CHAPTER 3

MAIN RESULTS

Since we will prove the existence of nonnegative symmetric solutions, we will consider the equivalent boundary value problem

$$y''(x) + f(x,y) = 0, \qquad a \le x \le \frac{b+a}{2}, \tag{3.1}$$

$$y(a) = 0, \quad y'(\frac{b+a}{2}) = 0. \tag{3.2}$$

We use $||y||$ to denote the sup-norm on $a \le x \le \frac{b+a}{2}$. By IVP(m), we denote the initial value problem consisting of (3.1) with the initial conditions

$$y(a) = 0, \quad y'(a) = m \ge 0. \tag{3.3}$$

We use $y_m(x)$ to denote the solution of (3.1) and (3.3).

3.1 Four Basic Lemmas

In this section, we introduced the main idea that will be used in section 3.2. That is, to provide conditions on the nonlinear function $f(x,y)$ which guarantees that the BVP (1.3), (1.4) has an odd nonnegative symmetric solutions.

Lemma 3.1.1 *Suppose $f : [a, x_1] \times \mathbb{R} \to [0,\infty)$ is continuous and $y = \phi(x)$ is a nonnegative solution of*

$$y'' + f(x,y) = 0, \quad y(a) = 0, \quad \text{for} \quad a \le x \le x_1.$$

Then

$$\frac{(\phi'(a))^2}{2} - \frac{(\phi'(x_1))^2}{2} = F(x, \phi(x_1))$$

where $\quad F(x,y) \equiv \int_0^y f(x,u)du.$

Proof. Since $\phi''(x) = -f(x, \phi(x))$,

$$\phi'(x)\phi''(x) = -f(x, \phi(x))\phi'(x)$$

on $[a, x_1]$, and integrating with respect to x from a to x_1, we have

$$\begin{aligned}
\int_a^{x_1} \phi'(x)\phi''(x)dx &= -\int_a^{x_1} f(x, \phi(x))\phi'(x)dx \\
\int_{\phi'(a)}^{\phi'(x_1)} \phi'(x)d(\phi'(x)) &= -\int_{\phi(a)}^{\phi(x_1)} f(x, \phi(x))d(\phi(x)) \\
\frac{(\phi'(x_1))^2}{2} - \frac{(\phi'(a))^2}{2} &= -\int_0^{\phi(x_1)} f(x, u)du \\
\frac{(\phi'(a))^2}{2} - \frac{(\phi'(x_1))^2}{2} &= \int_0^{\phi(x_1)} f(x, u)du \quad = \quad F(x, \phi(x_1)).
\end{aligned}$$

Lemma 3.1.2 *Suppose for some* $0 < d$ *and* $f : [a, b] \times \mathbb{R} \to [0, 8d]$ *is continuous. Then*

(a) *Any solution* $y_m(x)$ *of IVP(m) exists for* $a \leq x < \infty$ *and satisfies*

$$m - 8d(x - a) \leq y'_m(x) \leq m \tag{3.4}$$

$$m(x - a) - 4d(x - a)^2 \leq y_m(x) \leq m(x - a) \tag{3.5}$$

for $x \geq a$;

(b) *Any solution* $y(x)$ *of BVP (3.1) and (3.2) satisfies* $y\left(\frac{b+a}{2}\right) \leq d(b - a)^2$; *and if* $f(x, y) \not\equiv 8d$ *on* $\left[a, \frac{b+a}{2}\right] \times \left[0, d(b - a)^2\right]$, *then* $y\left(\frac{b+a}{2}\right) < d(b - a)^2$.

The notation $f(x, y) \not\equiv 8d$ on $\left[a, \frac{b+a}{2}\right] \times \left[0, d(b - a)^2\right]$ means that $f(x, y)$ not equal to $8d$ throughout an interval $\left[a, \frac{b+a}{2}\right] \times \left[0, d(b - a)^2\right]$.

Proof. Since $0 \leq f(x, y_m(x)) \leq 8d$, then

$$-8d \leq y''_m(x) = -f(x, y_m(x)) \leq 0, \qquad a \leq x \leq \frac{b + a}{2}.$$

Integrating with respect to x from a to x, we have

$$\begin{aligned}
-\int_a^x 8d ds &\leq \int_a^x y''_m(s)ds \leq 0 \\
-8d(x - a) &\leq y'_m(x) - y'_m(a) \leq 0 \\
m - 8d(x - a) &\leq y'_m(x) \leq m, \quad \text{for } x > a.
\end{aligned}$$

Integrating with respect to x from a to x again, we have

$$\begin{aligned} m(x-a)-4d(x-a)^2 &\le y_m(x)-y_m(a) \le m(x-a) \\ m(x-a)-4d(x-a)^2 &\le y_m(x) \le m(x-a), \quad \text{for } x>a. \end{aligned}$$

These bounds on $y_m(x), y_m'(x)$, together with standard theorems on the maximum interval of existence imply that the solution $y_m(x)$ exists for all $x \ge a$.

For part (b), since $v(x) = -4dx^2 + 4d(b+a)x - 4dab$ is a solution of the problem

$$v''(x) + 8d = 0, \quad v(a) = 0,\ v'\left(\left(\frac{b+a}{2}\right)\right) = 0.$$

Letting $u = v - y$ and since $f(x,y) \le 8d$, then we get

$$u'' = v'' - y'' = f(x, y(x)) - 8d \le 0.$$

By integrating with respect to x from x to $(b+a)/2$, we have

$$\begin{aligned} \int_x^{(b+a)/2} u''(s)ds &= \int_x^{(b+a)/2} \left[v''(s) - y''(s)\right] ds \le 0 \\ u'((b+a)/2) - u'(x) &= v'((b+a)/2) - v'(x) - y'((b+a)/2) + y'(x) \le 0 \\ u'(x) &= v'(x) - y'(x) \ge 0. \end{aligned}$$

By integrating with respect to x from a to $(b+a)/2$, we get

$$\begin{aligned} u((b+a)/2) - u(a) &= v((b+a)/2) - v(a) - y((b+a)/2) + y(a) \ge 0 \\ u((b+a)/2) &= v((b+a)/2) - y((b+a)/2) \ge 0 \\ y\left(\frac{b+a}{2}\right) \le v\left(\frac{b+a}{2}\right) &= -4d\left(\frac{b+a}{2}\right)^2 + 4d(b+a)\left(\frac{b+a}{2}\right) - 4dab \\ &= d(b-a)^2. \end{aligned}$$

Since $u'(x)$ is non-increasing on $[a, (b+a)/2]$ and from $u'((b+a)/2) = 0$, hence if $f(x,y) \not\equiv 8d$ on $[a,(b+a)/2] \times \left[0, d(b-a)^2\right]$, then we get $u'(a) > 0$. So $u(x)$ is strictly increasing at $x = a$, and from $u(a) = 0$, we have $u((b+a)/2) = v((b+a)/2) - y((b+a)/2) > 0$, therefore $y\left(\frac{b+a}{2}\right) < v\left(\frac{b+a}{2}\right) = d(b-a)^2$.

