CHAPTER III
PRE-NORMAL SPACES

In this chapter, we study generalized preclosed sets and generalized preclosed functions and give some characterizations of pre-normal spaces.

3.1 Preclosed Sets and Generalized Preclosed Sets

In this section, we study preclosed and generalized preclosed sets and give some relation among them.

Definition 3.1.1 Let A be a subset of a topological space X. Then A is said to be a **preopen** if $A \subseteq Int(Cl(A))$. The complement of a preopen set is said to be **preclosed.**

Remark 3.1.2 A subset A of a topological space X is preclosed if $Cl(Int(A)) \subseteq A$.
Proof. Let A be preclosed. Then $X - A$ is preopen. By Definition 3.1.1 and Theorem 2.1.9, we have that $X - A \subseteq Int(Cl(X - A)) = X - Cl(X - Cl(X - A))$, so that $Cl(X - Cl(X - A)) \subseteq A$. Hence $Cl(Int(A)) \subseteq A$.

Definition 3.1.3 Let A be a subset of a topological space X. The **preclosure** of A in X denoted by $pCl(A)$, is the set

$$pCl(A) = \cap \{F \subseteq X : F \text{ is preclosed and } A \subseteq F\}.$$

Definition 3.1.4 Let A be a subset of a topological space X. The **preinterior** of A in X denoted by $pInt(A)$, is the set

$$pInt(A) = \cup \{G \subseteq X : G \text{ is preopen and } G \subseteq A\}.$$

Remark 3.1.5 $Int(A) \subseteq pInt(A) \subseteq A \subseteq pCl(A) \subseteq Cl(A)$

Theorem 3.1.6 Let $\{G_\alpha : \alpha \in \mathcal{A}\}$ be a family of preopen subsets of a topological space X. Then $\bigcup_{\alpha \in \mathcal{A}} G_\alpha$ is preopen in X.
Proof. Let G_α is preopen in X for all $\alpha \in \mathcal{A}$, so that $G_\alpha \subseteq Int(Cl(G_\alpha))$. By Theorem 2.1.6 and 2.1.8(h), we have that $\bigcup_{\alpha \in \mathcal{A}} G_\alpha \subseteq \bigcup_{\alpha \in \mathcal{A}} Int(Cl(G_\alpha)) \subseteq Int\,(\bigcup_{\alpha \in \mathcal{A}} Cl(G_\alpha)) \subseteq Int(Cl\,(\bigcup_{\alpha \in \mathcal{A}} G_\alpha))$. Hence $\bigcup_{\alpha \in \mathcal{A}} G_\alpha$ is preopen in X.

Corollary 3.1.7 Let $\{F_\alpha : \alpha \in \mathcal{A}\}$ be a family of preclosed subsets of a topological space X. Then $\bigcap_{\alpha\in\mathcal{A}} F_\alpha$ is preclosed in X.

Proof. Let F_α is preclosed in X for all $\alpha \in \mathcal{A}$. Then $X - F_\alpha$ is preopen in X. By De Morgan' s Law, we have that $\bigcup_{\alpha\in\mathcal{A}} (X - F_\alpha) = X - \bigcap_{\alpha\in\mathcal{A}} F_\alpha$ and by Theorem 3.1.6, $\bigcup_{\alpha\in\mathcal{A}} (X - F_\alpha)$ is preopen in X. Hence $\bigcap_{\alpha\in\mathcal{A}} F_\alpha$ is preclosed in X.

Remark 3.1.8

(1) Finite intersection of preopen sets in X need not be preopen.

(2) Finite union of preclosed sets in X need not be preclosed.

The following example shows above statements.

Example 3.1.9 Let $X = \{a,b,c,d\}$ and $\Im = \{\emptyset, \{a,c\}, \{a,b,c\}, \{a,c,d\}, X\}$. Since $\{b,c\}$ and $\{a,b,d\}$ are preopen sets in X, but $\{b,c\} \cap \{a,b,d\} = \{b\}$ is not preopen in X because $\{b\} \not\subseteq \emptyset = Int(Cl(\{b\}))$. Since $X - \{b,c\} = \{a,d\}$ and $X - \{a,b,d\} = \{c\}$ are preclosed sets in X, but $\{a,d\} \cup \{c\} = \{a,c,d\}$ is not preclosed in X because $Cl(Int(\{a,c,d\})) = X \not\subseteq \{a,c,d\}$.

