Chapter 3

MAIN RESULTS

3.1 Fundamental relations and solution method

Let us first write the Eq.(1.4) in the form

$$D(x) = f(x) + \lambda_1 V(x) + \lambda_2 F(x) \qquad \text{or} \qquad D(x) = I(x) \tag{3.1}$$

where

$$D(x) = \sum_{k=0}^{m} P_k(x)\, y^{(k)}(x)$$

$$V(x) = \int_a^x \sum_{i=0}^{p} A_i(x,t)\, y^{(i)}(t)\, dt$$

$$F(x) = \int_a^b \sum_{j=0}^{q} B_j(x,t)\, y^{(j)}(t)\, dt$$

and

$$I(x) = f(x) + \lambda_1 V(x) + \lambda_2 F(x).$$

Here the expressions $D(x)$ and $I(x)$, respectively, are called the differential and integral parts of Eq.(3.1). To obtain the solution of the given problem in the form of expression (1.3) we first differentiate eqution (3.1) n times with respect to x to obtain

$$D^{(n)}(x) = f^{(n)}(x) + \lambda_1 V^{(n)}(x) + \lambda_2 F^{(n)}(x) \qquad \text{or} \qquad D^{(n)}(x) = I^{(n)}(x) \tag{3.2}$$

where $n = 0, 1, 2, ..., N$ and then analyse the expressions $D^{(n)}(x)$ and $I^{(n)}(x)$.

3.1.1 Matrix representation for the differential part

The expression $D^{(n)}(x)$ can be more clearly written as

$$\begin{aligned} D^{(n)}(x) &= [P_0(x)\, y(x)]^{(n)} + \left[P_1(x)\, y^{(1)}(x)\right]^{(n)} + .. + \left[P_m(x)\, y^{(m)}(x)\right]^{(n)} \\ n &= 0, 1, 2, ..., N, \ N \geq m \end{aligned} \tag{3.3}$$

Using the Leibnitz's rule (deal with differentiation of products of functions), simplifying and then substituting $x = c$ into the resulting relation, we have

$$\begin{aligned} D^{(n)}(c) &= \sum_{k=0}^{n} \binom{n}{k} \left\{ P_0^{(n-k)}(c)\, y^{(k)}(c) + P_1^{(n-k)}(c)\, y^{(k+1)}(c) + ... + P_m^{(n-k)}(c)\, y^{(k+m)}(c) \right\} \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} \binom{n}{k} P_j^{(n-k)}(c)\, y^{(k+j)}(c), \qquad (n = 0, 1, 2, ..., N). \end{aligned} \tag{3.4}$$

Here the $N+1$ unknown coefficients $y^{(0)}(c), y^{(1)}(c), ..., y^{(N)}(c)$ are Taylor coefficients to be determined and $P_0^{(i)}(c), P_1^{(i)}(c), ..., P_m^{(i)}(c)$ $(i = 0, 1, 2, ...)$, respectively, denote the values of the ith derivatives of the known functions $P_0, P_1, ..., P_m$ at $x = c$.

We now write the matrix form of expression (3.4) as

$$\mathbf{D} = \mathbf{W}\mathbf{Y}, \tag{3.5}$$

where

$$\mathbf{Y} = \begin{bmatrix} y^{(0)}(c) & y^{(1)}(c) & y^{(2)}(c) & . & . & . & y^{(N)}(c) \end{bmatrix}^{\mathrm{T}}$$

and

$$\mathbf{W} = [w_{nk}], \quad n, k = 0, 1, 2, ..., N$$

the elements of which are defined by

$$\begin{aligned} w_{nk} &= \begin{pmatrix} n \\ k-m \end{pmatrix} P_m^{(n-k+m)}(c) + \begin{pmatrix} n \\ k-m+1 \end{pmatrix} P_{m-1}^{(n-k+m-1)}(c) \\ &\quad + ... + \begin{pmatrix} n \\ k-1 \end{pmatrix} P_1^{(n-k+1)}(c) + \begin{pmatrix} n \\ k \end{pmatrix} P_0^{(n-k)}(c) \\ &= \sum_{q=0}^{m} \begin{pmatrix} n \\ k-m-q \end{pmatrix} P_{m-q}^{(n-k+m-q)}(c). \end{aligned} \tag{3.6}$$

Note in Eq.(3.6) that for $l < 0$

$$P_0^{(l)}(c) = P_1^{(l)}(c) = ... = P_m^{(l)}(c) = 0$$

and for $j < 0$ and $j > i$, $\begin{pmatrix} i \\ j \end{pmatrix} = 0$, where i, j and l are integers. In this case, in Eq.(3.6), for $n < k - m$ $(n = 0, 1, 2, ..., N - m - 1;\ k = m + 1,\ m + 2, ..., N)$,

$$w_{nk} = 0.$$

Hence, the matrix $\mathbf{W}$ becomes, clearly,

$$\mathbf{W} = \begin{bmatrix} w_{00} & \cdots & w_{0m} & 0 & \cdots & 0 \\ w_{10} & \cdots & w_{1m} & w_{1,m+1} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ w_{N-m-1,0} & \cdots & w_{N-m-1,m} & w_{N-m-1,m+1} & \cdots & 0 \\ w_{N-m,0} & \cdots & w_{N-m,m} & w_{N-m,m+1} & \cdots & w_{N0} \\ \vdots & & \vdots & \vdots & & \vdots \\ w_{N0} & \cdots & w_{Nm} & w_{N,m+1} & \cdots & w_{NN} \end{bmatrix} \tag{3.7}$$

3.1.2 Matrix representation for the integral part

The expression $I^{(n)}(x)$ can be more clearly written as

$$I^{(n)}(x) = f^{(n)}(x) + \lambda_1 V^{(n)}(x) + \lambda_2 F^{(n)}(x) \tag{3.8}$$

where

$$V^{(n)}(x) = \frac{d^n}{dx^n} \int_a^x \sum_{i=0}^{p} A_i(x,t)\, y^{(i)}(t)\, dt \tag{3.9}$$

For $n = 0$

$$V^{(0)}(x) = V(x) = \int_a^x \sum_{i=0}^{p} A_i(x,t)\, y^{(i)}(t)\, dt$$

By applying the Leibnitz's rule for differentiation of integral to the integral $V(x)$, we have

$$V^{(1)}(x) = \int_a^x \sum_{i=0}^{p} \frac{\partial A_i(x,t)}{\partial x} y^{(i)}(t)\, dt + \left[\sum_{i=0}^{p} A_i(x,t)\, y^{(i)}(t) \right]_{t=x}$$

$$V^{(2)}(x) = \int_a^x \sum_{i=0}^{p} \frac{\partial^2 A_i(x,t)}{\partial x^2} y^{(i)}(t)\,dt + \left[\sum_{i=0}^{p} \frac{\partial A_i(x,t)}{\partial x} y^{(i)}(t)\right]_{t=x} + \left[\sum_{i=0}^{p} A_i(x,t)\, y^{(i)}(t)\right]_{t=x}^{(1)}$$

$$V^{(3)}(x) = \int_a^x \sum_{i=0}^{p} \frac{\partial^3 A_i(x,t)}{\partial x^3} y^{(i)}(t)\,dt + \left[\sum_{i=0}^{p} \frac{\partial^2 A_i(x,t)}{\partial x^2} y^{(i)}(t)\right]_{t=x} + \left[\sum_{i=0}^{p} \frac{\partial A_i(x,t)}{\partial x} y^{(i)}(t)\right]_{t=x}^{(1)} + \left[\sum_{i=0}^{p} A_i(x,t)\, y^{(i)}(t)\right]_{t=x}^{(2)}$$

.

