CHAPTER IV
GENERALIZATION OF PRE-NORMAL SPACES

In this chapter, we define a new concept of separation axiom of topological spaces, called gp-normal which is a generalization of pre-normal and give characterizations of this space.

4.1 Generalized Pre-normal Space

In this section, we define gp-normal space and give some relationships among g-normal, pre-normal and gp-normal spaces.

Definition 4.1.1 A topological space X is called **generalized pre-normal** (brifly $gp-normal$) if for each closed set A in X and each regular closed set B in X with $A \cap B = \emptyset$, there exist gp-open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Example 4.1.2

(1) Let $X = \{a, b, c, d\}$ and $\Im = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. Since $\{a\}, \{b, c, d\}, \emptyset$ and X are all regular closed sets in X and $\{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, \emptyset$ and X are all closed sets in X, and $\{a\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset$ and X are all gp-open sets in X. We can easily see that X is gp-normal.

(2) Let $X = \{a, b, c, d\}$ and $\Im = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, X\}$. Since $\{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \emptyset$ and X are all gp-open sets in X and $\{a, d\}$ is a regular closed set, $\{c\}$ is a closed set in X with $\{a, d\} \cap \{c\} = \emptyset$, we easily see that $\{a, d\}$ and $\{c\}$ can not be separated by any two gp-open sets in X. Hence X is not gp-normal.

Theorem 4.1.3 If a topological space X is pre-normal, then X is gp-normal, but not conversely.

Proof. Assume that X is pre-normal. We shall show that X is gp-normal. Let A be a closed set in X and B a regular closed set in X such that $A \cap B = \emptyset$. Since every regular closed set is closed in X and X is pre-normal, there exist preopen sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Since every preopen set is gp-open in X, it follows that X is gp-normal.

The converse of this theorem is not true as seen by the following example.

Example 4.1.4 Let $X = \{a, b, c, d\}$ and $\Im = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. Since $\{a\}, \{b, c, d\}$, $\emptyset$ and X are all regular closed sets in X, by Example 4.1.2 (1), X is gp-normal. Since $\{a\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}$, $\{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \emptyset$ and X are all preopen sets in X, and $\{b\}$, $\{d\}$ are disjoint closed sets in X, we see that $\{b\}$ and $\{d\}$ can not be separated by two preopen sets in X. Hence X is not pre-normal.

Theorem 4.1.5 If a topological space X is gp-normal and regular $T_{\frac{1}{2}}$-space, then X is pre-normal, but not conversely.

Proof. Assume that X is gp-normal and a regular $T_{\frac{1}{2}}$-space. Let A and B are closed sets in X with $A \cap B = \emptyset$, so that B is rg-closed set in X. Since X is a regular $T_{\frac{1}{2}}$-space, we have that B is regular closed. Since X is gp-normal, there exist gp-open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. By Theorem 3.3.4, we obtain that X is pre-normal.

The converse of this theorem is not true as seen by the following example.

Example 4.1.6 Let $X = \{a, b, c\}$ and $\Im = \{\emptyset, \{a\}, \{a, b\}, X\}$. Since regular closed sets in X are only $\emptyset$ and X, and $\{a\}, \{a, b\}, \{a, c\}, \emptyset$ and X are all preopen sets in X, and $\{c\}, \{b, c\}, \emptyset$ and X are all closed sets in X, we can easily see that X is pre-normal. Since $\{a\}$ is rg-closed in X which is not regular closed, we obtain that X is not a regular $T_{\frac{1}{2}}$-space.

Theorem 4.1.7 If a topological space X is gp-normal which is a T_g-space, then X is g-normal, but not conversely.

Proof. Assume that X is gp-normal and a T_g-space. Let A be a closed set in X and B a regular closed set in X with $A \cap B = \emptyset$. Since X is gp-normal, there exist gp-open sets U and V in X such that $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$. Since X is a T_g-space, it implies that U and V are g-open sets in X, so we have that X is g-normal.

The converse of this theorem is not true as seen by the following example.

Example 4.1.8 Let $X = \{a, b, c\}$ and $\Im = \{\emptyset, \{a\}, \{a, b\}, X\}$. It is obvious that X is normal, so it is g-normal. Since $\{b\}$ is gp-closed set in X which is not g-closed, we obtain that X is not a T_g-space.

4.2 Characterization of gp-Normal Spaces

In this section, we give characterization of being gp-normal spaces and we show that a subspace of a gp-normal space need not be gp-normal.

Theorem 4.2.1 The following are equivalent for a topological space X :

(a) X is gp-normal.

(b) For each closed set A in X and each regular open set U in X containing A, there exists a gp-open set V in X such that $A \subseteq V \subseteq pCl(V) \subseteq U$.

(c) For each closed set A in X and each regular closed set B in X such that $A \cap B = \emptyset$, there exists a gp-open set V in X such that $A \subseteq V$ and $pCl(V) \cap B = \emptyset$.

(d) For each closed set A in X and each regular closed set B in X such that $A \cap B = \emptyset$, there exist gp-open sets U and V in X such that $A \subseteq U, B \subseteq V$ and $pCl(U) \cap V = \emptyset$.