Lemma 3.1.3 *Suppose for some* $0 < d$ *and* $f : [a,b] \times [0, d(b-a)^2] \to [0, 8d]$ *is continuous with* $f(x,y) \not\equiv 8d$ *on* $[a, (b+a)/2] \times [0, d(b-a)^2]$. *Then there exist* s_1 *and* m_1 *such that* $0 \le s_1 < m_1 < 4d(b-a)$ *and*

$$y'_{s_1}((b+a)/2) = 0,\ y_{s_1}((b+a)/2) < d(b-a)^2,$$
$$y'_m((b+a)/2) > 0,\ y_m((b+a)/2) < d(b-a)^2 \quad \text{for all } m \in (s_1, m_1].$$

Moreover, there is no solution $y(x)$ *of the BVP (3.1), (3.2) for which* $||y|| \le d(b-a)^2$ *and* $y'(a) > s_1$.

Proof. Extend the function $f(x,y)$ from $[a,b] \times [0, d(b-a)^2]$ to $[a,b] \times \mathbb{R}$ by $f(x,y) = f(x,0)$ for $y \le 0$ and $f(x,y) = f(x, d(b-a)^2)$ for $y \ge d(b-a)^2$, then $f : [a,b] \times \mathbb{R} \to [0, 8d]$. Let

$$S = \{s \ge 0 : y'_s((b+a)/2) = 0\}.$$

From Lemma 3.1.2(a) we know that $y'_m((b+a)/2) > 0$ and $y_m((b+a)/2) > d(b-a)^2$ if $m > 4d(b-a)$. Thus S is bounded above. Since $y'_0((b+a)/2) \le 0$, then by continuous dependence of a solution on initial value m there is $s \in [0, 4d(b-a)]$ such that $y'_s((b+a)/2) = 0$ so $S \ne \emptyset$. Let $s_1 = \sup S$. Then by continuous dependence of a solution on initial value m, we get

$$y'_{s_1}((b+a)/2) = 0.$$

So that $y_{s_1}(x)$ is a solution of the BVP (3.1), (3.2). Then by Lemma 3.1.2(b) implies that $y_{s_1}((b+a)/2) < d(b-a)^2$ and by Lemma 3.1.2(a) we have $s_1 \left(\frac{b+a}{2} - a\right) - 4d\left(\frac{b+a}{2} - a\right)^2 \le y_{s_1}\left(\frac{b+a}{2}\right)$, so $s_1 < 4d(b-a)$. By the definition of s_1 guarantees that there is no solution $y(x)$ of the BVP (3.1), (3.2) with $||y|| \le d(b-a)^2$ and $y'(a) > s_1$. (If $||y_m|| \le d(b-a)^2$, then y_m is a solution of IVP(m) with the unextended f.)

Since s_1 is the supremum of S, then there is no $m > s_1$ such that $y'_m((b+a)/2) = 0$, and from $y'_m((b+a)/2) > 0$ for $m > 4d(b-a)$ implies that $y'_m((b+a)/2) > 0$ for $m > s_1$. Thus by continuous dependence of a solution on initial value m, we can choose $m_1 \in (s_1, 4d(b-a))$ very close to s_1 such that

$$y'_m((b+a)/2) > 0 \text{ and } y_m((b+a)/2) < d(b-a)^2 \quad \text{for all } m \in (s_1, m_1].$$

Lemma 3.1.4 *Suppose* $0 < c < \frac{d}{2}(b-a)^2$, $f : [a,b] \times [0, d(b-a)^2] \to [0, 8d]$, *is continuous with* $f(x,y) \not\equiv 8d$, *on* $[a, (b+a)/2] \times [0, d(b-a)^2]$, *and* $f(x,y) \geq \frac{16c}{(b-a)^2}$ *for* $(x,y) \in [a,b] \times [c, 3c/2]$. *Then there exist* m_2, s_3, m_3 *such that* $0 < m_2 < s_3 < m_3 < 4d(b-a)$ *and*

$$y'_{m_2}((b+a)/2) < 0, \quad y_{m_2}((b+3a)/4) = c,$$
$$y_m((b+3a)/4) > c \quad \text{for all } m > m_2 \text{ for which } ||y_m|| \leq d(b-a)^2,$$
$$y'_{s_3}((b+a)/2) = 0, \quad ||y_{s_3}|| < d(b-a)^2,$$
$$y'_m((b+a)/2) > 0, \quad y_m((b+a)/2) < d(b-a)^2 \text{ for all } m \in (s_3, m_3].$$

Moreover, there is no solution $y(x)$ *of the BVP (3.1), (3.2) for which* $||y|| \leq d(b-a)^2$ *and* $y'(a) > s_3$.

Proof. Extend the function $f(x,y)$ from $[a,b] \times [0, d(b-a)^2]$ to $[a,b] \times \mathbb{R}$ by $f(x,y) = f(x,0)$ for $y \leq 0$ and $f(x,y) = f\left(x, d(b-a)^2\right)$ for $y \geq d(b-a)^2$, then $f : [a,b] \times \mathbb{R} \to [0, 8d]$. Let

$$S = \{s \geq 0 : y_s((b+3a)/4) \leq c\}.$$

For $s > \frac{4c}{b-a} + d(b-a)$, by Lemma 3.1.2(a) implies that

$$\begin{aligned} y_s((b+3a)/4) &\geq s((b+3a)/4 - a) - 4d((b+3a)/4 - a)^2 \\ &= s((b-a)/4) - \frac{d}{4}(b-a)^2 > c. \end{aligned}$$

Thus $s \notin S$, so S is bounded above. By Lemma 3.1.2(a) implies that

$$y_0((b+3a)/4) \leq 0.$$

Thus $0 \in S$, so $S \neq \emptyset$. Let $m_2 = \sup S$, so that $m_2 \leq \frac{4c}{b-a} + d(b-a)$. Then by continuous dependence of a solution on initial value m, we get

$$y_{m_2}((b+3a)/4) = c.$$

Now, we will show that $y'_{m_2}\left(\frac{b+3a}{4}\right) < \frac{4c}{b-a}$. Supposing the contrary, we have $y'_{m_2}\left(\frac{b+3a}{4}\right) \geq \frac{4c}{b-a}$. Since $f\left(\frac{b+3a}{4}, y_{m_2}\left(\frac{b+3a}{4}\right)\right) = f\left(\frac{b+3a}{4}, c\right) > 0$, then $y''_{m_2}\left(\frac{b+3a}{4}\right) < 0$. And since $y'_{m_2}(x)$ is non-increasing on $[a, (b+a)/2]$, thus $y'_{m_2}(x)$

is strictly decreasing at $x = \frac{b+3a}{4}$. Hence $y'_{m_2}(x) > y'_{m_2}\left(\frac{b+3a}{4}\right) \geq \frac{4c}{b-a}$ for all $a \leq x < \frac{b+3a}{4}$. Then we have

$$\begin{aligned} y_{m_2}((b+3a)/4) &= y_{m_2}((b+3a)/4) - y_{m_2}(a) \\ &= y'_{m_2}(x)\,((b+3a)/4 - a) \\ &> \left(\frac{4c}{b-a}\right)\left(\frac{b-a}{4}\right) = c \qquad \text{for some } a < x < (b+3a)/4 \end{aligned}$$

which contradicting the definition of m_2. Therefore $y'_{m_2}(\frac{b+3a}{4}) < \frac{4c}{b-a}$.