.

.

$$V^{(n)}(x) = \int_a^x \sum_{i=0}^{p} \frac{\partial^n A_i(x,t)}{\partial x^n} y^{(i)}(t)\,dt + \left[\sum_{i=0}^{p} \frac{\partial^{n-1} A_i(x,t)}{\partial x^{n-1}} y^{(i)}(t)\right]_{t=x} + \left[\sum_{i=0}^{p} \frac{\partial^{n-2} A_i(x,t)}{\partial x^{n-2}} y^{(i)}(t)\right]_{t=x}^{(1)} + \ldots + \left[\sum_{i=0}^{p} \frac{\partial A_i(x,t)}{\partial x} y^{(i)}(t)\right]_{t=x}^{(n-2)} + \left[\sum_{i=0}^{p} A_i(x,t)\, y^{(i)}(t)\right]_{t=x}^{(n-1)}.$$

Thus

$$V^{(n)}(x) = \int_a^x \sum_{i=0}^{p} \frac{\partial^n A_i(x,t)}{\partial x^n} y^{(i)}(t)\,dt + \sum_{r=0}^{n-1} \left[\sum_{i=0}^{p} \frac{\partial^r A_i(x,t)}{\partial x^r} y^{(i)}(t)\right]_{t=x}^{(n-r-1)}. \tag{3.10}$$

From the Leibnitz's rule (deal with differentiation of products of functions) we can evaluate

$$\left[\sum_{i=0}^{p} \frac{\partial^r A_i(x,t)}{\partial x^r} y^{(i)}(t)\right]_{t=x}^{(n-r-1)} = \sum_{k=0}^{n-r-1} \begin{pmatrix} n-r-1 \\ k \end{pmatrix} \left(\sum_{i=0}^{p} \left[\frac{\partial^r A_i(x,t)}{\partial x^r}\right]_{t=x}^{(n-r-k-1)} \left[y^{(i)}(x)\right]^{(k)}\right)$$

$$= \sum_{k=0}^{n-r-1} \begin{pmatrix} n-r-1 \\ k \end{pmatrix} \left(\sum_{i=0}^{p} \left[\frac{\partial^r A_i(x,t)}{\partial x^r}\right]_{t=x}^{(n-r-k-1)} y^{(k+i)}(x)\right)$$

substitute in Eq.(3.10). Thus Eq.(3.9) becomes

$$V^{(n)}(x) = \int_a^x \sum_{i=0}^{p} \frac{\partial^n A_i(x,t)}{\partial x^n} y^{(i)}(t)\,dt + \sum_{k=0}^{n-1}\sum_{r=0}^{n-k-1} \begin{pmatrix} n-r-1 \\ k \end{pmatrix} \left(\sum_{i=0}^{p} \left[\frac{\partial^r A_i(x,t)}{\partial x^r} \right]_{t=x}^{(n-r-k-1)} y^{(k+i)}(x) \right). \tag{3.11}$$

Note that in Eq.(3.11)

$$\sum_{k=0}^{n-1}\sum_{r=0}^{n-k-1} (...) = \sum_{r=0}^{n-1}\sum_{k=0}^{n-r-1} (...).$$

For $F^{(n)}(x)$ in Eq.(3.8), we have

$$F^{(n)}(x) = \int_a^b \sum_{j=0}^{q} \frac{\partial^n B_j(x,t)}{\partial x^n} y^{(j)}(t)\,dt. \tag{3.12}$$

First we put $x = c$ in relation (3.8), thereby in expressions (3.11) and (3.12), and then substitute the Taylor expansion of $y^{(i)}(t)$ and $y^{(j)}(t)$ at $t = c$, i.e.

$$y(t) = \sum_{k=0}^{\infty} \frac{1}{k!} y^{(k)}(c)(t-c)^k \tag{3.13}$$

$$y^{(i)}(t) = \sum_{k=0}^{\infty} \frac{1}{(k-i)!} y^{(k)}(c)(t-c)^{k-i} \tag{3.14}$$

$$y^{(j)}(t) = \sum_{k=0}^{\infty} \frac{1}{(k-j)!} y^{(k)}(c)(t-c)^{k-j} \tag{3.15}$$

in the resulting relation. Thus, expressions (3.11) and (3.12), respectively, become

$$V^{(n)}(c) = \int_a^c \left\{ \sum_{i=0}^{p} \left[\frac{\partial^n A_i(x,t)}{\partial x^n} \right]_{x=c} \left(\sum_{k=0}^{\infty} \frac{1}{(k-i)!} y^{(k)}(c)(t-c)^{k-i} \right) \right\} dt \tag{3.16}$$
$$+ \sum_{k=0}^{n-1}\sum_{r=0}^{n-k-1} \begin{pmatrix} n-r-1 \\ k \end{pmatrix} \left(\sum_{i=0}^{p} \left[\frac{\partial^r A_i(c,t)}{\partial x^r} \right]_{t=x}^{(n-r-k-1)} y^{(k+i)}(c) \right)$$

and

$$F^{(n)}(c) = \int_a^b \left\{ \sum_{j=0}^{q} \left[\frac{\partial^n B_j(x,t)}{\partial x^n} \right]_{x=c} \left(\sum_{k=0}^{\infty} \frac{1}{(k-j)!} y^{(k)}(c)(t-c)^{k-j} \right) \right\} dt. \tag{3.17}$$

When the obtained equations are substituted in Eq.(3.8) we get

$$\begin{aligned} I^{(n)}(c) &= f^{(n)}(c) + \lambda_1 \sum_{k=0}^{n-1} \sum_{r=0}^{n-k-1} \begin{pmatrix} n-r-1 \\ k \end{pmatrix} \left(\sum_{i=0}^{p} \left[\frac{\partial^r A_i(c,t)}{\partial x^r} \right]_{t=x}^{(n-r-k-1)} y^{(k+i)}(c) \right) \\ &\quad + \lambda_1 \int_a^c \left\{ \sum_{i=0}^{p} \left[\frac{\partial^n A_i(x,t)}{\partial x^n} \right]_{x=c} \left(\sum_{k=0}^{\infty} \frac{1}{(k-i)!} y^{(k)}(c)(t-c)^{k-i} \right) \right\} dt \\ &\quad + \lambda_2 \int_a^b \left\{ \sum_{j=0}^{q} \left[\frac{\partial^n B_j(x,t)}{\partial x^n} \right]_{x=c} \left(\sum_{k=0}^{\infty} \frac{1}{(k-j)!} y^{(k)}(c)(t-c)^{k-j} \right) \right\} dt \end{aligned}$$

or briefly

$$I^{(n)}(c) = f^{(n)}(c) + \lambda_1 \left(\sum_{k=0}^{n-1} H_{nki} y^{(k+i)}(c) + \sum_{k=0}^{\infty} T_{nk} y^{(k)}(c) \right) + \lambda_2 \sum_{k=0}^{\infty} K_{nk} y^{(k)}(c) \tag{3.18}$$

where for $n = 0$

$$\sum_{k=0}^{n-1} H_{nki} y^{(k+i)}(c) = 0$$

for $n = 1, 2, 3, ...;\quad k = 0, 1, 2, ... n-1 \quad (n > k)$

$$H_{nki} = \sum_{r=0}^{n-k-1} \left\{ \begin{pmatrix} n-r-1 \\ k \end{pmatrix} \sum_{i=0}^{p} \left[\frac{\partial^r A_i(c,t)}{\partial x^r} \right]_{t=x}^{(n-r-k-1)} \right\} \tag{3.19}$$

for $n \leq k$

$$H_{nki} = 0$$

and for $n, k = 0, 1, 2, ...$

$$T_{nk} = \int_a^c \left\{ \sum_{i=0}^{p} \frac{1}{(k-i)!} \left[\frac{\partial^n A_i(x,t)}{\partial x^n} \right]_{x=c} (t-c)^{k-i} \right\} dt \tag{3.20}$$

$$K_{nk} = \int_a^b \left\{ \sum_{j=0}^{q} \frac{1}{(k-j)!} \left[\frac{\partial^n B_j(x,t)}{\partial x^n} \right]_{x=c} (t-c)^{k-j} \right\} dt. \tag{3.21}$$

