CHAPTER 2

PRELIMINARIES

2.1 Differential Equations

Definition 2.1.1 *An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables is said to be a differential equation (DE).*

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is said to be an *ordinary differential equation (ODE).* For example,

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

are ordinary differential equations. An equation involving the partial derivatives of one or more dependent variables of two or more independent variables is called a *partial differential equation (PDE).* For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are partial differential equations.

The highest order derivative present in a differential equation is called the *order* of the differential equation. Therefore,

$$(1) \quad \frac{d^2y}{dx^2} + xy = x\sin x$$

$$(2) \quad \frac{d^3y}{dx^3} + 3\left(\frac{dy}{dx}\right)^5 + 6y = 0$$

$$(3) \quad \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - u^2 = y - x$$

$$(4) \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

differential equation (1) and (4) are of the second order while (2) and (3) are of the third and first orders, respectively.

We can express an nth order ordinary differential equation in one dependent variable by a general form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{2.1}$$

where F is a real valued function of $n+2$ variables, $x, y, y', \ldots, y^{(n)}$, and where $y^{(n)} = \frac{d^n y}{dx^n}$.

An nth order ordinary differential equation (2.1) is said to be linear if F is linear in $y, y', \ldots, y^{(n)}$. This means that an nth order ODE is linear when (2.1) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{2.2}$$

From (2.2) we see two characteristic properties of linear differential equations. First, the dependent variable and all its derivatives are of the first degree. That is, the power of each term involving y is 1. Second, each coefficient depends at most on the independent variable x. The equations

$$(y-x)dx + 4xdy = 0, \quad y'' - 2y' + y = 0, \quad \text{and} \quad \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, linear first, second, and third order ordinary differential equation. A *nonlinear* ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, must not appear in a linear equation. Therefore,

$$(1-y)y' + 2y = e^x, \quad \frac{d^2y}{dx^2} + \sin y = 0, \quad \text{and} \quad \frac{d^4y}{dx^4} + y^2 = 0$$

are examples of nonlinear first, second, and fourth order ordinary differential equations, respectively.

Definition 2.1.2 *Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth order* *ordinary differential equation reduces the equation to an identity, is said to be a solution of the equation on the interval.*

The interval I in Definition 2.1.2 is also called the *interval of definition*, the *interval of existence*, the *interval of validity*, or the *domain* of the solution. It can be an open interval (a, b), a closed interval $[a, b]$, or an infinite interval (a, ∞).

The graph of a solution ϕ of an ODE is called a *solution curve.* Since ϕ is a differentiable function, it is continuous on its interval I by definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ. That is, the domain of the function ϕ need not be the same as the interval I of the definition of the solution ϕ.

2.2 Initial and Boundary Value Problems

2.2.1 Initial Value Problems

We are often interested in solving a differential equation subject to prescribed side conditions. That is the condition that are imposed on the unknown solution $y = y(x)$ or its derivatives. On some interval I containing x_0, the problem

$$\begin{gathered} \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \\ y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}, \end{gathered} \tag{2.3}$$

where $y_0,\ y_1,\ \ldots,\ y_{n-1}$ are arbitrarily specified real constants, is called an *initial value problem (IVP).* The values of $y(x)$ and its first $n-1$ derivatives at a single point $x_0 : y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$ are called *initial conditions.* The problem given in (2.3) is also called an nth order initial value problem. For example,

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0, \tag{2.4}$$

and

$$\frac{d^2 y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0,\ y'(x_0) = y_1, \tag{2.5}$$

are first- and second-order initial value problems, respectively. These two problems are easy to interpret in geometric terms. For (2.4) we are seeking a solution of the differential equation on an interval I containing x_0 so that a solution curve passes through the prescribed point (x_0, y_0). For (2.5) we want to find a solution of the differential equation whose graph not only passes through (x_0, y_0) but also, the slope of the curve at this point is y_1.