Next we will show that $y'_{m_2}((b+a)/2) < 0$. Supposing the contrary, then $y'_{m_2}((b+a)/2) \geq 0$, so $y'_{m_2}(x) \geq 0$ on $[(b+3a)/4, (b+a)/2]$. Let

$$e = \sup\left\{x \in \left[\frac{b+3a}{4}, \frac{b+a}{2}\right] : y_{m_2}(x) \leq \frac{3c}{2}\right\}.$$

Since $y_{m_2}\left(\frac{b+3a}{4}\right) = c$, $y_{m_2}(e) \leq \frac{3c}{2}$, and $y_{m_2}(x)$ is non-decreasing, we have

$$y''_{m_2}(x) = -f(x, y_{m_2}(x)) \leq -(16c)/(b-a)^2 \quad \text{for } (b+3a)/4 \leq x \leq e.$$

Integrating with respect to x from $(b+3a)/4$ to x, we get

$$y'_{m_2}(x) - y'_{m_2}((b+3a)/4) \leq -\frac{16}{(b-a)^2}\left(x - \frac{b+3a}{4}\right).$$

Thus

$$y'_{m_2}(x) < \frac{4c}{b-a} - \frac{16c}{(b-a)^2}\left(x - \frac{b+3a}{4}\right). \tag{3.6}$$

Integrating with respect to x from $(b+3a)/4$ to x again, we get

$$y_{m_2}(x) - y_{m_2}\left(\frac{b+3a}{4}\right) < \frac{4c}{b-a}\left(x - \frac{b+3a}{4}\right) - \frac{8c}{(b-a)^2}\left(x - \frac{b+3a}{4}\right)^2.$$

Therefore

$$y_{m_2}(x) < c + \frac{4c}{b-a}\left(x - \frac{b+3a}{4}\right) - \frac{8c}{(b-a)^2}\left(x - \frac{b+3a}{4}\right)^2.$$

Since the right hand side of this inequality is strictly increasing on $\left[\frac{b+3a}{4}, \frac{b+a}{2}\right]$, it can be concluded that

$$y_{m_2}(e) < c + \frac{4c}{b-a}\left(\frac{b+a}{2} - \frac{b+3a}{4}\right) - \frac{8c}{(b-a)^2}\left(\frac{b+a}{2} - \frac{b+3a}{4}\right)^2 = \frac{3c}{2}.$$

By the definition of e, it implies that $e = (b+a)/2$ and by inequality (3.6) we have

$$y'_{m_2}\left(\frac{b+a}{2}\right) < \frac{4c}{b-a} - \frac{16c}{(b-a)^2}\left(\frac{b+a}{2} - \frac{b+3a}{4}\right) = 0$$

which contradicting with $y'_{m_2}((b+a)/2) \geq 0$. Therefore $y'_{m_2}((b+a)/2) < 0$.

Since m_2 is supremum of $S = \{s \geq 0 : y_s((b+3a)/4) \leq c\}$, then by continuous dependence of a solution on initial value m implies that $y_m((b+3a)/4) > c$ for all $m > m_2$. Let

$$T = \{s > m_2 : y'_s((b+a)/2) = 0\}.$$

By Lemma 3.1.2(a), $y'_m((b+a)/2) > 0$ and $y_m((b+a)/2) > d(b-a)^2$ for $m > 4d(b-a)$. Thus T is bounded above. Since $y'_{m_2}\left(\frac{b+a}{2}\right) < 0$ and $y'_m\left(\frac{b+a}{2}\right) > 0$ for $m > 4d(b-a)$, then by continuous dependence of a solution on initial value m there exists $s \in (m_2, m)$ such that $y'_s((b+a)/2) = 0$ so $T \neq \emptyset$. Let $s_3 = \sup T$. By continuous dependence of a solution on initial value m, we have

$$y'_{s_3}((b+a)/2) = 0.$$

Therefore $y_{s_3}(x)$ is a solution of the BVP (3.1), (3.2). Then from Lemma 3.1.2(b), $y_{s_3}((b+a)/2) < d(b-a)^2$ and from Lemma 3.1.2(a), $s_3\left(\frac{b+a}{2} - a\right) - 4d\left(\frac{b+a}{2} - a\right)^2 \leq y_{s_3}((b+a)/2)$, so $s_3 < 4d(b-a)$. By the definition of s_3 guarantees that there is no solution of the BVP (3.1), (3.2) with $||y|| \leq d(b-a)^2$ and $y'(a) > s_3$. By continuous dependence of a solution on initial value m, we can choose $m_3 \in (s_3, 4d(b-a))$ very close to s_3 such that

$$y'_m((b+a)/2) > 0 \text{ and } y_m((b+a)/2) < d(b-a)^2 \quad \text{for all } m \in (s_3, m_3].$$

3.2 Existence of Odd Number of Solutions

For odd number of solutions of the BVP (1.3), (1.4), we guarantee the existence of nonnegative symmetric solutions by the following theorem.

Theorem 3.2.1 *Suppose $k \geq 0$ is an integer, $0 < d_0(b-a)^2 < c_1 < \frac{d_1}{2}(b-a)^2 < \frac{c_2}{2} < \frac{d_2}{4}(b-a)^2 < \cdots < \frac{c_k}{2^{k-1}} < \frac{d_k}{2^k}(b-a)^2$, and $f : [a,b] \times [0, d_k(b-a)^2] \to [0, 8d_k]$*

is continuous with $f(x,y) \not\equiv 8d_k$ *on* $[a,(b+a)/2] \times [0, d_k(b-a)^2]$ *and satisfies*

$$f(x,y) \le 8d_0 \quad \text{for } a \le x \le b,\ 0 \le y \le d_0(b-a)^2;$$
$$f(x,y) \ge \frac{16c_j}{(b-a)^2} \quad \text{for } a \le x \le b,\ c_j \le y \le \frac{3c_j}{2},\ \text{for } j = 1,2,\ldots,k;$$
$$f(x,y) \le 8d_j \quad \text{for } a \le x \le b,\ 0 \le y \le d_j(b-a)^2,\ \text{for } j = 1,2,\ldots,k.$$

Then the boundary value problem (1.3), (1.4) has at least $2k+1$ *nonnegative symmetric solutions* $y_1, y_2, \ldots, y_{2k+1}$ *with* $0 \le y_1'(a) < y_2'(a) < \cdots < y_{2k+1}'(a)$*. Moreover,* $y_j(x)$ *is strictly positive on* (a,b) *for* $j = 2,3,\ldots,2k+1$, $y_{2j+1}((b+3a)/4) > c_j$ *for* $j = 1,2,\ldots,k$, *and* $y_1((b+a)/2) < d_0(b-a)^2 < y_2((b+a)/2) < y_3((b+a)/2) < d_1(b-a)^2 < \cdots < d_{k-1}(b-a)^2 < y_{2k}((b+a)/2) < y_{2k+1}((b+a)/2) < d_k(b-a)^2$*. Finally, if* $f(a,0) > 0$*, then* y_1 *is strictly positive on* (a,b)*.*

Now, we use Lemmas 3.1.3 and 3.1.4 to prove Theorem 3.2.1. We illustrate the proof for the case $k = 2$, where there are at least five nonnegative symmetric solutions. The proof for the general case is very similar.