The relation (3.18) gives us infinite linear equations. If we take $n = k = 0, 1, 2, ...N$ then relation (3.18) reduces to a system of $N+1$ linear equations for the $N+3$ unknown coefficients $y^{(0)}(c), y^{(1)}(c), ..., y^{(N)}(c), y^{(N+1)}(c), y^{(N+2)}(c)$. But in this study we want to find $N+1$ coefficients $y^{(0)}(c), y^{(1)}(c), ..., y^{(N)}(c)$. The system can be put in matrix form as

$$\mathbf{I} = \mathbf{F} + \lambda_1 \mathbf{T}\mathbf{Y} + \lambda_2 \mathbf{K}\mathbf{Y} \tag{3.22}$$

where the matrices $\mathbf{Y}$, $\mathbf{F}$, $\mathbf{K}$ and $\mathbf{T}$ are defined by

$$\mathbf{Y} = \begin{bmatrix} y^{(0)}(c) \\ y^{(1)}(c) \\ y^{(2)}(c) \\ . \\ . \\ . \\ y^{(N)}(c) \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} f^{(0)}(c) \\ f^{(1)}(c) \\ f^{(2)}(c) \\ . \\ . \\ . \\ f^{(N)}(c) \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} K_{00} & K_{01} & K_{02} & . & . & . & K_{0N} \\ K_{10} & K_{11} & K_{12} & . & . & . & K_{1N} \\ K_{20} & K_{21} & K_{22} & . & . & . & K_{2N} \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ K_{N0} & K_{N1} & K_{N2} & . & . & . & K_{NN} \end{bmatrix}$$

$$
\mathbf{T} = \begin{bmatrix}
T_{00} & T_{01} & \cdots & T_{0N} \\
(H_{100}+T_{10}) & (H_{101}+T_{11}) & \cdots & T_{1N} \\
(H_{200}+T_{20}) & (H_{210}+H_{201}+T_{21}) & \cdots & T_{2N} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
(H_{N00}+T_{N0}) & (H_{N10}+H_{N01}+T_{N1}) & \cdots & (H_{N,N-1,1}+\ldots+H_{N,N-p,p}+T_{NN})
\end{bmatrix}
$$

3.1.3 Fundamental matrix equation

Substituting the matrix forms (3.5) and (3.22) in expression of Eq.(3.1) at the point $x = c$ we get the matrix form of Eq.(3.1) as

$$
\mathbf{WY} = \mathbf{F} + \lambda_1 \mathbf{TY} + \lambda_2 \mathbf{KY} \tag{3.23}
$$

or

$$
(\mathbf{W} - \lambda_1 \mathbf{T} - \lambda_2 \mathbf{K})\,\mathbf{Y} = \mathbf{F}
$$

which is a fundamental equation for the integro-differential equation (1.1).

If we take $\mathbf{M} = \mathbf{W} - \lambda_1 \mathbf{T} - \lambda_2 \mathbf{K}$, then we have

$$
\mathbf{MY} = \mathbf{F} \tag{3.24}
$$

where

$$
\mathbf{M} = \begin{bmatrix}
w_{00} - \lambda_1 T_{00} - \lambda_2 K_{00} & . \; . \; . & -\lambda_1 T_{0N} - \lambda_2 K_{0N} \\
w_{10} - \lambda_1 \left(H_{100} + T_{10}\right) - \lambda_2 K_{10} & . \; . \; . & -\lambda_1 T_{1N} - \lambda_2 K_{1N} \\
w_{20} - \lambda_1 \left(H_{200} + T_{20}\right) - \lambda_2 K_{20} & . \; . \; . & -\lambda_1 T_{2N} - \lambda_2 K_{2N} \\
. & & . \\
. & & . \\
. & & . \\
w_{N0} - \lambda_1 \left(H_{N00} + T_{N0}\right) - \lambda_2 K_{N0} & . \; . \; . & w_{NN} - \lambda_1 \left(H_{N,N-1,1} + \dots + T_{NN}\right) - \lambda_2 K_{NN}
\end{bmatrix}
$$

where $\mathbf{Y}$ and $\mathbf{F}$ are defined in Eq.(3.22) .

The augmented matrix of the Eq.(3.24) becomes

$$\tilde{\mathbf{M}} = [\mathbf{M}; \mathbf{F}] ,$$

where $\mathbf{M}$ and $\mathbf{F}$ are defined in Eq.(3.24).

Then $\tilde{\mathbf{M}}$ can be written as

$$
\tilde{\mathbf{M}} = \begin{bmatrix}
w_{00} - \lambda_1 T_{00} - \lambda_2 K_{00} & \cdot \; \cdot \; \cdot & -\lambda_1 T_{0N} - \lambda_2 K_{0N} & : & f^{(0)}(c) \\
w_{10} - \lambda_1 \left(H_{100} + T_{10}\right) - \lambda_2 K_{10} & \cdot \; \cdot \; \cdot & -\lambda_1 T_{1N} - \lambda_2 K_{1N} & : & f^{(1)}(c) \\
w_{20} - \lambda_1 \left(H_{200} + T_{20}\right) - \lambda_2 K_{20} & \cdot \; \cdot \; \cdot & -\lambda_1 T_{2N} - \lambda_2 K_{2N} & : & f^{(2)}(c) \\
\cdot & & \cdot & : & \cdot \\
\cdot & & \cdot & : & \cdot \\
\cdot & & \cdot & : & \cdot \\
w_{N0} - \lambda_1 \left(H_{N00} + T_{N0}\right) - \lambda_2 K_{N0} & \cdot \; \cdot \; \cdot & w_{NN} - \lambda_1 \left(H_{N,N-2,2} + H_{N,N-1,1} + \dots + H_{N,N-p,p} + T_{NN}\right) - \lambda_2 K_{NN} & : & f^{(N)}(c)
\end{bmatrix}
\tag{3.25}
$$

