C_{66}\frac{n^2}{r^2}U\cos n\theta\cos(\xi z-\omega t)+(C_{12}+C_{66})\frac{n}{r}V'\cos n\theta\cos(\xi z-\omega t)$$
$$-(C_{11}+C_{66})\frac{n}{r^2}V\cos n\theta\cos(\xi z-\omega t)+(C_{13}+C_{44})\xi W'\cos n\theta\cos(\xi z-\omega t)$$
$$=\quad -\rho\omega^2U\cos n\theta\cos(\xi z-\omega t),$$

$$-C_{11}\frac{n^2}{r^2}V\sin n\theta\cos(\xi z-\omega t)-C_{44}\xi^2V\sin n\theta\cos(\xi z-\omega t)$$
$$-(C_{11}+C_{66})\frac{n}{r^2}U\sin n\theta\cos(\xi z-\omega t)-(C_{12}+C_{66})\frac{n}{r}U'\sin n\theta\cos(\xi z-\omega t)$$
$$-(C_{13}+C_{44})\frac{n\xi}{r}W\sin n\theta\cos(\xi z-\omega t)+C_{66}V''\sin n\theta\cos(\xi z-\omega t)$$
$$+C_{66}\frac{1}{r}V'\sin n\theta\cos(\xi z-\omega t)-C_{66}\frac{1}{r^2}V\sin n\theta\cos(\xi z-\omega t)$$
$$=\quad -\rho\omega^2V\sin n\theta\cos(\xi z-\omega t),$$

$$-C_{33}\xi^2W\cos n\theta\sin(\xi z-\omega t)+C_{44}W''\cos n\theta\sin(\xi z-\omega t)$$
$$-C_{44}\frac{n^2}{r^2}W\cos n\theta\sin(\xi z-\omega t)+C_{44}\frac{1}{r}W'\cos n\theta\sin(\xi z-\omega t)$$
$$-(C_{13}+C_{44})\xi U'\cos n\theta\sin(\xi z-\omega t)-(C_{13}+C_{44})\frac{\xi}{r}U\cos n\theta\sin(\xi z-\omega t)$$
$$-(C_{13}+C_{44})\frac{n\xi}{r}V\cos n\theta\sin(\xi z-\omega t)$$
$$=\quad -\rho\omega^2W\cos n\theta\sin(\xi z-\omega t), \tag{7}$$

where prime denotes $\frac{d}{dr}$. By omitting sine and cosine terms, Eqs. (7) can be rewritten as:

$$C_{11}U''+C_{11}\frac{1}{r}U'-C_{11}\frac{1}{r^2}U-C_{44}\xi^2U-C_{66}\frac{n^2}{r^2}U+(C_{12}+C_{66})\frac{n}{r}V'$$
$$-(C_{11}+C_{66})\frac{n}{r^2}V+(C_{13}+C_{44})\xi W' \quad = \quad -\rho\omega^2U,$$

$$-c_{11}\frac{n^2}{r^2}v - c_{44}\xi^2 v - (c_{11}+c_{66})\frac{n}{r^2}u - (c_{12}+c_{66})\frac{n}{r}u'$$
$$-(c_{13}+c_{44})\frac{n\xi}{r}w + c_{66}v'' + c_{66}\frac{1}{r}v' - c_{66}\frac{1}{r^2}v \quad = \quad -\rho\omega^2 v,$$

$$-c_{33}\xi^2 w + c_{44}w'' - c_{44}\frac{n^2}{r^2}w + c_{44}\frac{1}{r}w' - (c_{13}+c_{44})\xi u'$$
$$-(c_{13}+c_{44})\frac{\xi}{r}u - (c_{13}+c_{44})\frac{n\xi}{r}v \quad = \quad -\rho\omega^2 w. \tag{8}$$

These equations govern the radial variation of the displacement components. When n in Eqs. (8) is set to zero, the motion is independent of θ. The three equations of motion can then be separated in two sets of governing equations which are uncoupled. One set contains two coupled equations of motion describing the coupling between u and w, while the other set contains the equation describing the motion in the θ direction. The coupled equations govern what is commonly called the axisymmetric vibration and are given by:

$$c_{11}u'' + c_{11}\frac{1}{r}u' - c_{11}\frac{1}{r^2}u - c_{44}\xi^2 u + (c_{13}+c_{44})\xi w' \quad = -\rho\omega^2 u,$$

$$-c_{33}\xi^2 w + c_{44}w'' + c_{44}\frac{1}{r}w' - (c_{13}+c_{44})\xi u' - (c_{13}+c_{44})\frac{\xi}{r}u \quad = -\rho\omega^2 w. \tag{9}$$