From Lemma 3.1.3 (with $d = d_0$) gives us s_1, m_1 such that $0 \le s_1 < m_1 < 4d_0(b-a)$ and

$$y_{s_1}'((b+a)/2) = 0,\ y_{s_1}((b+a)/2) < d_0(b-a)^2,$$
$$y_{m_1}'((b+a)/2) > 0,\ y_{m_1}((b+a)/2) < d_0(b-a)^2.$$

By using Lemma 3.1.4 (with $c = c_1$, $d = d_1$), we obtain m_2, s_3, m_3 such that $0 < m_2 < s_3 < m_3 < 4d_1(b-a)$ and

$$y_{m_2}'((b+a)/2) < 0, \quad y_{m_2}((b+3a)/4) = c_1,$$
$$y_m((b+3a)/4) > c_1 \ \text{ for all } \ m > m_2 \ \text{ for which } ||y_m|| \le d_1(b-a)^2,$$
$$y_{s_3}'((b+a)/2) = 0, \quad ||y_{s_3}|| < d_1(b-a)^2,$$
$$y_{m_3}'((b+a)/2) > 0, \quad y_{m_3}((b+a)/2) < d_1(b-a)^2.$$

By applying Lemma 3.1.4 again (with $c = c_2$, $d = d_2$), we obtain m_4, s_5, m_5 such that $0 < m_4 < s_5 < m_5 < 4d_2(b-a)$ and

$$y'_{m_4}((b+a)/2) < 0, \quad y_{m_4}((b+3a)/4) = c_2,$$
$$y_m((b+3a)/4) > c_2 \text{ for all } m > m_4 \text{ for which } ||y_m|| \le d_2(b-a)^2,$$
$$y'_{s_5}((b+a)/2) = 0, \quad ||y_{s_5}|| < d_2(b-a)^2,$$
$$y'_{m_5}((b+a)/2) > 0, \quad y_{m_5}((b+a)/2) < d_2(b-a)^2.$$

We claim that $m_1 < m_2$. Supposing the contrary, that is $m_1 \ge m_2$, from above we have $y_{m_1}((b+3a)/4) \ge c_1$ and $y_{m_1}((b+a)/2) < d_0(b-a)^2$. But $y_{m_1}(x)$ is increasing on $[(b+3a)/4, (b+a)/2]$, so $c_1 \le y_{m_1}((b+3a)/4) < y_{m_1}((b+a)/2) < d_0(b-a)^2$, then it contradicts with $d_0(b-a)^2 < c_1$. Therefore $m_1 < m_2$. Since $y'_{m_1}\left(\frac{b+a}{2}\right) > 0$ and $y'_{m_2}\left(\frac{b+a}{2}\right) < 0$, then by continuous dependence of a solution on initial value m there exists $s_2 \in (m_1, m_2)$ such that

$$y'_{s_2}((b+a)/2) = 0.$$

Similarly, we have $m_3 < m_4$ because if $m_3 \ge m_4$ then $y_{m_3}((b+3a)/4) \ge c_2$ and $y_{m_3}((b+a)/2) < d_1(b-a)^2$. But $y_{m_3}(x)$ is increasing on $[(b+3a)/4, (b+a)/2]$, so $c_2 \le y_{m_3}((b+3a)/4) < y_{m_3}((b+a)/2) < d_1(b-a)^2$, which contradicts with $d_1(b-a)^2 < c_2$. Since $y'_{m_3}((b+a)/2) > 0$ and $y'_{m_4}((b+a)/2) < 0$, then by continuous dependence of a solution on initial value m there exists $s_4 \in (m_3, m_4)$ such that

$$y'_{s_4}((b+a)/2) = 0.$$

Since $0 \le s_1 < m_1$, $s_2 \in (m_1, m_2)$, $0 < m_2 < s_3 < m_3$, $s_4 \in (m_3, m_4)$, and $0 < m_4 < s_5 < m_5$, so $0 \le s_1 < s_2 < s_3 < s_4 < s_5$. Therefore, $y_{s_1}, y_{s_2}, y_{s_3}, y_{s_4}, y_{s_5}$ are five nonnegative solutions of the BVP (3.1), (3.2). Since $y'_{s_k}(a) = s_k$, we have

$$0 \le y'_{s_1}(a) < y'_{s_2}(a) < y'_{s_3}(a) < y'_{s_4}(a) < y'_{s_5}(a).$$

Next, we will show that y_{s_j} is strictly positive on (a,b) for $j = 2,3,4,5$. Since we know that $y_{s_j}(x)$ is non-decreasing on $[a, (b+a)/2]$, and $y'_{s_j}(a) > s_1 \ge 0$, $y_{s_j}(a) = 0$ for $j = 2,3,4,5$, then $y_{s_j}(x) > 0$ on $(a, (b+a)/2]$ for $j = 2,3,4,5$. By the symmetry about $x = (b+a)/2$, hence $y_{s_j}(x)$ is strictly positive on (a,b) for $j = 2,3,4,5$.

Next, we show that $y_{s_3}((b+3a)/4) > c_1$ and $y_{s_5}((b+3a)/4) > c_2$. Since $s_3 > m_2$ and $||y_{s_3}|| < d_1(b-a)^2$, by Lemma 3.1.4 $y_{s_3}((b+3a)/4) > c_1$. And since $s_5 > m_4$ and $||y_{s_5}|| < d_2(b-a)^2$, by Lemma 3.1.4 $y_{s_5}((b+3a)/4) > c_2$. Hence $y_{s_{2j+1}}((b+3a)/4) > c_j$ for $j = 1, 2$.

Next, we show that $y_{s_1}((b+a)/2) < d_0(b-a)^2 < y_{s_2}((b+a)/2) < y_{s_3}((b+a)/2) < d_1(b-a)^2 < y_{s_4}((b+a)/2) < y_{s_5}((b+a)/2) < d_2(b-a)^2$. Since $y'_{s_2}(a) = s_2 > s_1$, by the last line in Lemma 3.1.3 and by $y_{s_2}(x)$ is non-decreasing on $[a, (b+a)/2]$, then we have $y_{s_2}((b+a)/2) = ||y_{s_2}|| > d_0(b-a)^2$. Since $y'_{s_4}(a) = s_4 > s_3$, by the last line in Lemma 3.1.4 and the fact that $y_{s_4}(x)$ is non-decreasing on $[a, (b+a)/2]$, then we have $y_{s_4}((b+a)/2) = ||y_{s_4}|| > d_1(b-a)^2$. Hence, we have

$$\begin{aligned} &y_{s_1}((b+a)/2) < d_0(b-a)^2 < y_{s_2}((b+a)/2),\\ &y_{s_3}((b+a)/2) < d_1(b-a)^2 < y_{s_4}((b+a)/2),\\ &y_{s_5}((b+a)/2) < d_2(b-a)^2. \end{aligned}$$

Therefore, we must show that $y_{s_2}((b+a)/2) < y_{s_3}((b+a)/2)$ and $y_{s_4}((b+a)/2) < y_{s_5}((b+a)/2)$. Suppose that $y_{s_2}((b+a)/2) \geq y_{s_3}((b+a)/2)$, by Lemma 3.1.1

$$\begin{aligned} F\left(x, y_{s_2}\left(\frac{b+a}{2}\right)\right) - F\left(x, y_{s_3}\left(\frac{b+a}{2}\right)\right) &= \left[\frac{\left(y'_{s_2}(a)\right)^2}{2} - \frac{\left(y'_{s_2}\left(\frac{b+a}{2}\right)\right)^2}{2}\right] \\ &\quad - \left[\frac{\left(y'_{s_3}(a)\right)^2}{2} - \frac{\left(y'_{s_3}\left(\frac{b+a}{2}\right)\right)^2}{2}\right] \\ &= \frac{(s_2)^2}{2} - \frac{(s_3)^2}{2} \quad \geq 0, \end{aligned}$$

but $0 < s_2 < s_3$ such that $(s_2)^2 - (s_3)^2 < 0$, so we have a contradiction. Therefore $y_{s_2}((b+a)/2) < y_{s_3}((b+a)/2)$.