3.1.4 Relations for conditions

We can obtain the corresponding matrix forms for the conditions (1.2) as the following. The expression (1.3) and its derivatives are equivalent to the matrix equations

$$
\begin{aligned}
y^{(0)}(x) &= \begin{bmatrix} \frac{1}{0!} & \frac{(x-c)}{1!} & \frac{(x-c)^2}{2!} & \frac{(x-c)^3}{3!} & \cdot & \cdot & \cdot & \frac{(x-c)^N}{N!} \end{bmatrix} \mathbf{Y}, \\
y^{(1)}(x) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{(x-c)}{1!} & \frac{(x-c)^2}{2!} & \cdot & \cdot & \cdot & \frac{(x-c)^{N-1}}{(N-1)!} \end{bmatrix} \mathbf{Y}, \\
& \;\;\vdots \\
y^{(m-1)}(x) &= \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & \frac{1}{0!} & \frac{(x-c)}{1!} & \cdot & \cdot & \cdot & \frac{(x-c)^{N-m+1}}{(N-m+1)!} \end{bmatrix} \mathbf{Y},
\end{aligned}
$$

where $\mathbf{Y}$ is defined in Eq.(3.24). By using these equations, the quantities $y^{(j)}(a)$, $y^{(j)}(b)$ and $y^{(j)}(c)$, $j = 0, 1, 2, ..., m-1$, can be written as

$$
\begin{aligned}
y^{(0)}(c) &= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \mathbf{Y}, \\
y^{(0)}(a) &= \begin{bmatrix} 1 & \frac{h}{1!} & \frac{h^2}{2!} & \frac{h^3}{3!} & \cdot & \cdot & \cdot & \frac{h^N}{N!} \end{bmatrix} \mathbf{Y}, \\
y^{(0)}(b) &= \begin{bmatrix} 1 & \frac{k}{1!} & \frac{k^2}{2!} & \frac{k^3}{3!} & \cdot & \cdot & \cdot & \frac{k^N}{N!} \end{bmatrix} \mathbf{Y}, \\
y^{(1)}(c) &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \mathbf{Y}, \\
y^{(1)}(a) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{h}{1!} & \frac{h^2}{2!} & \cdot & \cdot & \cdot & \frac{h^{N-1}}{(N-1)!} \end{bmatrix} \mathbf{Y}, \\
y^{(1)}(b) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{k}{1!} & \frac{k^2}{2!} & \cdot & \cdot & \cdot & \frac{k^{N-1}}{(N-1)!} \end{bmatrix} \mathbf{Y}, \\
& \;\;\vdots \\
y^{(m-1)}(c) &= \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \mathbf{Y}, \\
y^{(m-1)}(a) &= \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & \frac{1}{0!} & \frac{h}{1!} & \frac{h^2}{2!} & \cdot & \cdot & \cdot & \frac{h^{N-m+1}}{(N-m+1)!} \end{bmatrix} \mathbf{Y}, \\
y^{(m-1)}(b) &= \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & \frac{1}{0!} & \frac{k}{1!} & \frac{k^2}{2!} & \cdot & \cdot & \cdot & \frac{k^{N-m+1}}{(N-m+1)!} \end{bmatrix} \mathbf{Y},
\end{aligned} \tag{3.26}
$$

where $h = a - c$ and $k = b - c$.

Substituting quantities (3.26) into Eq. (1.2) and then simplifying, we obtain the matrix forms corresponding to the first, second, third, ... and mth conditions defined in Eq.(1.2), respectively, as

$$\mathbf{U}_0\mathbf{Y} = [\mu_0], \quad \mathbf{U}_1\mathbf{Y} = [\mu_1], \ldots, \quad \mathbf{U}_{m-1}\mathbf{Y} = [\mu_{m-1}],$$

or the augmented matrices, more clearly,

$$\begin{aligned}
\widetilde{\mathbf{U}}_0 &= \left[\begin{array}{cccccccc} u_{00} & u_{01} & u_{02} & . & . & . & u_{0N} & \vdots & \mu_0 \end{array}\right], \\
\widetilde{\mathbf{U}}_1 &= \left[\begin{array}{cccccccc} u_{10} & u_{11} & u_{12} & . & . & . & u_{1N} & \vdots & \mu_1 \end{array}\right], \\
\widetilde{\mathbf{U}}_2 &= \left[\begin{array}{cccccccc} u_{20} & u_{21} & u_{22} & . & . & . & u_{2N} & \vdots & \mu_2 \end{array}\right], \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
\widetilde{\mathbf{U}}_{m-1} &= \left[\begin{array}{cccccccc} u_{m-1,0} & u_{m-1,1} & u_{m-1,2} & . & . & . & u_{m-1,N} & ; & \mu_{m-1} \end{array}\right],
\end{aligned} \tag{3.27}$$

where the constants u_{ij} are constants related to the cofficients a_{ij}, b_{ij}, c_{ij} in Eq.(1.2), and h and k in Eq.(3.26).

Note that the same method can be used when the conditions (1.2) are given at the points $x = a_r$ $(r = 0, 1, 2, ..., p)$, $(a = a_0 \leq a_r \leq b = a_p,\ a \leq c \leq b)$, that is,

$$\sum_{j=0}^{m-1} \left[\sum_{r=0}^{p} a_{ij}^{r} y^{(j)}(a_r) + c_{ij} y^{(j)}(c) \right] = \lambda_i, \quad (i = 0, 1, 2, ..., m-1).$$

3.1.5 Method of solution

We now consider the fundamental matrix equation (3.24), the augmented matrix (3.25) and the row matrices (3.27). By replacing the m rows matrices (3.27) by the last m rows of augmented matrix (3.25), we have the new augmented matrix

$$\tilde{\mathbf{M}}^{*} = \begin{bmatrix} w_{00}-\lambda_1 T_{00}-\lambda_2 K_{00} & \cdots & -\lambda_1 T_{0,N}-\lambda_2 K_{0,N} \\ w_{10}-\lambda_1\left(H_{100}+T_{10}\right)-\lambda_2 K_{10} & \cdots & -\lambda_1 T_{1,N}-\lambda_2 K_{1,N} \\ w_{20}-\lambda_1\left(H_{200}+T_{20}\right)-\lambda_2 K_{20} & \cdots & -\lambda_1 T_{2,N}-\lambda_2 K_{2,N} \\ \vdots & & \vdots \\ w_{N-m,0}-\lambda_1\left(H_{N-m,00}+T_{N-m,0}\right)-\lambda_2 K_{N-m,0} & \cdots & w_{N-m,N}-\lambda_1\left(H_{N-m,N-1,1}+H_{N-m,N-2,2}+\ldots+H_{N-m,N-p,p}+T_{N-m,N}\right)-\lambda_2 K_{N-m,N} \\ u_{00} & \cdots & u_{0,N} \\ u_{10} & \cdots & u_{1,N} \\ \vdots & & \vdots \\ u_{m-1,0} & \cdots & u_{m-1,N} \end{bmatrix} \tag{3.28}$$

or corresponding matrix equation

$$\mathbf{M}^{*}\mathbf{Y} = \mathbf{F}^{*} \tag{3.29}$$

so that

$$\mathbf{M}^{*} = \left[\begin{array}{ccc|c} w_{00}-\lambda_1 T_{00}-\lambda_2 K_{00} & \cdots & -\lambda_1 T_{0N}-\lambda_2 K_{0N} & f^{(0)}(c) \\ w_{10}-\lambda_1\left(H_{100}+T_{10}\right)-\lambda_2 K_{10} & \cdots & -\lambda_1 T_{1N}-\lambda_2 K_{1N} & f^{(1)}(c) \\ w_{20}-\lambda_1\left(H_{200}+T_{20}\right)-\lambda_2 K_{20} & \cdots & -\lambda_1 T_{2N}-\lambda_2 K_{2N} & f^{(2)}(c) \\ \vdots & & \vdots & \vdots \\ w_{N-m,0}-\lambda_1\left(H_{N-m,00}+T_{N-m,0}\right)-\lambda_2 K_{N-m,0} & \cdots & w_{N-m,N}-\lambda_1\left(H_{N-m,N-1,1}+H_{N-m,N-2,2}+\ldots+H_{N-m,N-p,p}+T_{N-m,N}\right)-\lambda_2 K_{N-m,N} & f^{(N-m)}(c) \\ u_{00} & \cdots & u_{0N} & \mu_0 \\ u_{10} & \cdots & u_{1N} & \mu_1 \\ \vdots & & \vdots & \vdots \\ u_{m-1,0} & \cdots & u_{m-1,N} & \mu_{m-1} \end{array}\right] \tag{3.30}$$

$\mathbf{F}^* = \begin{bmatrix} f^{(0)}(c) & f^{(1)}(c) & . & . & . & f^{(N-m)}(c) & \mu_0 & \mu_1 & . & . & . & \mu_{m-1} \end{bmatrix}^{\mathrm{T}}$,

$\mathbf{Y} = \begin{bmatrix} y^{(0)}(c) & y^{(1)}(c) & . & . & . & y^{(N)}(c) \end{bmatrix}^{\mathrm{T}}$.