2.2.2 Boundary Value Problems

Another type of problem consists of solving a differential equation of order two or greater in which the dependent variable y or its derivatives are specified at different points. A problem such as

$$\frac{d^2y}{dx^2} = f(x, y, y'),$$

$$y(a) = y_0, \quad y(b) = y_1,$$

is called a *boundary value problem (BVP)*. The prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called *boundary conditions.* A solution of the previous problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through the two points (a, y_0) and (b, y_1).

For a second order differential equation, other pairs of boundary conditions could be

$$y'(a) = y_0, \quad y(b) = y_1$$

$$y(a) = y_0, \quad y'(b) = y_1$$

$$y'(a) = y_0, \quad y'(b) = y_1$$

where y_0 and y_1 denote arbitrary constants. These three pairs of conditions are just special cases of the general boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1$$

$$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2$$

The problem consisting of nonlinear differential equation with initial conditions is called a *nonlinear initial value problem.* And the problem consisting of nonlinear differential equation with boundary conditions is called a *nonlinear boundary value problem.*

2.3 Systems of First Order Differential Equations

Consider the following system of n first order differential equations :

$$\begin{aligned} y_1' &= f_1(x, y_1, y_2, \ldots, y_n) \\ y_2' &= f_2(x, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, y_2, \ldots, y_n) \end{aligned} \tag{2.6}$$

where y_i, $(i = 1, 2, \ldots, n)$ are real valued functions of the independent variable x and f_i, $(i = 1, 2, \ldots, n)$ are real valued functions of $x, y_1, \ldots, y_n$. Let

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

be a vector of real valued functions. Then the vector

$$\mathbf{y}' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} \quad \text{and} \quad \int_a^b \mathbf{y}\,dx = \begin{pmatrix} \int_a^b y_1 dx \\ \int_a^b y_2 dx \\ \vdots \\ \int_a^b y_n dx \end{pmatrix}.$$

Let

$$\mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \\ \vdots \\ f_n(x, \mathbf{y}) \end{pmatrix}.$$

Then the system (2.6) may be represented more concisely by the vector differential equation

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}) \tag{2.7}$$

and a *solution* of (2.7) on an interval I is a vector of functions which has a continuous first derivative and satisfies (2.7) on I.

2.4 Existence and Uniqueness Theorems

The vector initial value problem which corresponds to the scalar initial value problem $y' = f(x, y)$; $y(x_0) = c_0$ is

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}); \quad \mathbf{y}(x_0) = \mathbf{c}_0, \tag{2.8}$$

where $\mathbf{c}_0$ is a constant vector. That is a vector with components which are real numbers. Hence, an initial value problem for a system of equations is to find a solution of (2.7) subject to the conditions $\mathbf{y}(x_0) = \mathbf{c}_0$.

Theorem 2.4.1 *Let $\mathbf{f}(x, \mathbf{y})$ be a vector whose components, $f_1, f_2, \ldots, f_n$, are all continuous functions of $x, y_1, y_2, \ldots, y_n$. That is, let $\mathbf{f}(x, \mathbf{y}) \in C$ where C is the set of all continuous real valued functions on $\mathbb{R}$. Then $\mathbf{y}$ is a solution of the IVP (2.8) on an interval I about x_0 if and only if $\mathbf{y}$ satisfies the integral equation*

$$\mathbf{y}(x) = \mathbf{c}_0 + \int_{x_0}^{x} \mathbf{f}(t, \mathbf{y}(t))dt$$

for all $x \in I$.

See [4] for more details.

There is a close relationship between nth order differential equations which are written in normal form and a system of n first order differential equations. We wish to exhibit this relationship and then exploit it. Consider the nth order differential equation.