Similarly, suppose that $y_{s_4}((b+a)/2) \geq y_{s_5}((b+a)/2)$. By Lemma 3.1.1, we have

$$\begin{aligned} F\left(x, y_{s_4}\left(\frac{b+a}{2}\right)\right) - F\left(x, y_{s_5}\left(\frac{b+a}{2}\right)\right) &= \left[\frac{\left(y'_{s_4}(a)\right)^2}{2} - \frac{\left(y'_{s_4}\left(\frac{b+a}{2}\right)\right)^2}{2}\right] \\ &\quad - \left[\frac{\left(y'_{s_5}(a)\right)^2}{2} - \frac{\left(y'_{s_5}\left(\frac{b+a}{2}\right)\right)^2}{2}\right] \end{aligned}$$

$$= \frac{(s_4)^2}{2} - \frac{(s_5)^2}{2} \geq 0,$$

but $0 < s_4 < s_5$ such that $(s_4)^2 - (s_5)^2 < 0$ which is a contradiction, therefore $y_{s_4}((b+a)/2) < y_{s_5}((b+a)/2)$.

Finally, we will show that if $f(a,0) > 0$, then $y_{s_1}(x)$ is strictly positive on (a,b). Since $y_s'(x)$ is non-increasing on $[a,(b+a)/2]$, hence if $f(a,0) > 0$, then $f(a,y_s(a)) = f(a,0) > 0$ and $y_s''(a) = -f(a,y_s(a)) < 0$. Thus $y_s'(x)$ is strictly decreasing at $x = a$ and since $y_0'(a) = 0$, hence $y_0'(x) < 0$ on $(a,(b+a)/2]$. Thus $y_0'((b+a)/2) < 0$. Let

$$\bar{S} = \{s \geq 0 : y_s'((b+a)/2) = 0\}.$$

By Lemma 3.1.2(a) $y_m'((b+a)/2) > 0$ if $m > 4d_0(b-a)$, thus $\bar{S}$ is bounded above. Hence by continuous dependence of a solution on initial value m there exists $s \in (0, 4d_0(b-a)]$ such that $y_s'((b+a)/2) = 0$. Then $\bar{S} \neq \emptyset$. Let $s_1 = \sup \bar{S}$. Then by continuous dependence of a solution on initial value m, we get

$$y_{s_1}'((b+a)/2) = 0.$$

Therefore $y_{s_1}(x)$ is a solution of the BVP (3.1), (3.2). Since $y_{s_1}'(a) = s_1 > 0$ and $y_{s_1}(a) = 0$, then $y_{s_1}(x) > 0$ on $(a,(b+a)/2]$. By the symmetry about $x = (b+a)/2$ it is clear that $y_{s_1}(x)$ is strictly positive on (a,b). Thus we proved the Theorem 3.2.1 (case $k = 2$).

3.3 Existence of Even Number of Solutions

For even number of solutions of the BVP (1.3), (1.4) we guarantee the existence of nonnegative symmetric solutions by the following theorem.

Theorem 3.3.1 *In addition to the hypotheses for Theorem 3.2.1, also assume $f : [a,b] \times [0,\infty) \to [0,\infty)$ is continuous and*

$$\frac{f(x,y)}{y} \to \infty \quad as \quad y \to \infty.$$

Then the boundary value problem (1.3), (1.4) has at least $2k+2$ nonnegative symmetric solutions $y_1, y_2, \cdots, y_{2k+2}$; the $2k+1$ solutions of Theorem 3.2.1 and one additional solution y_{2k+2} satisfying $y_{2k+2}'(a) > y_{2k+1}'(a)$.

Here, we illustrate the proof of Theorem 3.3.1 for the case $k = 2$. Five solutions of the BVP (1.3), (1.4) are guaranteed by the Theorem 3.2.1, we will now find the sixth solution. Since $\frac{f(x,y)}{y} \to \infty$ as $y \to \infty$, then there exists $Q > 0$ such that $f(x,y) \geq y$ if $y \geq Q(b-a)$. Consider $u < Q(b-a)$ gives

$$\begin{aligned}\int_0^u f(x,y)dy &< \int_0^{Q(b-a)} f(x,y)dy \\ &\leq M^* \int_0^{Q(b-a)} dy \\ &= M^* Q(b-a) \\ &= \frac{M_1^2}{4},\end{aligned}$$

where

$$M^* = \sup_{y\in[0,Q(b-a)]} f(x,y) \quad \text{and} \quad M_1 = 2\sqrt{M^*Q(b-a)}.$$

Hence there exists M_1 such that

$$u < Q(b-a) \quad \text{implies} \quad \int_0^u f(x,y)dy < \frac{M_1^2}{4},$$

or

$$\int_0^u f(x,y)dy \geq \frac{M_1^2}{4} \quad \text{implies} \quad u \geq Q(b-a). \tag{3.7}$$

Let $M = \max\{\sqrt{2}Q, M_1\}$. Now we show that for $m \geq M$ there exists $\beta(m)$ for which $y_m'(\beta(m))$, where y_m is a solution of IVP(m). Suppose no such $\beta(m)$ exists such that $y_m'(\beta(m)) = 0$, then $y_m'(x) > 0$ for $x \geq a$. By Lemma 3.1.1

$$\begin{aligned}\frac{(y_m'(a))^2}{2} - \frac{(y_m'(b))^2}{2} &= \int_0^{y_m(b)} f(x,y)dy \\ \frac{m^2}{2} &= \frac{(y_m'(b))^2}{2} + \int_0^{y_m(b)} f(x,y)dy.\end{aligned}$$

Then either

$$(1) \quad \frac{(y_m'(b))^2}{2} \geq \frac{m^2}{4} \qquad \text{or} \qquad (2) \quad \int_0^{y_m(b)} f(x,y)dy \geq \frac{m^2}{4}.$$

In case (1),

$$(y_m'(b))^2 \geq \frac{m^2}{2},$$

or

$$y'_m(b) \geq \frac{m}{\sqrt{2}} \geq \frac{M}{\sqrt{2}} \geq Q.$$

Since $y'_m(x)$ is non-increasing, then $y'_m(x) \geq y'_m(b) \geq Q$ on $[a,b]$. Then

$$\int_a^b y'_m(s)ds \geq \int_a^b Qds$$
$$y_m(b) - y_m(a) \geq Q(b-a)$$
$$y_m(b) \geq Q(b-a).$$

In case (2),

$$\int_0^{y_m(b)} f(x,y)dy \geq \frac{m^2}{4} \geq \frac{M^2}{4} \geq \frac{M_1^2}{4},$$

so (3.7) implies that $y_m(b) \geq Q(b-a)$. Therefore, in either case, $y_m(b) \geq Q(b-a)$ for $m \geq M$. Since $y'_m(x) > 0$ for $x \geq a$, then $y_m(x)$ is increasing. Thus $y_m(x) \geq y_m(b) \geq Q(b-a)$ for $x \geq b$ implies

$$f(x, y_m(x)) \geq y_m(x) \geq Q(b-a)$$

and hence

$$y''_m(x) = -f(x, y_m(x)) \leq -Q(b-a).$$

Integrating with respect to x from a to x, we get

$$y'_m(x) - y'_m(a) \leq -Q(b-a)(x-a)$$
$$y'_m(x) - m \leq -Q(b-a)(x-a),$$

so $y'_m(x) \leq m - Q(b-a)(x-a) \to -\infty$ as $x \to \infty$, contradicting with $y'_m(x) > 0$ for $x \geq a$. Therefore, there exists $\beta(m)$ such that $y'_m(\beta(m)) = 0$.

