CHAPTER II
PRELIMINARIES

In this chapter, we give some definitions, notations and theorems which will be used in the later chapters.

2.1 Topological Spaces

Definition 2.1.1 Let X be a set. A topology (or *topology structure*) in X is a family $\Im$ of subsets of X satisfies :

(a) Each Union of members of $\Im$ is also a member of $\Im$.

(b) Each finite intersection of members of $\Im$ is also a member of $\Im$.

(c) $\emptyset$ and X are members of $\Im$.

Each members of $\Im$ is called open. A subset A of X is closed in X if $X - A$ is open.

Theorem 2.1.2 If $\mathcal{F}$ is the family of closed sets in a topological space X, then

(a) Each intersection of members of $\mathcal{F}$ belongs to $\mathcal{F}$.

(b) Each finite union of members of $\mathcal{F}$ belongs to $\mathcal{F}$.

(c) $\emptyset$ and X both belong to $\mathcal{F}$.

Proof. See [10] page 24.

Definition 2.1.3 A couple $(X, \Im)$ consisting of a set X and a topology $\Im$ in X is called a topological space. Note that in the set $X \neq \emptyset$, let $\Im = P(X)$. Then $(X, \Im)$ is a topological space and it is called **the discrete topological space**.

Definition 2.1.4 Let X be a topological space and $A \subseteq X$. The closure of A in X denoted by $Cl(A)$, is the set

$$Cl(A) = \cap\{F \subseteq X : F \textit{ is closed and } A \subseteq F\}.$$

Theorem 2.1.5 Let X be a topological space and A, $B \subseteq X$, then

(a) $A \subseteq Cl(A)$

(b) If $A \subseteq B$, then $Cl(A) \subseteq Cl(B)$

(c) A is closed in X if and only if $A = Cl(A)$

(d) $Cl(A)$ is the smallest closed set in X with $A \subseteq Cl(A)$

(e) $Cl(Cl(A)) = Cl(A)$

(f) $Cl(A \cup B) = Cl(A) \cup Cl(B)$

(g) $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$.

Proof. See [4] page 69-70.

Theorem 2.1.6 Let $\{B_\alpha : \alpha \in \mathcal{A}\}$ be a family of subsets of a topological space X. Then $\bigcup_{\alpha \in \mathcal{A}} Cl(B_\alpha) \subseteq Cl\,(\bigcup_{\alpha \in \mathcal{A}} B_\alpha)$.

Proof. See [4] page 70.

Definition 2.1.7 Let X be a topological space and $A \subseteq X$. The interior of A in X denoted by $Int(A)$, is the set

$$Int(A) = \cup\{G \subseteq X : G \text{ is open and } G \subseteq A\}.$$

Theorem 2.1.8 Let X be a topological space and $A, B \subseteq X$, then

(a) $Int(A) \subseteq A$

(b) If $A \subseteq B$, then $Int(A) \subseteq Int(B)$

(c) A is open if and only if $Int(A) = A$

(d) $Int(A)$ is the largest open in X with $Int(A) \subseteq A$

(e) $Int(Int(A)) = Int(A)$

(f) $Int(A) \cup Int(B) \subseteq Int(A \cup B)$

(g) $Int(A) \cap Int(B) = Int(A \cap B)$

(h) If $A_\alpha \subseteq X$ for all $\alpha \in \mathcal{A}$, then $\bigcup_{\alpha \in \mathcal{A}} Int(A_\alpha) \subseteq Int\,(\bigcup_{\alpha \in \mathcal{A}} A_\alpha)$.

Proof. See [10] page 27.

Theorem 2.1.9 Let A be a subset of a topological space X, then $Int(A) = X - Cl(X - A)$.

Proof. See [4] page 71.

Definition 2.1.10 Let $(X, \Im)$ be a topological space and $Y \subseteq X$. The collection $\Im_Y = \{G \cap Y : G \in \Im\}$ is a topology for Y, called **the relative topology for Y**. The fact that a subset of X is being given this topology is signified by referring to it as a subspace of X.

2.2 Generalized closed and Regular generalized closed sets

Definition 2.2.1 Let A be a subset of a topological space X. Then A is said to be a **regular open** if $A = Int(Cl(A))$, and A is said to be a **regular closed** if $A = Cl(Int(A))$.

Definition 2.2.2 Let A be a subset of a topological space X. Then A is said to be **generalized closed** (briefly $g-closed$) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in X. The complement of a g-closed set is said to be **generalized open** (briefly $g-open$).

Theorem 2.2.3 Let X be a topological space. A subset A of X is g-open if and only if $F \subseteq Int(A)$ whenever $F \subseteq A$ and F is closed in X.
Proof. See [2] page 195.

Definition 2.2.4 Let A be a subset of a topological space X. Then A is said to be **regular generalized closed** (briefly $rg-closed$) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in X. The complement of a rg-closed set is said to be **regular generalized open** (briefly $rg-open$).

Theorem 2.2.5 Let X be a topological space. A subset A of X is rg-open if and only if $F \subseteq Int(A)$ whenever $F \subseteq A$ and F is regular closed in X.
Proof. See [7] page 215.

Note that we obtain the following diagram from their definitions.

DIAGRAM I

regular closed ⇒ closed ⇒ g-closed ⇒ rg-closed
regular open ⇒ open ⇒ g-open ⇒ rg-open

Theorem 2.2.6 Let $(X, \Im)$ be a topological space and $(Y, \Im_Y)$ a subspace of $(X, \Im)$. Then $A \subseteq Y$ is a closed set in Y if and only if $A = Y \cap F$, for some closed set F in X.
Proof. See [4] page 77.

Theorem 2.2.7 Let Y be a subspace of X. If $A \subseteq Y$ is closed (*open*) in Y and Y is closed (*open*) in X, then A is closed (*open*) in X.
Proof. See [4] page 78.

Theorem 2.2.8 Let Y be a subspace of X. If $A \subseteq Y$ is regular closed in Y and Y is regular closed in X, then A is regular closed in X.
Proof. See [9] page 31.

Definition 2.2.9 A function $f : X \to Y$ is said to be **open** if for each open set U in X, $f(U)$ is open in Y.

Definition 2.2.10 A function $f : X \to Y$ is said to be **closed** if for each closed set F in X, $f(F)$ is closed in Y.

Definition 2.2.11 A function $f : X \to Y$ is said to be **regular closed** if for each closed set F in X, $f(F)$ is regular closed in Y.

Definition 2.2.12 A function $f : X \to Y$ is said to be **continuous** if for each closed set U in Y, $f^{-1}(U)$ is closed in X.

Definition 2.2.13 A function $f : X \to Y$ is said to be **regular continuous** if for each closed set F in Y, $f^{-1}(F)$ is regular closed in X.

Definition 2.2.14 A function $f : X \to Y$ is said to be **gc-irresolute** if for each g-closed set F in Y, $f^{-1}(F)$ is g-closed in X.

Definition 2.2.15 A topological space X is called a $T_{\frac{1}{2}}$**-space** if every g-closed set in X is closed in X.

Definition 2.2.16 A topological space X is called a **regular** $T_{\frac{1}{2}}$**-space** if every rg-closed set in X is regular closed in X.

Definition 2.2.17 A topological space X is called **normal** if for each closed sets A and B in X with $A \cap B = \emptyset$, there exist open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Definition 2.2.18 A topological space X is called **g-normal** if for each closed set A in X and each regular closed set B in X with $A \cap B = \emptyset$, there exist g-open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Theorem 2.2.19 If a topological space X is normal, then X is g-normal, but not conversely.

Proof. See [3] page 23.