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}). \tag{2.9}$$

Let $y_1 = y$, $y_2 = y'$, $\ldots$, $y_n = y^{(n-1)}$. Then the nth order differential equation (2.9) may be rewritten as

$$\begin{aligned} y_1' &= y_2 &&= f_1(x, \mathbf{y}) \\ y_2' &= y_3 &&= f_2(x, \mathbf{y}) \\ &\vdots && \vdots \\ y_{n-1}' &= y_n &&= f_{n-1}(x, \mathbf{y}) \\ y_n' &= f(x, y_1, y_2, \ldots, y_n) &&= f_n(x, \mathbf{y}). \end{aligned} \tag{2.10}$$

Hence, the nth order differential equation (2.9) is equivalent to the system of n first order differential equations (2.10). Observe that (2.10) is a special case of (2.7). Recall that the initial value problem associated with (2.7) is

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad \mathbf{y}(x_0) = \mathbf{c}_0. \tag{2.11}$$

Making use of the correspondence between (2.9) and (2.10), we see that the initial conditions for (2.9) which are equivalent to the initial conditions $\mathbf{y}(x_0) = \mathbf{c}_0$ for (2.10) are

$$y_1(x_0) = y(x_0) = c_1,\ y_2(x_0) = y'(x_0) = c_2,\ \ldots,\ y_n(x_0) = y^{(n-1)}(x_0) = c_n.$$

Hence, the initial value problem associated with (2.9) which is equivalent to (2.11) is

$$\begin{gathered} y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \\ y(x_0) = c_1,\ y_2(x_0) = y'(x_0) = c_2, \ldots, y_n(x_0) = y^{(n-1)}(x_0) = c_n. \end{gathered} \tag{2.12}$$

It is easy to see that $f_1(x, \mathbf{y}) = y_2$, $f_2(x, \mathbf{y}) = y_3, \ldots, f_{n-1}(x, \mathbf{y}) = y_n$ are all continuous in any domain D of $(x, \mathbf{y})$-space and for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n$

$$\frac{\partial f_i}{\partial y_j} = \begin{cases} 0, & j \neq i+1 \\ 1, & j = i+1. \end{cases}$$

Theorem 2.4.2 *If $f(x, y, y', \ldots, y^{(n-1)}) \in C$ and has continuous first partial derivatives with respect to $y, y', \ldots, y^{(n-1)}$ on some bounded domain D and if $(x_0, c_1, \ldots, c_n) \in D$, then there exists a unique solution to the IVP (2.12) on some interval about x_0 and this solution can be extended uniquely until the boundary of D is reached.*

See [4] for more details.

2.5 Interval of Definition and Extension of Solutions

The interval of definition and extension of solution, were treated in details in [3]. The domain of the function $y_1(x), y_2(x), \ldots, y_n(x)$ that constitute the solution of the given initial value problem can be extended beyond the bounds guaranteed by the existence theorem. In fact, the functions $y_1(x), y_2(x), \ldots, y_n(x)$ determined by the theorem in the interval $[x_0 - \delta, x_0 + \delta]$ assume for $x = x_0 + \delta$ values $y_1(x_0 + \delta), y_2(x_0 + \delta), \ldots, y_n(x_0 + \delta)$, and these can be considered initial conditions of an initial value problem relative to the same system. The solution of this problem will be defined in an interval $[x_0 + \delta - \delta_1, x_0 + \delta + \delta_1]$ where δ_1 will depend on the maximum value that the functions $|f_i(x, y_1, y_2, \ldots, y_n)|$ assume in a rectangular domain of $\mathbb{R}^{n+1}$ of the form

$$\begin{aligned} x_0 + \delta - a_1 \ &\leq \ x \ \leq \ x_0 + \delta + a_1, \\ y_i(x_0 + \delta) - b_1 \ &\leq \ y_i \ \leq \ y_i(x_0 + \delta) + b_1 \quad (i = 1, 2, \ldots, n). \end{aligned}$$