Next, we will prove that

$$\beta(m) \to a \quad \text{as} \quad m \to \infty.$$

Let $\varepsilon > 0$ be given and choose $k > \frac{8}{\varepsilon^2}$. Since $\frac{f(x,y)}{y} \to \infty$ as $y \to \infty$, there exists y_k such that

$$\frac{f(x,y)}{y} \geq k \quad \text{if} \quad y \geq y_k.$$

By Lemma 3.1.1 we have (here, y means y_m)

$$\frac{(y'(a))^2}{2} - \frac{(y'(\beta(m)))^2}{2} = \int_0^{y(\beta(m))} f(x,u)du$$
$$\frac{m^2}{2} = \int_0^{y(\beta(m))} f(x,u)du,$$

so $y(\beta(m)) \to \infty$ as $m \to \infty$. Thus there exists $N > 0$ so that $y(\beta(m)) \geq 2y_k$ if $m \geq N$. Since $y''(x) \leq 0$ on $a \leq x \leq \beta(m)$, then $y(x)$ is concave function on $a \leq x \leq \beta(m)$. Therefore, by Definition 2.8.2 with $c = a$, $d = \beta(m)$, $\lambda = 1/2$, and $\mu = 1/2$, we have

$$y\left(\frac{a+\beta(m)}{2}\right) \geq \frac{y(\beta(m))}{2}. \tag{3.8}$$

Hence,

$$y(x) \geq \frac{y(\beta(m))}{2} \quad \text{for} \quad \frac{a+\beta(m)}{2} \leq x \leq \beta(m). \tag{3.9}$$

Then for $m \geq N$, we have $y(x) \geq \frac{y(\beta(m))}{2} \geq y_k$, so

$$\frac{f(x,y(x))}{y(x)} \geq k \quad \text{for} \quad \frac{a+\beta(m)}{2} \leq x \leq \beta(m)$$

and hence, by (3.9), $f(x,y(x)) \geq ky(x) \geq \frac{k}{2}y(\beta(m))$. Thus

$$y''(x) = -f(x,y(x)) \leq -\frac{k}{2}y(\beta(m)).$$

Integrating with respect to x from x to $\beta(m)$, we get

$$y'(\beta(m)) - y'(x) \leq -\frac{k}{2}y(\beta(m))[\beta(m) - x]$$
$$y'(x) \geq \frac{k}{2}y(\beta(m))[\beta(m) - x].$$

Integrating with respect to x from $\frac{a+\beta(m)}{2}$ to $\beta(m)$, we get

$$y(\beta(m)) - y\left(\frac{a+\beta(m)}{2}\right) \geq -\frac{k}{2}y(\beta(m))\frac{(\beta(m)-x)^2}{2}\Bigg|_{x=\frac{a+\beta(m)}{2}}^{\beta(m)}$$
$$y(\beta(m)) \geq y\left(\frac{a+\beta(m)}{2}\right) + \frac{k}{16}y(\beta(m))(\beta(m)-a)^2. \tag{3.10}$$

Combining the inequalities (3.8) and (3.10), we get

$$y(\beta(m)) \geq \frac{y(\beta(m))}{2} + \frac{k}{16}y(\beta(m))(\beta(m)-a)^2$$
$$\frac{y(\beta(m))}{2} \geq \frac{k}{16}y(\beta(m))(\beta(m)-a)^2.$$

Then,

$$(\beta(m)-a)^2 \le \frac{8}{k} < \varepsilon^2 \quad \text{for} \quad m \ge N$$

or

$$|\beta(m)-a| < \varepsilon \quad \text{if} \quad m \ge N.$$

Thus

$$\beta(m) \to a \quad \text{as} \quad m \to \infty.$$

Consequently there exists $m_6 > m_5$ so that $\beta(m_6) < \frac{b+a}{2}$. Since y'_{m_6} is non-increasing and $y'_{m_6}(\beta(m_6)) = 0$, then $y'_{m_6}((b+a)/2) < 0$. And since $y'_{m_5}((b+a)/2) > 0$, then by continuous dependence of a solution on initial value m there exists $s_6 \in (m_5, m_6)$ such that

$$y'_{s_6}((b+a)/2) = 0.$$

Therefore $y_{s_6}(x)$ is the sixth solution of the BVP (1.3), (1.4). Since $s_6 > m_5 > s_5$ and $y'_{s_6}(a) = s_6, y'_{s_5}(a) = s_5$, then $y'_{s_6}(a) > y'_{s_5}(a)$. Thus we proved the Theorem 3.3.1 for the case $k = 2$.

3.4 Example of Existence of Multiple Solutions

Example 1 Consider the nonlinear BVP

$$\begin{gathered} y''(x) + f(x,y) = 0, \quad -1 \le x \le 1 \\ y(-1) = 0, \; y(1) = 0 \end{gathered} \tag{3.11}$$

when $\quad f(x,y) = x^2 + \frac{15}{64}(y-8)^2$.

Theorem 3.2.1 with $a = -1, b = 1$ can be used to guarantee that the BVP (3.11) has nonnegative symmetric solution but we must check that the nonlinear function $f(x,y)$ satisfies the hypotheses of the Theorem. It is easy to see that $f(x,y)$ is continuous and $f(x,y) \ge 0$ for all $x, y \in \mathbb{R}$. Let $d_0 = 2$. Since $\frac{15}{64}(y-8)^2$ is a concave upward parabola which has a value of 15 at $y = 0$ and its vertex is at (8,0), then $f(x,y) \le 8d_0 = 16$ for $-1 \le x \le 1$, $0 \le y \le 8$ and $f(x,y) \not\equiv 8d_0$ for $-1 \le x \le 0$, $0 \le y \le 8$. Therefore $f(x,y)$ satisfies the hypotheses of the Theorem 3.2.1 (with $a = -1$, $b = 1$, $k = 0$, $d_0 = 2$). By this theorem, it guarantees that the BVP (3.11) has at least one nonnegative symmetric solution. Moreover, we see that $f(x,y)$ satisfied the hypotheses of Theorem 3.3.1 (with $k = 0$), when $f : [-1,1] \times [0,\infty) \to [0,\infty)$ is continuous and $\frac{f(x,y)}{y} \to \infty$ as $y \to \infty$. Hence by this theorem, we guarantee that the BVP (3.11) has at least two nonnegative symmetric solutions.

Now we also use numerical techniques to find the solution of the BVP (3.11) by the shooting method. The numerical solutions of the BVP (3.11) computed by the computer program (See Appendix) gave two solutions:

- First solution; we used the initial guess equal to 0, the number of subintervals $N = 20$. The computer program computed 9 iterations and gave the solution which agrees with the tolerance 10^{-10}. See Table 3.1 and Figure 3.1.

- Second solution; we used the initial guess equal to 30, the number of subintervals $N = 20$. The computer program computed 10 iterations and gave the solution which agrees with the tolerance 10^{-10}. See Table 3.2 and Figure 3.2.