If rank $\tilde{\mathbf{M}}^* =$ rank $\mathbf{M}^* = N + 1$, then we can write

$$\mathbf{Y} = (\mathbf{M}^*)^{-1}\mathbf{F}^*. \tag{3.31}$$

Thus, the coefficients $y^{(n)}(c)$, $n = 0, 1, 2, ..., N$ are uniquely determined by Eq.(3.31). Thereby, the Volterra-Fredholm integro-differential equation (1.4) and (3.1) with condition (1.2) has only one unique solution.

This solution is given by the Taylor polynomial

$$y(x) \cong \sum_{n=0}^{N} \frac{1}{n!} y^{(n)}(c)(x-c)^n. \tag{3.32}$$

3.2 Accuracy of solution

We can easily check the accuracy of the solution obtained in form (3.32) as follows. Since the truncated Taylor series (3.32) or the corresponding polynomial expansion is an approximate solution of Eq. (1.4) and (3.1), when the solution $y(x)$ and its derivatives are substituted in Eq. (1.4) and (3.1), the resulting equation must be satisfied approximately; that is, for $x = x_r \in [a, b]$, $r = 0, 1, 2, 3, ...$

$$E(x_r) = \left| \sum_{n=0}^{m} P_n(x_r) - f(x_r) - \lambda_1 V(x_r) - \lambda_2 F(x_r) \right| \cong 0, \tag{3.33}$$

where

$$V(x_r) = \int_a^{x_r} \sum_{i=0}^{p} A_i(x_r, t) y^{(i)}(t) dt, \qquad F(x_r) = \int_a^{x_r} \sum_{j=0}^{q} B_j(x_r, t) y^{(j)}(t) dt$$

or

$E(x_r) \leq 10^{-k_r}$ (k_r is any positive integer).

If max $\left|10^{-k_r}\right| = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $|E(x_r)|$ at each of the points becomes smaller than the prescribed 10^{-k}.

3.3 Examples

Example 1 Let us consider the linear Volterra integro-differential equation

$$y'(x) = 1 - \int_0^x \left[y(t) + y''(t) \right] dt\ , \qquad 0 \le x, t \le 1$$

with condition $y(0) = 0$ and approximate the solution $y(x)$ by Taylor polynomial

$$y(x) = \sum_{n=0}^{5} \frac{1}{n!} y^{(n)}(0) x^n$$

where $N = 5$, $a = 0$. $b = 1$, $c = 0$.

Since $P_0(x) = 0$ and $P_1(x) = 1$, we have

$$\begin{aligned} P_0^{(0)}(0) &= 0,\ P_0^{(1)}(0) = P_0^{(2)}(0) = ... = 0 \\ P_1^{(0)}(0) &= 1,\ P_1^{(1)}(0) = P_1^{(2)}(0) = ... = 0. \end{aligned}$$

and obtain the matrix representation for the differential part as

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The coefficients H_{nki}, T_{nk} and K_{nk} (n, k = 0, 1, 2, 3, 4, 5) are determined by using (3.19) – (3.21). The values of derivatives of the function $f(x) = 1$ at $x = 0$ are

$$f^{(0)}(0) = 1,\ f^{(1)}(0) = f^{(2)}(0) = ... = f^{(5)}(0) = 0.$$

By means of these values, the matrices **T**, **K** and **F** are

$$\mathbf{F}=\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix},\qquad \mathbf{T}=\begin{bmatrix}0&0&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\0&0&0&0&1&0\end{bmatrix},\qquad \mathbf{K}=\begin{bmatrix}\bar{0}\end{bmatrix}.$$

Using the matrices **W**, **T**, **K** and **F**, we find the augmented matrix $\tilde{\mathrm{M}}$ as

$$\tilde{\mathrm{M}}=\left[\begin{array}{cccccc:c}0&1&0&0&0&0&1\\1&0&2&0&0&0&0\\0&1&0&2&0&0&0\\0&0&1&0&2&0&0\\0&0&0&1&0&2&0\\0&0&0&0&1&0&0\end{array}\right].$$

By Eq.(3.26) and (3.27), the augmented matrix corresponding to the condition $y(0)=0$ is obtained as

$$\tilde{\mathrm{U}}_0=\left[\begin{array}{cccccc:c}1&0&0&0&0&0&0\end{array}\right].$$

By replacing the last row of $\tilde{\mathrm{M}}$ by $\tilde{\mathrm{U}}_0$, we have the required augmented matrix

$$\tilde{\mathrm{M}}^*=\left[\begin{array}{cccccc:c}0&1&0&0&0&0&1\\1&0&2&0&0&0&0\\0&1&0&2&0&0&0\\0&0&1&0&2&0&0\\0&0&0&1&0&2&0\\1&0&0&0&0&0&0\end{array}\right].$$

From this system, we obtain the coefficients $y^{(n)}(0)$ $(n = 0, 1, 2, 3, 4, 5)$ as

$$\begin{bmatrix} y^{(0)}(0) \\ y^{(1)}(0) \\ y^{(2)}(0) \\ y^{(3)}(0) \\ y^{(4)}(0) \\ y^{(5)}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -0.5 \\ 0 \\ 0.25 \end{bmatrix}.$$

Thus, by substituting these coefficients in Taylor polynomial, we get the solution of this problem as

$$y(x) = x - \frac{0.5}{3!}x^3 + \frac{0.25}{5!}x^5.$$

The Taylor coefficients $(y^{(0)}(c), y^{(1)}(c), y^{(2)}(c), ..., y^{(N)}(c))$ for $c = 0$ and $N = 6, 8, 10, 12, 15, 18$ are shown in Table 1. The comparison of approximate solutions with exact solution is presented in Table 2.

Table 1 The Taylor coeffients

Taylor coefficients	N					
	6	8	10	12	15	18
$y^{(0)}(0)$	0	0	0	0	0	0
$y^{(1)}(0)$	1	1	1	1	1	1
$y^{(2)}(0)$	0	0	0	0	0	0
$y^{(3)}(0)$	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
$y^{(4)}(0)$	0	0	0	0	0	0
$y^{(5)}(0)$	0.25	0.25	0.25	0.25	0.25	0.25
$y^{(6)}(0)$	0	0	0	0	0	0
$y^{(7)}(0)$		-0.125	-0.125	-0.125	-0.125	-0.125
$y^{(8)}(0)$		0	0	0	0	0
$y^{(9)}(0)$			0.0625	0.0625	0.0625	0.0625
$y^{(10)}(0)$			0	0	0	0
$y^{(11)}(0)$				-0.03125	-0.03125	-0.03125
$y^{(12)}(0)$				0	0	0
$y^{(13)}(0)$					0.015625	0.015625
$y^{(14)}(0)$					0	0
$y^{(15)}(0)$					-0.0078125	-0.0078125
$y^{(16)}(0)$						0
$y^{(17)}(0)$						0.00390625
$y^{(18)}(0)$						0