This domain is contained in the set I where the functions $f_i(x, y)$ are defined. By the uniqueness theorem, both the primitive solution and the solution just now obtained must coincide in the interval $[x_0 + \delta - \delta_1, x_0 + \delta + \delta_1]$. The new solution is an *extension* of the primitive one into a bigger interval. By repeating this extension, we arrive at an interval larger than the original interval in which the solution is defined. It is clear that this may not happen in the case of

$$y' = y^2.$$

The solution of this for which $y(x_0) = y_0 > 0$ is the function

$$y(x) = \frac{y_0}{1 - y_0(x - x_0)}.$$

It is defined in the interval $\left[x_0, x_0 + \frac{1}{y_0}\right)$ and may not be extended beyond this interval to the right because at the right-hand endpoint the solution becomes infinite.

2.6 Shooting Method

One way to approximate a solution of a BVP $y'' = f(x, y, y'), y(a) = \alpha$, $y(b) = \beta$ is by the *shooting method* [7]. In this method, the starting point is the replacement of the boundary value problem by an initial value problem

$$y'' = f(x, y, y'),\ y(a) = \alpha,\ y'(a) = m_1. \tag{2.13}$$

The number m_1 in (2.13) is simply a guess for the unknown slope of the solution curve at the known point $(a, y(a))$. We then apply one of the step-by-step numerical techniques to the second order equation in (2.13) to find an approximation β_1 for the value of $y(b)$. If β_1 agrees with the given value $y(b) = \beta$ to some pre-assigned tolerance, we stop; otherwise the calculations are repeated, starting with a different guess $y'(a) = m_2$ to obtain a second approximation β_2 for $y(b)$. The process is repeated until the solution of the initial value problem agrees with the specified tolerance $y(b) = \beta$.

2.7 Continuous Dependence

Saperstone mentioned the use of continuous dependence on initial condition in [5]. The solution $y = \phi(x)$ of the initial value problem

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0,$$

may be thought of as a function of the initial point (x_0, y_0), as well as a function of x. Consider, for instance, the linear initial value problem

$$\frac{dy}{dx} + \frac{1}{2}y = \frac{1}{4}\sin(x/2), \qquad y(0) = y_0,$$

whose solution is

$$y = e^{-x/2}y_0 + \frac{1}{4}\left(\sin\frac{x}{2} - \cos\frac{x}{2} + e^{-x/2}\right).$$

Notice how the solution depends on y_0 as well as x. Indeed, the solution depends continuously on y_0. This dependence is readily visualized by considering y as a function of both x and y_0 and graphing the resulting surface. Figure 2.1 illustrates such a surface. The near edge of the surface represents the solution corresponding to $y_0 = -1$. As y_0 increases from -1 to 1, the graphs of the resulting solutions sweep out the surface from the near edge to the rear edge. The rear edge represents the solution corresponding to $y_0 = 1$. The x-interval for all solutions is fixed at $0 \leq x \leq 8\pi$. The important thing to observe from Figure 2.1 is the smoothness of the surface. It is this smoothness that illustrates the continuity of y with respect to both x and y_0.