Proof.(a) $\Rightarrow$ (b) Assume that (a) holds. Let A be a closed set in X and U a regular open set in X with $A \subseteq U$. Then $X - U$ is regular closed set in X and $A \cap (X - U) = \emptyset$. Since X is gp-normal, there exist gp-open sets W and V in X such that $A \subseteq V, X - U \subseteq W$ and $W \cap V = \emptyset$. Then $X - W \subseteq U$ and $V \subseteq X - W$. Since $X - W$ is gp-closed set in X, we have $pCl(X - W) \subseteq U$. Hence $A \subseteq V \subseteq pCl(V) \subseteq pCl(X - W) \subseteq U$.

(b) $\Rightarrow$ (c) Assume that (b) holds. Let A be a closed set in X and B a regular closed set in X with $A \cap B = \emptyset$. By assumption, there exists a gp-open set V in X such that $A \subseteq V \subseteq pCl(V) \subseteq X - B$. Hence $A \subseteq V$ and $pCl(V) \cap B = \emptyset$.

(c) $\Rightarrow$ (d) Assume that (c) holds. Let A be a closed set in X and B a regular closed set in X with $A \cap B = \emptyset$. By assumption, there exists a gp-open set U in X such that $A \subseteq U$ and $pCl(U) \cap B = \emptyset$. Since $pCl(U)$ is preclosed set in X. Put $V = X - pCl(U)$, so that V is a preopen set in X. It implies that V is gp-open set in X with $B \subseteq X - pCl(U) = V$ and $pCl(U) \cap V = \emptyset$.

(d) $\Rightarrow$ (a) It is obvious.

The following example shows that a subspace of a gp-normal space need not be gp-normal.

Example 4.2.2 Let $X = \{a, b, c, d, e\}$ and $\Im = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, X\}$. Since $\{e\}, \{c, e\}, \{d, e\}, \{a, d, e\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}, \emptyset$ and X are all closed sets in X. Since $\emptyset$ is only one closed set in X which is disjoint with another, it is obvious that X is normal, so it is gp-normal. Let $Y = \{a, b, c, d\}$ and $\Im_Y = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, Y\}$. By Example 4.1.2(2), we have that Y is not gp-normal.

4.3 Preservation Theorems of gp-Normal Spaces

In this section, we give some preservation theorems concerning gp-normal spaces.

Theorem 4.3.1 If $f : X \to Y$ is regular continuous, pre gp-closed and bijective and X is gp-normal, then Y is gp-normal.

Proof. Assume that X is gp-normal. Let A be a closed set and B a regular closed set in Y with $A \cap B = \emptyset$. Since f is regular continuous, we have that $f^{-1}(A)$ and $f^{-1}(B)$ are regular closed sets in X and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$, but every regular closed set is closed, so that $f^{-1}(A)$ is closed in X. Since X is gp-normal, there exist gp-open sets U and V in X such that $f^{-1}(A) \subseteq U, f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is surjective, $A = f(f^{-1}(A)) \subseteq f(U)$ and $B = f(f^{-1}(B)) \subseteq f(V)$. Since every regular continuous function is continuous, by Theorem 3.2.9, we obtain that $f(U)$ and $f(V)$ are gp-open sets in Y, and since f is injective, $f(U) \cap f(V) = \emptyset$. Hence Y is gp-normal.

Theorem 4.3.2 If $f : X \to Y$ is regular closed, open, pre-irresolute and bijective and Y is gp-normal, then X is gp-normal.

Proof. Assume that Y is gp-normal. Let A be a closed set and B a regular closed set in X with $A \cap B = \emptyset$. Since f is regular closed, we have that $f(A)$ and $f(B)$ are regular closed sets in Y and f is injective, $f(A) \cap f(B) = \emptyset$, but every regular closed set is closed, so that $f(A)$ is closed in Y. Since Y is gp-normal, there exist gp-open sets U and V in Y such that $f(A) \subseteq U, f(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is injective, $A = f^{-1}(f(A)) \subseteq f^{-1}(U)$ and $B = f^{-1}(f(B)) \subseteq f^{-1}(V)$. By Theorem

3.2.13, $f^{-1}(U)$ and $f^{-1}(V)$ are gp-open sets in X, and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence X is gp-normal.

Theorem 4.3.3 If $f : X \to Y$ is regular closed, gp-irresolute and injective and Y is gp-normal, then X is gp-normal.

Proof. Assume that Y is gp-normal. Let A be a closed set and B a regular closed set in X with $A \cap B = \emptyset$. Since f is regular closed, we have that $f(A)$ and $f(B)$ are regular closed sets in Y and since f is injective, $f(A) \cap f(B) = \emptyset$, but every regular closed set is closed, so that $f(A)$ is closed in Y. Since Y is gp-normal, there exist gp-open sets U and V in Y such that $f(A) \subseteq U, f(B) \subseteq V$ and $U \cap V = \emptyset$. Since f is injective, $A = f^{-1}(f(A)) \subseteq f^{-1}(U)$ and $B = f^{-1}(f(B)) \subseteq f^{-1}(V)$. Since f is gp-irresolute, $f^{-1}(U)$ and $f^{-1}(V)$ are gp-open sets in X, and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence X is gp-normal.

Corollary 4.3.4 If $f : X \to Y$ is regular closed, gc-irresolute and injective and Y is a T_g-space and gp-normal, then X is gp-normal.

Proof. Since f is gc-irresolute and Y is a T_g-space, by Theorem 3.2.14, we have that f is gp-irresolute, and by Theorem 4.3.3, it implies that X is gp-normal.