For the other case of vibration which is independent of θ, as mentioned, the motion only occurs in the θ direction and it is called the torsional free vibration. The governing equation for the torsional free vibration is:

$$-c_{44}\xi^2 v + c_{66}v'' + c_{66}\frac{1}{r}v' - c_{66}\frac{1}{r^2}v \quad = -\rho\omega^2 v. \tag{10}$$

2.3 Axisymmetric (Longitudinal) Free Vibration

The displacement functions U(r) and W(r) in Eqs. (9) can be rewritten in terms of the potential function $\Phi(r)$ (Keck and Armenakas (1971)):

$$U(r) = \left(C_{13} + C_{44}\right)\frac{d\Phi(r)}{dr},$$

$$W(r) = -\frac{1}{\xi}\left[C_{11}\nabla^2\Phi(r) + \xi^2\left(\frac{\rho\omega^2}{\xi^2} - C_{44}\right)\Phi(r)\right], \tag{11}$$

in which $\Phi(r)$ must satisfy

$$\left(\nabla^2 + p^2\right)\left(\nabla^2 + q^2\right)\Phi(r) = 0, \tag{12}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr},$$

and ω and ξ are the frequency and the axial wavenumber, respectively. The radial wavenumbers p and q are given by:

$$p^2 = \frac{\xi^2}{2C_{11}C_{44}}\left(A + \sqrt{A^2 - B}\right),$$

$$q^2 = \frac{\xi^2}{2C_{11}C_{44}}\left(A - \sqrt{A^2 - B}\right), \tag{13}$$

where

$$A = C_{11}\left(m - C_{33}\right) + C_{44}\left(m - C_{44}\right) + \left(C_{13} + C_{44}\right)^2,$$

$$B = 4C_{11}C_{44}\left(m - C_{33}\right)\left(m - C_{44}\right), \tag{14}$$

and $m = \dfrac{\rho\omega^2}{\xi^2}$.

Note that when the radicand $A^2 - B$ is negative, the radial wavenumbers p and q become complex. To avoid obtaining the complex roots of these radial wavenumbers, the materials used in this study are selected in such a way that the radicand $A^2 - B$ is positive for any real value of m. The mechanical properties of the materials which satisfy the above condition are shown in Appendix A.

On the basis of the foregoing analysis, the radial wavenumbers p and q may be written as:

$$p = \sqrt{\varepsilon_1}(\xi\alpha),$$
$$q = \sqrt{\varepsilon_2}(\xi\beta), \qquad (15)$$

where

$$\alpha = \frac{1}{\sqrt{2C_{11}C_{44}}}\left|A + \sqrt{A^2 - B}\right|^{1/2},$$

$$\beta = \frac{1}{\sqrt{2C_{11}C_{44}}}\left|A - \sqrt{A^2 - B}\right|^{1/2}, \qquad (16)$$

and the sign factors ε_i (i = 1, 2) are assigned at different ranges of m as shown in Table 2.1.

Table 2.1 The sign factors ε_i used for different ranges of m.

Ranges of m.	ε_1	ε_2
$m > C_{33}$	1	1
$C_{44} < m < C_{33}$	1	-1
$0 < m < C_{44}$	-1	-1

Finally, the solution for $\Phi(r)$ can be obtained as:

$$\Phi(r) = \frac{\xi}{c_{13}+c_{44}}\{c_1 z_0(\xi\alpha r)+c_2 w_0(\xi\alpha r)\}+c_3 z_0(\xi\beta r)+c_4 w_0(\xi\beta r), \quad (17)$$

in which Z_0 and W_0 are regular or modified Bessel functions of the first and second kind of orders zero, respectively, depending on whether ε_i is 1 or -1. C_1, C_2, C_3, and C_4 are unknown constants for the layer. Substitution of Eq. (17) into Eqs. (11), together with the sign factors and the recurrence relations of Bessel functions given in Appendix B, the closed-form solutions of U (r) and W(r) can be represented explicitly as:

$$\begin{aligned} U(r) = \; & c_1\{-\varepsilon_1\xi^2\alpha z_1(\xi\alpha r)\} + \\ & c_2\{-\xi^2\alpha w_1(\xi\alpha r)\} + \\ & c_3\{-\varepsilon_2(c_{13}+c_{44})\xi\beta z_1(\xi\beta r)\} + \\ & c_4\{-(c_{13}+c_{44})\xi\beta w_1(\xi\beta r)\}, \end{aligned}$$