Table 2 Numerical solution of Example 1 for $c = 0$

x	Exact solution $y = \sqrt{2}\sin\frac{1}{\sqrt{2}}x$	Approximate solution $N = 6$	$N = 8$	$N = 10$
0	0	0	0	0
0.1	0.099916687	0.099916688	0.099916687	0.099916687
0.2	0.199334000	0.199334000	0.199334000	0.199334000
0.3	0.297755057	0.297755063	0.297755057	0.297755057
0.4	0.394687959	0.394688000	0.394687959	0.394687959
0.5	0.489648244	0.489648438	0.489648244	0.489648244
0.6	0.582161307	0.582162000	0.582161306	0.582161307
0.7	0.671764777	0.671766813	0.671764770	0.671764777
0.8	0.758010822	0.758016000	0.758010799	0.758010822
0.9	0.840468391	0.840480188	0.840468325	0.840468392
1.0	0.918725370	0.918750000	0.918725198	0.918725371

Table 2 (continued)

x	Exact solution $y = \sqrt{2}\sin\frac{1}{\sqrt{2}}x$	Approximate solution		
		$N = 12$	$N = 15$	$N = 18$
0	0	0	0	0
0.1	0.099916687	0.099916687	0.099916687	0.099916687
0.2	0.199334000	0.199334000	0.199334000	0.199334000
0.3	0.297755057	0.297755057	0.297755057	0.297755057
0.4	0.394687959	0.394687959	0.394687959	0.394687959
0.5	0.489648244	0.489648244	0.489648244	0.489648244
0.6	0.582161307	0.582161307	0.582161307	0.582161307
0.7	0.671764777	0.671764777	0.671764777	0.671764777
0.8	0.758010822	0.758010822	0.758010822	0.758010822
0.9	0.840468391	0.840468391	0.840468391	0.840468391
1.0	0.918725370	0.918725370	0.918725370	0.918725370

Example 2 Let us consider the linear Volterra integro-differential equation

$$y'(x) = 1 - \int_0^x \left[y(t) + xy'(t)\right] dt \; . \qquad 0 \le x, t \le 1$$

with condition $y(0) = 0$.

Here $P_0(x) = 0$, $P_1(x) = 1$, $f(x) = 1$. $\lambda_1 = -1$, $\lambda_2 = 0$, $A_0(x,t) = 1$ and $A_1(x,t) = x$.

To find a Taylor polynomial solution of this problem, we take $c = 0$, $N = 5$. We obtain the matrix representation for the differential part as

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

and matrices **T**, **K** and **F** as

$$\mathbf{F} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} \bar{0} \end{bmatrix} .$$

From the matrices **W**, **T**, **K** and **F**, we find the augmented matrix $\tilde{\mathbf{M}}$ as

$$\tilde{\mathrm{M}}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & : & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & : & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & : & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & 5 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & : & 0 \end{bmatrix}.$$

Consider the condition, we have the augmented matrix corresponding the condition $y(0) = 0$, when $c = 0$, $a = 0$, $b = 1$, as

$$\tilde{\mathrm{U}}_0=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}.$$

Thus by the previous procedures, we obtain the desired augmented matrix

$$\tilde{\mathrm{M}}^{*}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & : & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & : & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 & : & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & 5 & 0 & 1 & : & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

and its solution

$$\mathbf{Y}=\begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 15 \end{bmatrix}^{\mathrm{T}}.$$

Substituting these elements into Taylor polynomial, we obtain the solution

$$y(x) = x - \frac{3}{3!}x^3 + \frac{15}{5!}x^5.$$

The Taylor coefficients $(y^{(0)}(c), y^{(1)}(c), y^{(2)}(c), ..., y^{(N)}(c))$ for $c = 0$ and $N = 6, 8, 10, 12, 15$ 18 are shown in Table 3. The Taylor solution and the accuracy of solution, which can be checked by means of $E(x_r)$ defined in relation (3.33) at any point x_r in the interval $[0,1]$, is shown in Table 4.

Table 3 Taylor coefficients

Taylor coefficients	N					
	6	8	10	12	15	18
$y^{(0)}(0)$	0	0	0	0	0	0
$y^{(1)}(0)$	1	1	1	1	1	1
$y^{(2)}(0)$	0	0	0	0	0	0
$y^{(3)}(0)$	-3	-3	-3	-3	-3	-3
$y^{(4)}(0)$	0	0	0	0	0	0
$y^{(5)}(0)$	15	15	15	15	15	15
$y^{(6)}(0)$	0	0	0	0	0	0
$y^{(7)}(0)$		-105	-105	-105	-105	-105
$y^{(8)}(0)$		0	0	0	0	0
$y^{(9)}(0)$			945	945	945	945
$y^{(10)}(0)$			0	0	0	0
$y^{(11)}(0)$				-10395	-10395	-10395
$y^{(12)}(0)$				0	0	0
$y^{(13)}(0)$					135135	135135
$y^{(14)}(0)$					0	0
$y^{(15)}(0)$					-2027025	-2027025
$y^{(16)}(0)$						0
$y^{(17)}(0)$						34459425
$y^{(18)}(0)$						0

Table 4 Numerical results and error in approximate solution for $c = 0$

x	$N = 6$		$N = 8$		$N = 10$	
	$y(x)$	$E(x)$	$y(x)$	$E(x)$	$y(x)$	$E(x)$
0	0	0	0	0	0	0
0.1	0.09950125	1.45833E-07	0.099501248	2.34375E-10	0.099501248	2.86458E-13
0.2	0.19604000	9.33333E-06	0.196039733	0.000000060	0.196039735	2.93333E-10
0.3	0.28680375	0.000106313	0.286799194	1.53773E-06	0.286799245	1.69151E-08
0.4	0.36928000	0.000597333	0.369245867	0.000015360	0.369246549	3.00373E-07
0.5	0.44140625	0.002278646	0.441243490	9.15527E-05	0.441248576	2.79744E-06
0.6	0.50172000	0.006804000	0.501136800	0.000393660	0.501163044	1.73210E-05
0.7	0.54950875	0.017157146	0.547793035	0.001351125	0.547898123	8.09174E-05
0.8	0.58496000	0.038229333	0.580590933	0.003932160	0.580940459	0.000307582
0.9	0.60931125	0.077501812	0.599346731	0.010089075	0.600355639	0.000998818
1.0	0.62500000	0.145833333	0.604166667	0.023437500	0.606770833	0.002864583

Table 4 (continued)

x	$N = 12$		$N = 15$		$N = 18$	
	$y(x)$	$E(x)$	$y(x)$	$E(x)$	$y(x)$	$E(x)$
0	0	0	0	0	0	0
0.1	0.099501248	2.82118E-16	0.099501248	2.32515E-19	0.099501248	1.64698E-22
0.2	0.196039735	1.15556E-12	0.196039735	3.80952E-15	0.196039735	1.07937E-17
0.3	0.286799245	1.49929E-10	0.286799245	1.11211E-12	0.286799245	7.08971E-15
0.4	0.369246538	4.73316E-09	0.369246539	6.24152E-11	0.369246539	7.07373E-13
0.5	0.441248449	6.88765E-08	0.441248451	1.41916E-09	0.441248451	2.51309E-11
0.6	0.501162099	6.14110E-07	0.501162127	1.82208E-08	0.501162127	4.64631E-10
0.7	0.547892974	3.90488E-06	0.547893177	1.57697E-07	0.547893177	5.47340E-09
0.8	0.580918089	1.93870E-05	0.580919228	1.02261E-06	0.58091923	4.63584E-08
0.9	0.600273917	7.96785E-05	0.600279114	5.31919E-06	0.60027913	3.05189E-07
1.0	0.606510417	0.000282118	0.606530568	2.32515E-05	0.606530665	1.64698E-06

Example 3 Let us consider the linear Volterra integro-differential equation

$$y'(x) = 1 - \int_0^x \left[y(t) + xy''(t)\right] dt\ , \qquad 0 \le x, t \le 1$$

with condition $y(0) = 0$ and approximate the solution $y(x)$ by the Taylor polynomial

$$y(x) = \sum_{n=0}^{5} \frac{1}{n!} y^{(n)}(0) x^n$$

where $N = 5$, $a = 0$, $b = 1$, $c = 0$.