Table 3.1: First numerical solution of the BVP (3.11)

i	x(i)	y(x)	y′(x)
0	-1.0	0.0000000000000000	8.1938896123390299
1	-0.9	0.7444661877195654	6.7431290715391982
2	-0.8	1.3568853860659056	5.5414511184825456
3	-0.7	1.8590519929121781	4.5295383884414781
4	-0.6	2.2676130229014019	3.6628556566256886
5	-0.5	2.5953087831662773	2.9071908450798168
6	-0.4	2.8518429396817270	2.2355976248904484
7	-0.3	3.0444955076214269	1.6262459671081751
8	-0.2	3.1785527088253476	1.0608667836032764
9	-0.1	3.2576021156628347	0.5235848805315594
10	0.0	3.2837244770099335	-0.0000000000147681
11	0.1	3.2576017412496402	-0.5235849359632309
12	0.2	3.1785519337811193	-1.0608670104139699
13	0.3	3.0444942765575132	-1.6262464976472477
14	0.4	2.8518411615676823	-2.2355986218607045
15	0.5	2.5953063200535598	-2.9071925209558204
16	0.6	2.2676096714584422	-3.6628583026785206
17	0.7	1.8590474545539198	-4.5295424204317951
18	0.8	1.3568792182146002	-5.5414571515918209
19	0.9	0.7444577227524444	-6.7431380478094479
20	1.0	-0.0000117929704186	-8.1939030253105505

Table 3.2: Second numerical solution of the BVP (3.11)

i	x(i)	y(x)	y'(x)
0	-1.0	0.0000000000000000	37.9165115652640932
1	-0.9	3.7326746625046474	36.9193748438277893
2	-0.8	7.4091359853225039	36.6824772141664247
3	-0.7	11.0730998422017035	36.5637814359823829
4	-0.6	14.7048221252360724	35.9344108715344949
5	-0.5	18.2228093076194845	34.1954029591430659
6	-0.4	21.4897711050421518	30.8443340569606466
7	-0.3	24.3271665863628789	25.5766061065010269
8	-0.2	26.5403145058625578	18.3885099409168754
9	-0.1	27.9519519246286875	9.6350808111660249
10	0.0	28.4374372998368664	0.0000000000080665
11	0.1	27.9518341505213612	-9.6350097066045095
12	0.2	26.5401107519550266	-18.3882534264061800
13	0.3	24.3269272440298793	-25.5761162288271762
14	0.4	21.4895473825207056	-30.8436302872132383
15	0.5	18.2226406608845775	-34.1945440336678978
16	0.6	14.7047329625130041	-35.9334601658768878
17	0.7	11.0731032606661878	-36.5627825484005668
18	0.8	7.4092399593334609	-36.6814426567150764
19	0.9	3.7328873444450589	-36.9182821950839497
20	1.0	0.0003329008697808	-37.9153004002536809

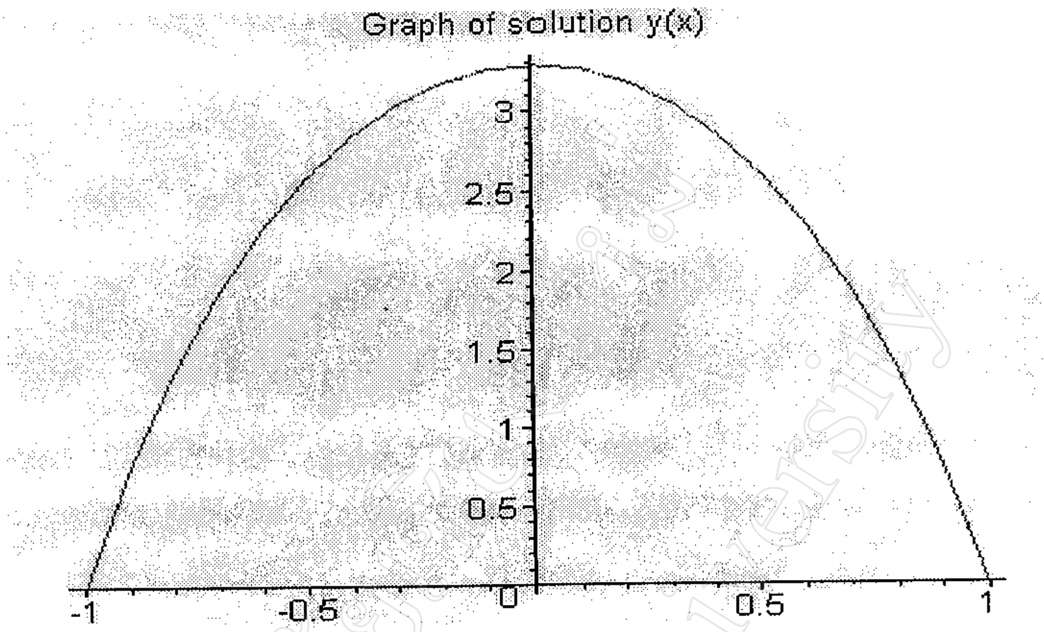

Figure 3.1: First numerical solution of the BVP (3.11)

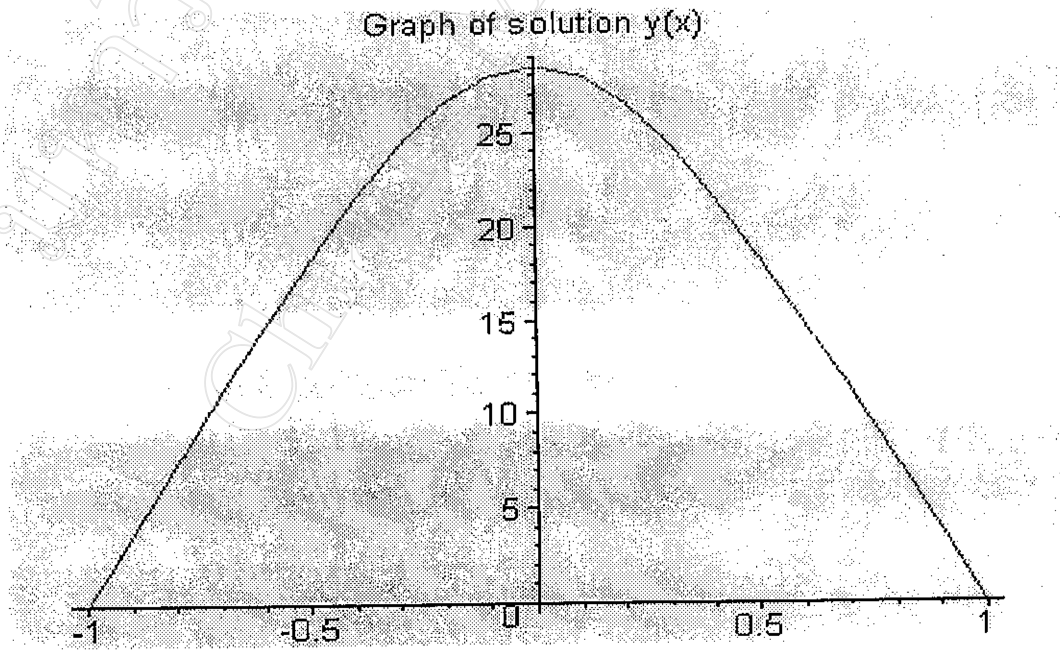

Figure 3.2: Second numerical solution of the BVP (3.11)