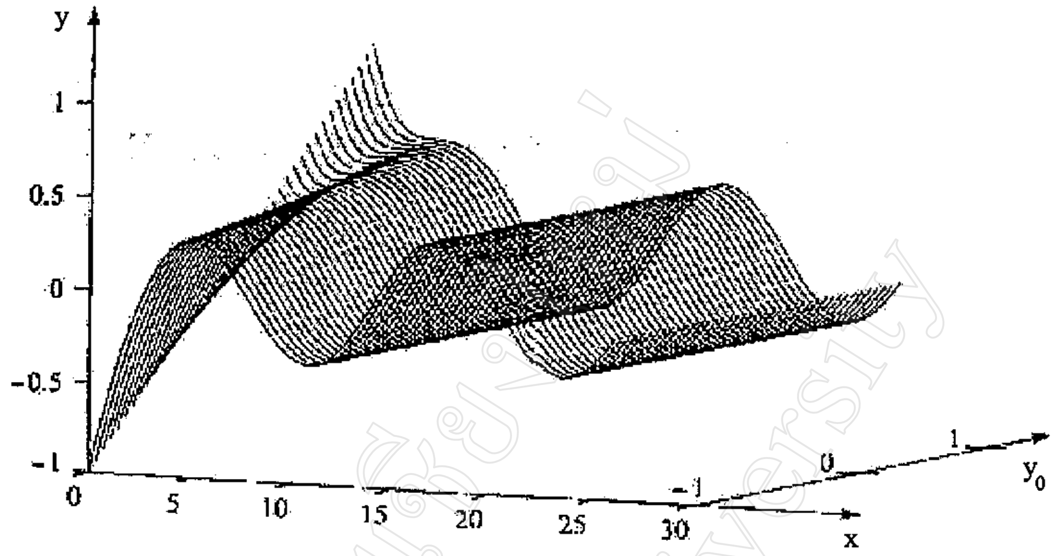

Figure 2.1: A surface representing $y = e^{-x/2}y_0 + \frac{1}{4}\left(\sin\frac{x}{2} - \cos\frac{x}{2} + e^{-x/2}\right)$

The subject of continuous dependence of solutions on initial values is an important topic for a number of reasons. When an IVP serves to model a real problem, the initial value y_0 is often measured experimentally. Small errors in this measurement should produce only small changes in the solution which is the essence of continuity.

Another illustration of the continuous dependence of a solution on initial values may be found in the (nonlinear) IVP

$$\frac{dy}{dx} = -y^2, \qquad y(0) = y_0,$$

whose solution is

$$y = \frac{y_0}{xy_0 + 1}.$$

Again, y is a continuous function of x and y_0, provided $xy_0 + 1 \neq 0$.

2.8 Convex and Concave Functions

We will now give some definitions and theorems of convex functions.

Definition 2.8.1 *Let I be an open interval. Let f be a function $I \to \mathbb{R}$. f is said to be convex if*

$$f(\lambda a + \mu b) \leq \lambda f(a) + \mu f(b)$$

for all $a, b \in I$ and all $\lambda, \mu \in \mathbb{R}$ such that $\lambda > 0$, $\mu > 0$, $\lambda + \mu = 1$.

Theorem 2.8.1 *Let I be an open interval. Let $f : I \to \mathbb{R}$ be twice differentiable. The f is convex if and only if $f''(x) \geq 0$ for all $x \in I$.*

A function g is said to be *concave* if $-g$ is convex. Hence, we have a definition and a theorem for concave function g.

Definition 2.8.2 *Let E be an open interval. Let g be a function $E \to \mathbb{R}$. g is said to be concave if*

$$g(\lambda c + \mu d) \geq \lambda g(c) + \mu g(d)$$

for all $c, d \in E$ and all $\lambda, \mu \in \mathbb{R}$ such that $\lambda > 0$, $\mu > 0$, $\lambda + \mu = 1$.

Theorem 2.8.2 *Let E be an open interval. Let $g : E \to \mathbb{R}$ be twice differentiable. The g is concave if and only if $g''(x) \leq 0$ for all $x \in E$.*

See [6] for more details.

2.9 Definition of Limits

We will now give some definitions for function $f(x)$.

Definition 2.9.1 *Let f be a real function whose domain D contains points in every interval of the form (B, ∞) where $B > 0$ and let $L \in \mathbb{R}$. Then*

$$\lim_{x \to \infty} f(x) = L$$

if and only if for each $\varepsilon > 0$ there exists a $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in D$ and $x > N$.

Definition 2.9.2 *Let f be a real function whose domain D contains points in every interval of the form (B, ∞) where $B > 0$. Then*

$$\lim_{x \to \infty} f(x) = \infty$$

if and only if for each $M > 0$ there exists a $N > 0$ such that $f(x) > M$ whenever $x \in D$ and $x > N$.

See [2] for more details.