$$\begin{aligned} W(r) = \; & c_1\left\{\left(\frac{\xi^2}{c_{13}+c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2-\frac{\rho\omega^2}{\xi^2}+c_{44}\right)z_0(\xi\alpha r)\right\} + \\ & c_2\left\{\left(\frac{\xi^2}{c_{13}+c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2-\frac{\rho\omega^2}{\xi^2}+c_{44}\right)w_0(\xi\alpha r)\right\} + \\ & c_3\left\{\xi\left(\varepsilon_2 c_{11}\beta^2-\frac{\rho\omega^2}{\xi^2}+c_{44}\right)z_0(\xi\beta r)\right\} + \\ & c_4\left\{\xi\left(\varepsilon_2 c_{11}\beta^2-\frac{\rho\omega^2}{\xi^2}+c_{44}\right)w_0(\xi\beta r)\right\}. \end{aligned} \quad (18)$$

2.4 Cut-Off Frequencies for Axisymmetric Vibration

The cut-off frequencies are the frequencies below in which some type of wave cannot travel and correspond to wave speeds which approach infinity as the wavenumber approaches zero, i.e. $\xi = 0$. When $\xi = 0$, the motion becomes independent of z which results in the uncoupling of the nontorsional equations of motion, Eqs. (9). The radial motion, or plane strain extensional vibration, is governed by

$$C_{11}U'' + C_{11}\frac{1}{r}U' - C_{11}\frac{1}{r^2}U = -\rho\omega^2 U. \tag{19}$$

Eq. (19) can be written in the form of Bessel equation as:

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left(\chi^2 - \frac{1}{r^2}\right)\right\}U = 0, \tag{20}$$

where $\chi^2 = \dfrac{\rho\omega^2}{C_{11}}$.

The displacement solution, U(r), to Eq. (20) in terms of Bessel functions is:

$$U(r) = C_1 J_1(\chi r) + C_2 Y_1(\chi r), \tag{21}$$

where J_1 and Y_1 are regular Bessel functions of the first kind and the second kind of orders one, respectively. C_1 and C_2 are unknown constants for the layer.

For the other case, the motion represents longitudinal shear vibration involving only axial displacement and is governed by:

$$C_{44}W'' + C_{44}\frac{1}{r}W' = -\rho\omega^2 W, \tag{22}$$

Eq. (22) can be written in the form of Bessel equation as:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \lambda^2\right)W = 0, \quad (23)$$

where $\lambda^2 = \dfrac{\rho\omega^2}{c_{44}}$.

The solution to Eq. (23) in terms of Bessel functions is:

$$W(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r), \quad (24)$$

in which J_0 and Y_0 are regular Bessel functions of the first kind and the second kind of orders zero, respectively. C_1 and C_2 are unknown constants for the layer.

2.5 Torsional Free Vibration

For the case of torsional free vibration, the governing equation, Eq. (10), the displacement can be rewritten in the form of Bessel equation as:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \left\{\frac{\left(\rho\omega^2 - c_{44}\xi^2\right)}{c_{66}} - \frac{1}{r^2}\right\}\right]V = 0. \quad (25)$$

The solution, V(r), to Eq. (25) is:

1) When $\rho\omega^2 > c_{44}\xi^2$

$$V(r) = C_1 J_1(\eta r) + C_2 Y_1(\eta r), \quad (26)$$

where $\eta^2 = \left(\rho\omega^2 - c_{44}\xi^2\right)/c_{66}$ and $J_1(\eta r)$ and $Y_1(\eta r)$ are the regular Bessel functions of the first and second kinds of orders one with argument ηr, respectively. C_1 and C_2 are unknown constants of the layer.

2) When $\rho\omega^2 < C_{44}\xi^2$

$$V(r) = C_1 I_1(\eta r) + C_2 K_1(\eta r), \tag{27}$$

where $\eta^2 = \left(C_{44}\xi^2 - \rho\omega^2\right)/C_{66}$ and $I_1(\eta r)$ and $K_1(\eta r)$ are the modified Bessel functions of the first and second kinds of orders one with argument ηr, respectively.