Theorem 3.1.10 Let A and B be subsets of a topological space X and $x \in X$. Then

(a) $x \in pCl(A)$ if and only if $A \cap U \neq \emptyset$ for every preopen set U containing x

(b) $pCl(A)$ is preclosed

(c) A is preclosed if and only if $A = pCl(A)$

(d) $X - pCl(X - A) = pInt(A)$

(e) $X - pInt(A) = pCl(X - A)$

(f) $pInt(A)$ is preopen

(g) If $A \subseteq B$, then $pCl(A) \subseteq pCl(B)$.

Proof. (a) ($\Rightarrow$) Let $x \in pCl(A)$ and U be preopen in X such that $x \in U$. Suppose that $A \cap U = \emptyset$, then $A \subseteq X - U$. Since $X - U$ is preclosed, so that $x \in X - U$ which is a contradiction. Hence $A \cap U \neq \emptyset$.

($\Leftarrow$) Suppose that $x \notin pCl(A)$, then there exists a preclosed subset W of X such that $A \subseteq W$ and $x \notin W$, then $x \in X - W$ is preopen. By assumption $(X - W) \cap A \neq \emptyset$ which is a contradiction. Hence $x \in pCl(A)$.

(b) Since $pCl(A) = \cap\{F \subseteq X : F \ \textit{is preclosed and} \ A \subseteq F\}$ and by Corollary 3.1.7, we have $pCl(A)$ is preclosed.

(c) ($\Rightarrow$) Assume that A is preclosed. Clearly $A \subseteq pCl(A)$ and it follows by the definition of preclosure that $pCl(A) \subseteq A$ because A itself is preclosed. Hence $A = pCl(A)$.

($\Leftarrow$) It follows directly from (b).

(d) Since
$$\begin{aligned} pInt(A) &= \cup\{G \subseteq X : G \text{ is preopen and } G \subseteq A\} \\ &= \cup\{G \subseteq X : X - G \text{ is preclosed and } X - A \subseteq X - G\} \\ &= \cup\{X - F \subseteq X : F \text{ is preclosed and } X - A \subseteq F\} \\ &= X - \cap\{F \subseteq X : F \text{ is preclosed and } X - A \subseteq F\} \\ &= X - pCl(X - A). \end{aligned}$$

(e) By (d), $pInt(A) = X - pCl(X - A)$. Thus $X - pInt(A) = X - (X - pCl(X - A)) = pCl(X - A)$.

(f) Since $pInt(A) = \cup\{G \subseteq X : G \text{ is preopen and } G \subseteq A\}$ and by Theorem 3.1.6, we get that $pInt(A)$ is preopen.

(g) Let $x \in pCl(A)$ and U be preopen in X such that $x \in U$. By (a), $A \cap U \neq \emptyset$ and since $A \subseteq B$, we have $B \cap U \neq \emptyset$. Thus $x \in pCl(B)$. Hence $pCl(A) \subseteq pCl(B)$.

Definition 3.1.11 A subset A of a topological space X is said to be **generalized preclosed** (briefly $gp - closed$) if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open set in X. The complement of a gp-closed set is said to be **generalized preopen** (briefly $gp - open$).

By the definition stated above, we obtain the following diagram :

DIAGRAM II

$$\begin{array}{ccc} \text{closed } (open) & \Rightarrow & \text{preclosed } (preopen) \\ \Downarrow & & \Downarrow \\ \text{g-closed } (g - open) & \Rightarrow & \text{gp-closed } (gp - open) \end{array}$$

Remark 3.1.12

(1) None of the implications in the diagram are reversible.

(2) g-closedness and preclosedness are independent.

Example 3.1.13

(1) Let $X = \{a, b, c\}$ and $\Im = \{\emptyset, \{b, c\}, X\}$. Then $\{c\}$ is preclosed but it is neither g-closed nor closed in X.

(2) Let $X = \{a, b, c\}$ and $\Im = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then $\{a, c\}$ is g-closed but it is neither preclosed nor closed in X.

Definition 3.1.14 A topological space X is called a T_g**-space** if every gp-closed set in X is g-closed in X.