Since $P_0(x) = 0$, and $P_1(x) = 1$, we have

$$\begin{aligned} P_0^{(0)}(0) &= 0,\ P_0^{(1)}(0) = P_0^{(2)}(0) = ... = 0 \\ P_1^{(0)}(0) &= 1,\ P_1^{(1)}(0) = P_1^{(2)}(0) = ... = 0. \end{aligned}$$

Then, by using relation in section 3.1.1, we obtain the matrix representation for the differential part as

$$\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the matrices $\mathbf{T}$, $\mathbf{K}$ and $\mathbf{F}$ as

$$\mathbf{F} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}, \qquad \mathbf{K} = \left[\bar{0}\right].$$

By using the matrices **W**, **T**, **K** and **F**, we find the augmented matrix $\widetilde{\mathbf{M}}$ as

$$\widetilde{\mathbf{M}}=\left[\begin{array}{cccccc:c} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

By taking $c = 0$, $a = 0$ and $b = 1$ and using relation for condition, the augmented matrix corresponding to the condition $y(0) = 0$ is obtained as

$$\widetilde{\mathrm{U}}_0=\left[\begin{array}{cccccc:c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the required augmented matrix $\widetilde{\mathbf{M}}^*$ and the unknown coefficients $y^{(n)}(0)$ $(n = 0, 1, 2, ..., 5)$ become

$$\widetilde{\mathbf{M}}^*=\left[\begin{array}{cccccc:c} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} y^{(0)}(0) \\ y^{(1)}(0) \\ y^{(2)}(0) \\ y^{(3)}(0) \\ y^{(4)}(0) \\ y^{(5)}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 3 \\ -11 \end{bmatrix}.$$

Then, we obtain the Taylor polynomial solution

$$y(x) = x - \frac{1}{3!}x^3 + \frac{3}{4!}x^4 - \frac{11}{5!}x^5.$$

The Taylor coefficients $(y^{(0)}(c), y^{(1)}(c), y^{(2)}(c), \ldots y^{(N)}(c))$ for $c = 0$ and $N = 6, 8, 10, 12, 15, 18$ are shown in Table 5. The Taylor solution and the accuracy of solution, which can be checked by means of $E(x_r)$ defined in relation (3.33) at any point x_r in the interval $[0, 1]$, is shown in Table 6.

Table 5 Taylorcoefficients

Taylor coefficients	N					
	6	8	10	12	15	18
$y^{(0)}(0)$	0	0	0	0	0	5.55E-17
$y^{(1)}(0)$	1	1	1	1	1	1
$y^{(2)}(0)$	0	0	0	0	0	0
$y^{(3)}(0)$	-1	-1	-1	-1	-1	-1
$y^{(4)}(0)$	3	3	3	3	3	3
$y^{(5)}(0)$	-11	-11	-11	-11	-11	-11
$y^{(6)}(0)$	52	52	52	52	52	52
$y^{(7)}(0)$		-301	-301	-301	-301	-301
$y^{(8)}(0)$		2055	2055	2055	2055	2055
$y^{(9)}(0)$			-16139	-16139	-16139	-16139
$y^{(10)}(0)$			143196	143196	143196	143196
$y^{(11)}(0)$				-1.42E+06	-1.42E+06	-1.42E+06
$y^{(12)}(0)$				15430835	15430835	15430835
$y^{(13)}(0)$					-1.84E+08	-1.84E+08
$y^{(14)}(0)$					2.37E+09	2.37E+09
$y^{(15)}(0)$					-3.30E+10	-3.30E+10
$y^{(16)}(0)$						4.93E+11
$y^{(17)}(0)$						-7.86E+12
$y^{(18)}(0)$						1.33E+14

Table 6 Numerical results and error in approximate solution for $c = 0$

x	$N = 6$		$N = 8$		$N = 10$	
	$y(x)$	$E(x)$	$y(x)$	$E(x)$	$y(x)$	$E(x)$
0	0	0	0	0	0	0
0.1	0.099844989	4.19087E-07	0.099844983	4.00839E-09	0.099844983	3.90521E-11
0.2	0.198841956	2.68876E-05	0.198841322	1.02760E-06	0.198841303	4.00261E-08
0.3	0.296342400	0.000307019	0.296332683	2.63734E-05	0.296332040	2.31022E-06
0.4	0.391890489	0.001729260	0.391826042	0.000263807	0.391818521	4.10619E-05
0.5	0.485243056	0.006612723	0.484975567	0.001574626	0.484927238	0.000382770
0.6	0.576441600	0.019793623	0.575625814	0.006780117	0.575416217	0.002372177
0.7	0.665936289	0.050033505	0.663956068	0.023303455	0.663276026	0.011092054
0.8	0.754761956	0.111754484	0.750788185	0.067914714	0.749055975	0.042201511
0.9	0.844766100	0.227106672	0.838140881	0.174498296	0.834669642	0.137166939
1.0	0.938888889	0.428373016	0.930133929	0.405935847	0.925120150	0.393749674

Table 6 (continued)

x	$N = 12$		$N = 15$		$N = 18$	
	$y(x)$	$E(x)$	$y(x)$	$E(x)$	$y(x)$	$E(x)$
0	0	0	0	0	0	5.55112E-17
0.1	0.099844983	2.57180E-13	0.099844983	1.27683E-13	0.099844983	1.27305E-13
0.2	0.198841302	1.43179E-09	0.198841302	1.56875E-10	0.198841302	1.44415E-10
0.3	0.296331995	1.95421E-07	0.296331991	1.47003E-08	0.296331991	9.14091E-09
0.4	0.391817574	6.28349E-06	0.391817422	5.90197E-07	0.391817429	1.59236E-07
0.5	0.484917784	9.22704E-05	0.484915072	1.34666E-05	0.484915343	5.19861E-07
0.6	0.575357659	0.000827131	0.575328571	0.000191265	0.575333594	2.42316E-05
0.7	0.663020574	0.005278370	0.662799339	0.001866811	0.662860144	0.000534811
0.8	0.748222954	0.026275164	0.746908941	0.013617585	0.747449535	0.006406387
0.9	0.832637348	0.108209872	0.826161935	0.079045008	0.829966320	0.054852835
1.0	0.921865428	0.383785255	0.894311738	0.382225400	0.916583262	0.369550460

Example 4 Let us consider the linear Fredholm integro-differential equation

$$y''(x) + xy'(x) - xy(x) = e^{x} - 2\sin x + \int_{-1}^{1} \left[(\sin x)\, e^{-t} y(t) + x y^{(3)}(t) \right] dt\ , \qquad -1 \le x, t \le 1$$

with conditions $y(0) = 1$ and $y'(0) = 1$, where $N = 6$, $a = -1$, $b = 1$, $c = 0$, $P_0(x) = -x$, $P_1(x) = x$, $P_2(x) = 1$, $f(x) = e^{x} - 2\sin x$, $\lambda_1 = 0$, $\lambda_2 = 1$, $B_0(x,t) = (\sin x)\, e^{-t}$ and $B_1(x,t) = B_2(x,t) = 0$, $B_3(x,t) = x$.