2.6 Propagator Matrix

For laminated cylinder, the propagator matrix or the transfer matrix (Pestel and Leckie, 1963) is employed to analyze the wave motion. The analytical frequency equation of laminated cylinder is obtained by applying the conditions of continuity of displacements and stresses at interfaces between adjoining layers. By substituting the closed-form displacement solutions into the strain-displacement relations, Eqs. (3), and stress-strain relations, Eqs. (2), the displacement and the stress components of the k^{th} layer at the interface $r = r_k$ can be presented in terms of unknown constants as:

$$\begin{Bmatrix} \{\tilde{U}_k\} \\ \{\tilde{S}_k\} \end{Bmatrix} = \left[D_k\right]\{C\}, \tag{28}$$

where

$\{\tilde{U}_k\}^T = \langle U_k \quad W_k \rangle$ in case of axisymmetric vibration

or $= \langle U_k \rangle$ in case of plane strain extensional vibration

or $= \langle W_k \rangle$ in case of longitudinal shear vibration

or $= \langle V_k \rangle$ in case of torsional vibration,

$\{\tilde{S}_k\}^T = \langle \sigma_{rr} \quad \sigma_{rz} \rangle$ in case of axisymmetric vibration

or $= \langle \sigma_{rr} \rangle$ in case of plane strain extensional vibration

or $= \langle \sigma_{rz} \rangle$ in case of longitudinal shear vibration

or $= \langle \sigma_{r\theta} \rangle$ in case of torsional vibration. (29)

$[D_k]$ is the matrix of known solutions, presented in section 2.7, and $\{C\}$ is the vector of unknown constants. Superscript T represent the transpose. Similarly, the displacement and the stress components at the interface $r = r_{k+1}$ of k^{th} layer can also be presented in terms of unknown constants.

By evaluating the displacement and stress components at the surface $r = r_{k+1}$ of the k^{th} layer using Eq. (28), the following relation can be obtained:

$$\begin{Bmatrix} \{\tilde{U}_{k+1}\} \\ \{\tilde{S}_{k+1}\} \end{Bmatrix} = [P_k] \begin{Bmatrix} \{\tilde{U}_k\} \\ \{\tilde{S}_k\} \end{Bmatrix}, \tag{30}$$

where

$$[P_k] = [D_{k+1}][D_k]^{-1}. \tag{31}$$

Superscript −1 denotes the inverse of matrix. The matrix $[P_k]$ is the propagator matrix for the k^{th} layer. Repeated application of Eq. (30) for every layer in the cylinder, composed of N layers, results in:

$$\begin{Bmatrix} \{\tilde{U}_{N+1}\} \\ \{\tilde{S}_{N+1}\} \end{Bmatrix} = [P] \begin{Bmatrix} \{\tilde{U}_1\} \\ \{\tilde{S}_1\} \end{Bmatrix}, \tag{32}$$

where

$$[P] = [P_N][P_{N-1}]\ldots[P_1]. \tag{33}$$

The matrix $[P]$ can be partitioned as:

$$[P] = \begin{bmatrix} [P]_{11} & [P]_{12} \\ [P]_{21} & [P]_{22} \end{bmatrix}. \tag{34}$$

Invoking the zero traction conditions at the inner and the outer surfaces of the laminated cylinder simplifies Eq. (32) to:

$$[P]_{21}\{\tilde{U}_1\} = \{0\}. \tag{35}$$

The exact dispersion relation can be obtained by equating the determinant of the coefficient matrix in Eq. (35) to zero. i.e.;

$$\left|[P]_{21}\right| = 0. \tag{36}$$

This relation is a characteristic equation and can be used to evaluate ω for a given ξ, or alternately, for a given ω it can be solved for ξ.

2.7 The Solution Matrices $[D_k]$ and $[D_{k+1}]$

2.7.1 Axisymmetric Free Vibration

The strain-displacement relations for axisymmetric free vibration are:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \quad ; \quad \varepsilon_{\theta\theta} = \frac{u}{r} \quad ; \quad \varepsilon_{zz} = \frac{\partial w}{\partial z},$$
$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \quad ; \gamma_{r\theta} = \gamma_{\theta z} = 0. \tag{37}$$