Example 3.1.15

(1) Let $X = \{a, b, c, d\}$ and $\Im = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. Since $\{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}$, $\emptyset$ and X are all gp-closed sets in X, we can easily see that every gp-closed sets are g-closed in X. Thus X is a T_g-space.

(2) Let $X = \{a, b, c\}$ and $\Im = \{\emptyset, \{a\}, \{a, b\}, X\}$. Since $\{b\}$ is gp-closed set in X, but $\{b\}$ is not g-closed set in X, we have that X is not a T_g-space.

Theorem 3.1.16 A subset A of a topological space X is gp-open in X if and only if $F \subseteq pInt(A)$ whenever $F \subseteq A$ and F is closed in X.

Proof.($\Rightarrow$) Let A be a gp-open set and F a closed set in X with $F \subseteq A$, then we have $X - A \subseteq X - F$. Since $X - A$ is gp-closed set, so that $pCl(X - A) \subseteq X - F$. By Theorem 3.1.10(d), $F \subseteq X - pCl(X - A) = pInt(A)$.

($\Leftarrow$) Let U be open set in X such that $X - A \subseteq U$, we have $X - U \subseteq A$. By assumption, we have $X - U \subseteq pInt(A)$. Thus $X - pInt(A) \subseteq U$. This shows by Theorem 3.1.10(e) that $pCl(X - A) \subseteq U$, then $X - A$ is gp-closed set. Hence A is gp-open in X.

3.2 Preclosed, Generalized Preclosed and Pre Generalized Preclosed Functions

In this section, preclosed, generalized preclosed and pre generalized preclosed functions are defined and studied and we also give some relationships between those functions.

Definition 3.2.1 A function $f : X \to Y$ is said to be **preclosed** if for each closed set F in X, $f(F)$ is preclosed in Y.

Definition 3.2.2 A function $f : X \to Y$ is said to be **generalized preclosed** (briefly $gp - closed$) if for each closed set F in X, $f(F)$ is gp-closed in Y.

Definition 3.2.3 A function $f : X \to Y$ is said to be **pre generalized preclosed** (briefly $pre\ \ gp - closed$) if for each preclosed set F in X, $f(F)$ is gp-closed in Y.

Remark 3.2.4 Every closed function is preclosed but not conversely. It is obvious that both preclosedness and pre gp-closedness imply gp-closed. However, the converses are false as the following example shows.

Example 3.2.5

(1) Let$X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is closed and hence preclosed, but it is not pre gp-closed because $\{b\}$ is preclosed in (X, τ) but $\{b\}$ is not gp-closed in (X, σ).

(2) Let (X, σ) and (X, τ) be the topological spaces defined as in (1) and let $g : (X, \sigma) \to (X, \tau)$ be the identity function. We have that g is pre gp-closed but not preclosed because $\{a, c\}$ is a closed subset of (X, σ) but $g(\{a, c\}) = \{a, c\}$ is not preclosed in (X, τ).

(3) Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, X\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is preclosed, but it is not closed because $\{c\}$ is closed in (X, τ) but $\{c\}$ is not closed in (X, σ).

Theorem 3.2.6 A surjective function $f : X \to Y$ is gp-closed (resp. $pre\ \ gp - closed$) if and only if for each subset B of Y and each open (resp. *preopen*) subset U of X containing $f^{-1}(B)$, there exists a gp-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof.($\Rightarrow$) Suppose that f is gp-closed (resp. $pre\ \ gp - closed$). Let B any subset of Y and U an open (resp. *preopen*) set of X containing $f^{-1}(B)$. Put $V = Y - f(X - U)$. Then V is gp-open in Y, $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

($\Leftarrow$) Let F be any closed (resp. *preclosed*) set of X. Put $B = Y - f(F)$, then we have $f^{-1}(B) \subseteq X - F$ and since $X - F$ is open (resp. *preopen*) in X, there exists a gp-open set V of Y such that $B = Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$.

Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is gp-closed in Y. This shows that f is gp-closed (resp. $pre\ gp-closed$).