Then, by using relation in section 3.1.1, we obtain the matrix representation for the differential part as

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 6 \end{bmatrix}$$

and the matrices $\mathbf{T}$, $\mathbf{K}$ and $\mathbf{F}$ as

$$\mathbf{F} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{T} = \left[\vec{0} \right],$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{872}{371} & -\frac{465}{632} & \frac{410}{933} & \frac{591}{307} & \frac{323}{14034} & \frac{367}{1110} & \frac{11}{19574} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{872}{371} & \frac{465}{632} & -\frac{410}{933} & \frac{365}{4872} & -\frac{323}{14034} & \frac{65}{24052} & -\frac{11}{19574} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{872}{371} & -\frac{465}{632} & \frac{410}{933} & -\frac{365}{4872} & \frac{323}{14034} & -\frac{65}{24052} & \frac{11}{19574} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using the matrices **W**, **T**, **K** and **F**, we find the augmented matrix $\widetilde{\mathbf{M}}$ as

$$\widetilde{\mathbf{M}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & : & 1 \\ -\frac{1243}{371} & \frac{1097}{632} & -\frac{410}{933} & -\frac{284}{307} & -\frac{323}{14034} & -\frac{367}{1110} & -\frac{11}{19574} & : & -1 \\ 0 & -2 & 2 & 0 & 1 & 0 & 0 & : & 1 \\ \frac{872}{371} & -\frac{465}{632} & -\frac{2389}{933} & \frac{898}{307} & \frac{323}{14034} & \frac{369}{370} & \frac{11}{19574} & : & 3 \\ 0 & 0 & 0 & -4 & 4 & 0 & 1 & : & 1 \\ -\frac{872}{371} & \frac{465}{632} & -\frac{410}{933} & \frac{365}{4872} & -\frac{1964}{391} & \frac{1851}{370} & -\frac{11}{19574} & : & -1 \\ 0 & 0 & 0 & 0 & 0 & -6 & 6 & : & 1 \end{bmatrix}.$$

Consider the condition, we have the augmented matrix corresponding the condition $y(0) = 1$ and $y^{'}(0) = 1$, when $c = 0$, $a = -1$, $b = 1$, as

$$\widetilde{\mathbf{U}}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & : & 1 \end{bmatrix}$$

$$\widetilde{\mathbf{U}}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & : & 1 \end{bmatrix}.$$

Thus, the required augmented matrix $\widetilde{\mathbf{M}}^*$ and the unknown coefficients $y^{(n)}(0)$ $(n = 0, 1, 2, ..., 6)$

become

$$\tilde{\mathrm{M}}^{*} = \left[\begin{array}{ccccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & : & 1 \\ -\frac{1243}{371} & \frac{1097}{632} & -\frac{410}{933} & -\frac{284}{307} & -\frac{323}{14034} & -\frac{367}{1110} & -\frac{11}{19574} & : & -1 \\ 0 & -2 & 2 & 0 & 1 & 0 & 0 & : & 1 \\ \frac{872}{371} & -\frac{465}{632} & -\frac{2389}{933} & \frac{898}{307} & \frac{323}{14034} & \frac{369}{370} & \frac{11}{19574} & : & 3 \\ 0 & 0 & 0 & -4 & 4 & 0 & 1 & : & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & : & 1 \end{array}\right].$$

From this system, we obtain the coefficients $y^{(n)}(0)$ ($n = 0, 1, 2, 3, 4, 5, 6$) as

$$\begin{bmatrix} y^{(0)}(0) \\ y^{(1)}(0) \\ y^{(2)}(0) \\ y^{(3)}(0) \\ y^{(4)}(0) \\ y^{(5)}(0) \\ y^{(6)}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 55.06427288 \\ 1 \\ -157.6928737 \\ 217.2570915 \end{bmatrix}.$$

Substituting these elements into Taylor polynomial, we obtain the solution

$$y(x) = 1 + x + \frac{1}{2!}x^2 - \frac{55.06427288}{3!}x^3 + \frac{1}{4!}x^4 - \frac{157.6928737}{5!}x^5 + \frac{217.2570915}{6!}x^6.$$

The Taylor coefficients $(y^{(0)}(c), y^{(1)}(c), y^{(2)}(c), ..., y^{(N)}(c))$ for $c = 0$ and $N = 6, 8, 10, 12, 15$ 18 are shown in Table 7. The Taylor solution and the accuracy of solution, which can be checked by means of $E(x_r)$ defined in relation (3.33) at any point x_r in the interval $[-1, 1]$, is shown in Table 8.

Table 7 Taylorcoefficients

Taylor coefficients	N			
	10	12	15	18
$y^{(0)}(0)$	1	1	1	1
$y^{(1)}(0)$	1	1	1	1
$y^{(2)}(0)$	1	1	1	1
$y^{(3)}(0)$	2.111236239	1.949669707	1.962437034	1.962496684
$y^{(4)}(0)$	1	1	1	1
$y^{(5)}(0)$	5.143159348	4.51696705	4.566449928	4.566681133
$y^{(6)}(0)$	5.444944958	4.798678826	4.849748137	4.849986756
$y^{(7)}(0)$	-27.19266504	-22.95081161	-23.28601087	-23.28757706
$y^{(8)}(0)$	-0.810713656	-0.690270661	-0.699789251	-0.699833745
$y^{(9)}(0)$	7823.233828	6604.500728	6700.807317	6701.257302
$y^{(10)}(0)$	-210.0556111	-177.0843276	-179.689773	-179.7019466
$y^{(11)}(0)$		-59452.08496	-60319.01771	-60323.06839
$y^{(12)}(0)$		67816.85056	68805.9709	68810.59249
$y^{(13)}(0)$			661540.0611	661584.4851
$y^{(14)}(0)$			-1549498.863	-1549602.931
$y^{(15)}(0)$			-7705548.627	-7706066.058
$y^{(16)}(0)$				30956624.82
$y^{(17)}(0)$				92346954.36
$y^{(18)}(0)$				-618603053

Table 8 Numerical results and error in approximate solution for $c = 0$

x	$N = 10$		$N = 12$	
	$y(x)$	$E(x)$	$y(x)$	$E(x)$
-1.0	0.138255365	4.490102413	0.173663714	4.364919038
-0.8	0.343074369	4.951292459	0.358749415	4.715000442
-0.6	0.506348646	4.123858846	0.512545038	3.929255615
-0.4	0.658142075	2.872093983	0.659914779	2.741187562
-0.2	0.817238512	1.466520447	0.817455537	1.401045669
0	1	0	1	0
0.2	1.222895789	1.466520452	1.222678649	1.401045670
0.4	1.504053171	2.872098626	1.502273124	2.741188179
0.6	1.865155643	4.124126564	1.858876322	3.929283448
0.8	2.335004675	4.956046530	2.318881444	4.715302983
1.0	2.960046831	4.534378152	2.923150603	4.365441055

Table 8 (continued)

x	$N = 15$		$N = 18$	
	$y(x)$	$E(x)$	$y(x)$	$E(x)$
-1.0	0.170900222	2.575664422	0.170888978	2.575665212
-0.8	0.357517211	2.138402731	0.357511518	2.138403357
-0.6	0.512055771	1.651144048	0.512053486	1.651144494
-0.4	0.659774702	1.123789779	0.659774047	1.123790065
-0.2	0.817438388	0.568829506	0.817438307	0.568829646
0	1	0	1	0
0.2	1.222695808	0.568914359	1.222695888	0.568914499
0.4	1.502413782	1.125129781	1.502414440	1.125130069
0.6	1.859372172	1.65778222	1.859374488	1.657782673
0.8	2.320149322	2.158760036	2.320155269	2.158760642
1.0	2.926022628	2.623498084	2.926037309	2.623497958