Substitution of the displacements u_k and w_k, Eqs. (6), with zero of n and the displacement functions U_k and W_k from Eqs. (18), into Eqs. (37) and the stress-strain relations, Eqs. (2), provides four by four of $[D_k]$ matrix in Eq. (28). Explicitly, the coefficients of $[D_k]$ at the interface $r = r_k$ for this case are as follows:

$$D_{11} = -\varepsilon_1 \xi^2 \alpha z_1(\xi\alpha r_k),$$
$$D_{12} = -\xi^2 \alpha w_1(\xi\alpha r_k),$$
$$D_{13} = -\varepsilon_2 (c_{13} + c_{44})\xi\beta z_1(\xi\beta r_k),$$
$$D_{14} = -(c_{13} + c_{44})\xi\beta w_1(\xi\beta r_k),$$
$$D_{21} = \left(\frac{\xi^2}{c_{13} + c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2 - \frac{\rho\omega^2}{\xi^2} + c_{44}\right) z_0(\xi\alpha r_k),$$
$$D_{22} = \left(\frac{\xi^2}{c_{13} + c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2 - \frac{\rho\omega^2}{\xi^2} + c_{44}\right) w_0(\xi\alpha r_k),$$

$$
\begin{aligned}
D_{23} &= \xi\left(\varepsilon_2 c_{11}\beta^2 - \frac{\rho\omega^2}{\xi^2} + c_{44}\right) z_0\left(\xi\beta r_k\right), \\
D_{24} &= \xi\left(\varepsilon_2 c_{11}\beta^2 - \frac{\rho\omega^2}{\xi^2} + c_{44}\right) w_0\left(\xi\beta r_k\right), \\
D_{31} &= \left\{\varepsilon_1\left(c_{11} - c_{12}\right)\xi^2 \frac{\alpha}{r}\right\} z_1\left(\xi\alpha r_k\right) \\
&\quad - \xi^3\left(\frac{c_{44}}{c_{13} + c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2 + \frac{c_{13}}{c_{44}}\frac{\rho\omega^2}{\xi^2} - c_{13}\right) z_0\left(\xi\alpha r_k\right), \\
D_{32} &= \left\{\left(c_{11} - c_{12}\right)\xi^2 \frac{\alpha}{r_k}\right\} w_1\left(\xi\alpha r_k\right) \\
&\quad - \xi^3\left(\frac{c_{44}}{c_{13} + c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2 + \frac{c_{13}}{c_{44}}\frac{\rho\omega^2}{\xi^2} - c_{13}\right) w_0\left(\xi\alpha r_k\right), \\
D_{33} &= \left\{\varepsilon_2\left(c_{11} - c_{12}\right)\left(c_{13} + c_{44}\right)\xi \frac{\beta}{r_k}\right\} z_1\left(\xi\beta r_k\right) \\
&\quad - \xi c_{44}\left(\varepsilon_2 c_{11}\beta^2 + \frac{c_{13}}{c_{44}}\frac{\rho\omega^2}{\xi^2} - c_{13}\right) z_0\left(\xi\beta r_k\right), \\
D_{34} &= \left\{\left(c_{11} - c_{12}\right)\left(c_{13} + c_{44}\right)\xi \frac{\beta}{r_k}\right\} w_1\left(\xi\beta r_k\right) \\
&\quad - \xi c_{44}\left(\varepsilon_2 c_{11}\beta^2 + \frac{c_{13}}{c_{44}}\frac{\rho\omega^2}{\xi^2} - c_{13}\right) w_0\left(\xi\beta r_k\right), \\
D_{41} &= -\varepsilon_1\left(\frac{c_{44}\xi^3\alpha}{c_{13} + c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2 - \frac{\rho\omega^2}{\xi^2} - c_{13}\right) z_1\left(\xi\alpha r_k\right), \\
D_{42} &= -\left(\frac{c_{44}\xi^3\alpha}{c_{13} + c_{44}}\right)\left(\varepsilon_1 c_{11}\alpha^2 - \frac{\rho\omega^2}{\xi^2} - c_{13}\right) w_1\left(\xi\alpha r_k\right), \\
D_{43} &= -\varepsilon_2 c_{44}\xi^2\beta\left(\varepsilon_2 c_{11}\beta^2 - \frac{\rho\omega^2}{\xi^2} - c_{13}\right) z_1\left(\xi\beta r_k\right), \\
D_{44} &= -c_{44}\xi^2\beta\left(\varepsilon_2 c_{11}\beta^2 - \frac{\rho\omega^2}{\xi^2} - c_{13}\right) w_1\left(\xi\beta r_k\right).
\end{aligned}
\tag{38}
$$

where ε_1 and ε_2 are given in Table 2.1. The coefficients of $\left[D_{k+1}\right]$ at interface $r = r_{k+1}$ can be obtained by replacing r_k in Eqs. (38) with r_{k+1}.