Corollary 3.2.7 If $f : X \to Y$ is gp-closed (resp. $pre\ gp-closed$), then for any closed set F of Y and for any open (resp. $preopen$) set U of X containing $f^{-1}(F)$, there exists a preopen set V of Y such that $F \subseteq V$ and $f^{-1}(V) \subseteq U$.
Proof. By Theorem 3.2.6, there exists a gp-open set W of Y such that $F \subseteq W$ and $f^{-1}(W) \subseteq U$. Since F is closed, by Theorem 3.1.16 we have $F \subseteq pInt(W)$. Put $V = pInt(W)$, then V is preopen in Y, $F \subseteq V$, and $f^{-1}(V) \subseteq U$.

Theorem 3.2.8 If $f : X \to Y$ is continuous, pre gp-closed and A is gp-closed in X, then $f(A)$ is gp-closed in Y.
Proof. Let V be any open set of Y containing $f(A)$. Then $A \subseteq f^{-1}(V)$ and $f^{-1}(V)$ is open in X. Since A is gp-closed in X, $pCl(A) \subseteq f^{-1}(V)$ and hence $f(A) \subseteq f(pCl(A)) \subseteq V$. Since f is pre gp-closed and $pCl(A)$ is preclosed in X, $f(pCl(A))$ is gp-closed in Y and hence $pCl(f(A)) \subseteq pCl(f(pCl(A))) \subseteq V$. This shows that $f(A)$ is gp-closed in Y.

Theorem 3.2.9 If $f : X \to Y$ is continuous, pre gp-closed and bijective and U is gp-open in X, then $f(U)$ is gp-open in Y.
Proof. Let U be a gp-open set in X. Then $X - U$ is gp-closed in X. By Theorem 3.2.8, $f(X - U)$ is gp-closed set in Y, but $f(X - U) = Y - f(U)$ because f is bijective. Hence $f(U)$ is gp-open in Y.

Definition 3.2.10 A function $f : X \to Y$ is said to be **pre-irresolute** if for each preopen set V in Y, $f^{-1}(V)$ is preopen in X.

Definition 3.2.11 A function $f : X \to Y$ is said to be **generalized pre-irresolute** (briefly $gp - irresolute$) if for each gp-open set V in Y, $f^{-1}(V)$ is gp-open in X.

By using the complement of preopen (resp. $gp - open$) set, we have that $f : X \to Y$ is pre-irresolute (resp. $gp - irresolute$) if and only if $f^{-1}(F)$ is preclosed (resp. $gp - closed$) in X for every preclosed (resp. $gp - closed$) set F in Y.

Theorem 3.2.12 If $f : X \to Y$ is open, pre-irresolute and bijective and B is gp-closed in Y, then $f^{-1}(B)$ is gp-closed in X.
Proof. Let U be any open set of X containing $f^{-1}(B)$. Then $B \subseteq f(U)$ and $f(U)$ is open in Y. Since B is gp-closed in Y, $pCl(B) \subseteq f(U)$ and hence $f^{-1}(B) \subseteq f^{-1}(pCl(B)) \subseteq U$. Since f is pre-irresolute, $f^{-1}(pCl(B))$ is preclosed in X and hence by Theorem 3.1.10(c) we have $pCl(f^{-1}(B)) \subseteq f^{-1}(pCl(B)) \subseteq U$. This shows that $f^{-1}(B)$ is gp-closed in X.

Theorem 3.2.13 If $f : X \to Y$ is open, pre-irresolute and bijective and A is gp-open in Y, then $f^{-1}(A)$ is gp-open in X.
Proof. Let A be a gp-open set in Y. We have $Y - A$ is gp-closed in Y. By Theorem 3.2.12, $f^{-1}(Y - A)$ is gp-closed set in X, but $f^{-1}(Y - A) = X - f^{-1}(A)$. Hence $f^{-1}(A)$ is gp-open in X.

Theorem 3.2.14 If a function $f : X \longrightarrow Y$ is gc-irresolute and Y is a T_g-space, then f is gp-irresolute.
Proof. Let A be a gp-closed set in Y. Since Y is a T_g-space, A is g-closed in Y. Since f is gc-irresolute, we have that $f^{-1}(A)$ is g-closed in X. But every g-closed set is gp-closed. Therefore, $f^{-1}(A)$ is gp-closed in X. Hence f is gp-irresolute.