2.7.2 Cut-off frequency for axisymmetric plane strain extensional vibration

Since the motion involves only radial displacement, the strain-displacement relations becomes:

$$\varepsilon_{zz} = \gamma_{\theta z} = \gamma_{rz} = \gamma_{r\theta} = 0 \; ; \varepsilon_{rr} = \frac{\partial u}{\partial r} \; ; \; \varepsilon_{\theta\theta} = \frac{u}{r}. \quad (39)$$

Repeat the procedure described in the preceding subsection using the displacement functions U_k from Eq. (21), the two by two $[D_k]$ matrix is obtained. The coefficients of $[D_k]$ at the interface $r = r_k$ for this case are as follows:

$$\begin{aligned} D_{11} &= J_1(\chi r_k), \\ D_{12} &= Y_1(\chi r_k), \\ D_{21} &= C_{11}\chi J_0(\chi r_k) - (C_{11} - C_{12})\frac{1}{r_k} J_1(\chi r_k), \\ D_{22} &= C_{11}\chi Y_0(\chi r_k) - (C_{11} - C_{12})\frac{1}{r_k} Y_1(\chi r_k). \end{aligned} \quad (40)$$

The coefficients of $[D_{k+1}]$ can be obtained by replacing r_k in Eqs. (40) with r_{k+1}.

2.7.3 Cut-off frequency of axisymmetric longitudinal shear vibration

In this case, the motion involves only the axial direction. Therefore, the strain-displacement relations becomes:

$$\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{zz} = \gamma_{r\theta} = \gamma_{\theta z} = 0 \; ; \; \gamma_{rz} = \frac{\partial w}{\partial r}. \quad (41)$$

Similarly, by substituting the displacement solution W_k, Eq. (24), into the strain-displacement relations in Eqs. (41) and stress-strain relations in Eqs. (2), the coefficients of the 2 x 2 $[D_k]$ matrix at the interface $r = r_k$ of the k^{th} layer are:

$$\begin{aligned} D_{11} &= J_0(\lambda r_k), \\ D_{12} &= Y_0(\lambda r_k), \end{aligned}$$

$$
\begin{aligned}
D_{21} &= -c_{44}\lambda J_1(\lambda r_k), \\
D_{22} &= -c_{44}\lambda Y_1(\lambda r_k).
\end{aligned}
\tag{42}
$$

The coefficients of $[D_{k+1}]$ can be obtained by replacing r_k in Eqs. (42) with r_{k-1}.

2.7.4 Torsional vibration

The strain-displacement relations in Eqs. (3) for torsional vibration are reduced as:

$$
\begin{aligned}
&\varepsilon_{rr} = \varepsilon_{\theta\theta} = \varepsilon_{zz} = \gamma_{rz} = 0, \\
&\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}\,; \quad \gamma_{\theta z} = \frac{\partial v}{\partial z}.
\end{aligned}
\tag{43}
$$

Repeat the procedure described in the preceding subsection using the displacement functions V_k from Eq. (26) or (27), the two by two $[D_k]$ matrix is obtained. The coefficients of $[D_k]$ at the interface $r = r_k$ for this case are as follows:

$$
\begin{aligned}
D_{11} &= Z_1(\eta r_k), \\
D_{12} &= W_1(\eta r_k), \\
D_{21} &= c_{66}\left\{\eta Z_0(\eta r_k) - \frac{2}{r_k} Z_1(\eta r_k)\right\}, \\
D_{22} &= c_{66}\left\{\delta\eta W_0(\eta r_k) - \frac{2}{r_k} W_1(\eta r_k)\right\},
\end{aligned}
\tag{44}
$$

where

$$
\begin{aligned}
\delta &= +1 \quad \text{if} \quad \rho\omega^2 > c_{44}\xi^2\,, \\
\delta &= -1 \quad \text{if} \quad \rho\omega^2 < c_{44}\xi^2.
\end{aligned}
\tag{45}
$$

Z_0 and W_0 are regular or modified Bessel functions of first kind and second kind of orders zero, respectively, depending on whether δ is +1 or -1. Z_1 and W_1 are regular or modified Bessel functions of first kind and second kind of orders one, respectively, depending on whether δ is +1 or -1. At interface $r = r_{k+1}$, the components of $[D_{k+1}]$ matrix can be obtained by replacing r_k in Eqs. (44) with r_{k+1}.