3.3 Pre-normal Spaces

In this section, we define a pre-normal space which is a generalization of normal space and give characterizations of pre-normal space. By the end of this section we give some preservation theorems concerning pre-normal spaces.

Definition 3.3.1 A topological space X is called **pre-normal** if for each disjoint closed sets A and B in X, there exist disjoint preopen sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

Theorem 3.3.2 If a topological space X is normal, then X is pre-normal, but not conversely.
Proof. Assume that X is normal. Let A and B are closed sets in X with $A \cap B = \emptyset$. Since X is normal, there exist open sets U and V in X such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$, but every open set is preopen in X. Hence X is pre-normal.

The converse of this theorem is not true as seen by the following example.

Example 3.3.3 Let $X = \{a, b, c, d\}$ and $\Im = \{\emptyset, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$. Since $\{b\}, \{d\}, \{b, d\}, \emptyset$ and X are all closed sets in X and since $\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset$ and X are all preopen sets in X, and $\{b\}$, $\{d\}$ are disjoint closed sets in X, we see that $\{a, d\}$ and $\{b, c\}$ are preopen sets in X such that $\{b\} \subseteq \{b, c\}, \{d\} \subseteq \{a, d\}$. Thus X is pre-normal. But $\{b\}$ and $\{d\}$ can not be separated by any two open sets in X. Hence X is not normal.

By using gp-open sets, we obtain some new characterizations of pre-normal spaces.

Theorem 3.3.4 The following are equivalent for a topological space X :

(a) X is pre-normal.

(b) For any pair of disjoint closed sets A and B of X, there exist disjoint gp-open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

(c) For any closed set A of X and any open set V of X containing A, there exists a gp-open set U of X such that $A \subseteq U \subseteq pCl(U) \subseteq V$.

Proof.(a) $\Rightarrow$ (b) It is obvious since every preopen set is gp-open.

(b) $\Rightarrow$ (c) Let A be any closed set of X and V an open set of X containing A. Since A and $X - V$ are disjoint closed set of X, there exist gp-open sets U and W of X such that $A \subseteq U$, $X - V \subseteq W$ and $U \cap W = \emptyset$. By Theorem 3.1.16, we have $X - V \subseteq pInt(W)$. Since $U \cap pInt(W) = \emptyset$, we have $pCl(U) \cap pInt(W) = \emptyset$, and hence $pCl(U) \subseteq X - pInt(W) \subseteq V$. Therefore, we obtain $A \subseteq U \subseteq pCl(U) \subseteq V$.

(c) $\Rightarrow$ (a) Let A and B be any disjoint closed sets of X. Since $X - B$ is an open set containing A, there exists a gp-open set G of X such that $A \subseteq G \subseteq pCl(G) \subseteq X - B$. By Theorem 3.1.16, we have $A \subseteq pInt(G)$. Put $U = pInt(G)$ and $V = X - pCl(G)$. Then U and V are disjoint preopen sets such that $A \subseteq U$ and $B \subseteq V$. Therefore, X is pre-normal.

Theorem 3.3.5 If $f : X \rightarrow Y$ is a continuous, gp-closed and surjective mapping and X is normal, then Y is pre-normal.

Proof. Let A and B be any disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X, since f is continuous. Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By Theorem

3.2.6, there exist gp-open sets G and H of Y such that $A \subseteq G$, $B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Then, we have $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H) = \emptyset$ which implies $G \cap H = \emptyset$. It follows from Theorem 3.3.4 that Y is pre-normal.

Corollary 3.3.6 If $f : X \to Y$ is a continuous, preclosed and surjective mapping and X is normal, then Y is pre-normal.
Proof. Since every preclosed function is gp-closed, the proof is obtained directly from Theorem 3.3.5 .

Theorem 3.3.7 If $f : X \to Y$ is a continuous, pre gp-closed and surjective mapping and X is pre-normal, then Y is pre-normal.
Proof. Let A and B be any disjoint closed sets in Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in X. Since X is pre-normal, there exist disjoint preopen sets U and V of X such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since f is pre gp-closed, by Corollary 3.2.7, there exist preopen sets G, H in Y such that $A \subseteq G$, $B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since U and V are disjoint, it implies $G \cap H = \emptyset$. This shows that Y is pre-normal.