Example 2 Consider the nonlinear BVP

$$\begin{aligned} y''(x) + f(x,y) &= 0, \quad 0 \le x \le 1 \\ y(0) = 0, \ y(1) &= 0 \end{aligned} \tag{3.12}$$

when

$$f(x,y) = x^2 + \frac{227}{27095040}(y-2)^6 - \frac{3191}{22579200}(y-2)^5 - \frac{66677}{33868800}(y-2)^4 - \frac{148853}{1128960}(y-2)^3 + \frac{14787587}{4233600}(y-2)^2.$$

Theorem 3.2.1 with $a = 0, b = 1$ can be used to guarantee that the BVP (3.12) has nonnegative symmetric solution but we must check that the nonlinear function $f(x,y)$ satisfies the hypotheses of the Theorem. It is easy to see that $f(x,y)$ is continuous for all $x, y \in \mathbb{R}$. Consider

$$g(y) = \frac{227}{27095040}(y-2)^6 - \frac{3191}{22579200}(y-2)^5 - \frac{66677}{33868800}(y-2)^4 - \frac{148853}{1128960}(y-2)^3 + \frac{14787587}{4233600}(y-2)^2.$$

and its graph (Figure 3.3).

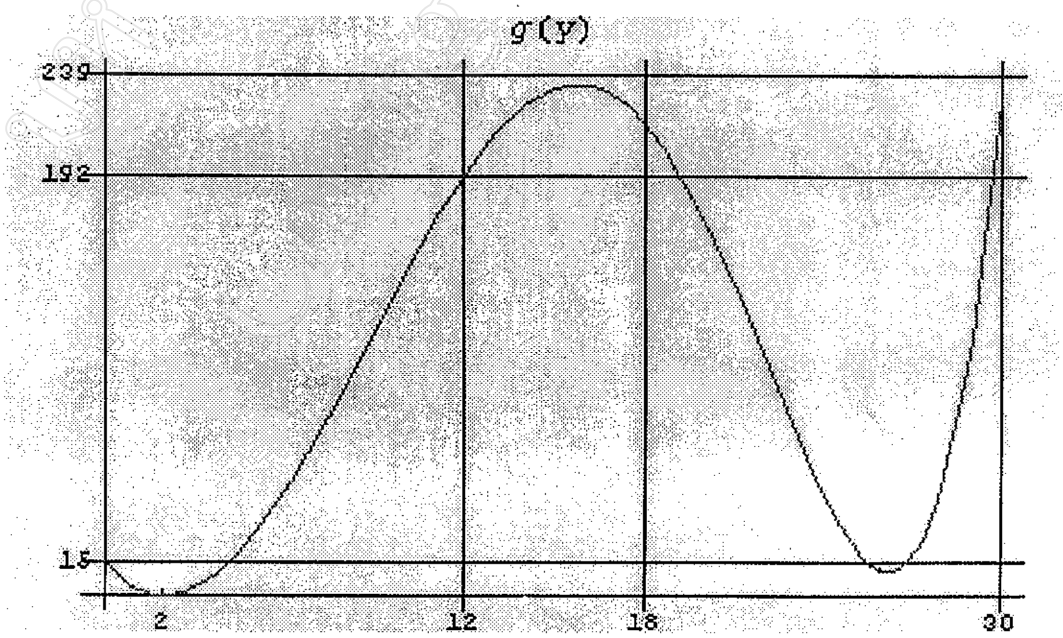

Figure 3.3: Graph of $g(y)$

Let $d_0 = 2, c_1 = 12$, and $d_1 = 30$, then $0 < d_0 < c_1 < \frac{d_1}{2}$. The graph of $g(y)$ shown that $f(x,y)$ satisfied the hypotheses of Theorem 3.2.1 (with $a = 0, b = 1, k = 1, d_0 = 2, c_1 = 12, d_1 = 30$), that is

$$f : [0,1]\times[0,d_j] \to [0, 8d_j] \text{ is continuous,}$$
$$f(x,y) \not\equiv 8d_j \text{ on } [0,1/2]\times[0,d_j] \quad \text{for } j = 0,1$$

and

$$f(x,y) \le 8d_0 \quad \text{for} \quad 0 \le x \le 1,\ 0 \le y \le d_0$$
$$f(x,y) \ge 16c_1 \quad \text{for} \quad 0 \le x \le 1,\ c_1 \le y \le \frac{3c_1}{2}$$
$$f(x,y) \le 8d_1 \quad \text{for} \quad 0 \le x \le 1,\ 0 \le y \le d_1.$$

Hence, by this theorem it guarantees that the BVP (3.12) has at least three nonnegative symmetric solutions. Moreover, we see that $f(x,y)$ satisfied the hypotheses of Theorem 3.3.1 (with $k = 1$), hence by this theorem, it guarantees that the BVP 3.12 has at least four nonnegative symmetric solutions.

Now we also use numerical techniques to find the solution of the BVP (3.12) by the shooting method. The numerical solutions of the BVP (3.12) computed by the same computer program used in Example 1 gave four solutions:

- First solution; we used the initial guess equal to 0, the number of subintervals $N = 20$. The computer program computed 9 iterations and gave the solution which agrees with the tolerance 10^{-10}. See Table 3.3 and Figure 3.4.

- Second solution; we used the initial guess equal to 20, the number of subintervals $N = 20$. The computer program computed 8 iterations and gave the solution which agrees with the tolerance 10^{-10}. See Table 3.4 and Figure 3.5.

- Third solution; we used the initial guess equal to 70, the number of subintervals $N = 20$. The computer program computed 16 iterations and gave the solution which agrees with the tolerance 10^{-10}. See Table 3.5 and Figure 3.6.

- Fourth solution; we used the initial guess equal to 92, the number of subintervals $N = 20$. The computer program computed 20 iterations and gave the solution which agrees with the tolerance 10^{-10}. See Table 3.6 and Figure 3.7.

Table 3.3: First numerical solution of the BVP (3.12)

i	x(i)	y(x)	y′(x)
0	0.00	0.0000000000000000	3.9596163143885726
1	0.05	0.1804171813537289	3.2792919580755563
2	0.10	0.3298394773277301	2.7142267576573571
3	0.15	0.4532750503978927	2.2357122128007289
4	0.20	0.5544924618709629	1.8224086324876682
5	0.25	0.6363285305837084	1.4580765389347280
6	0.30	0.7009025686136290	1.1300428548932863
7	0.35	0.7497664512771232	0.8281329822523630
8	0.40	0.7840093203559343	0.5439018863240614
9	0.45	0.8043289977095700	0.2700568786639476
10	0.50	0.8110777793475423	-0.0000000000148605
11	0.55	0.8042872375266436	-0.2725615300350985
12	0.60	0.7836743935743693	-0.5539771551904443
13	0.65	0.7486297429317500	-0.8510230065283991
14	0.70	0.6981858129381043	-1.1713171963627460
15	0.75	0.6309629280461581	-1.5238246227862801
16	0.80	0.5450863133712184	-1.9195103163148153
17	0.85	0.4380651227237220	-2.3722264437514528
18	0.90	0.3066187125367130	-2.8999623489830750
19	0.95	0.1464273043151483	-3.5266621777123780
20	1.00	-0.0482289910373764	-4.2849445233542976

Table 3.4: Second numerical solution of the BVP (3.12)

i	x(i)	y(x)	y′(x)
0	0.00	0.0000000000000000	23.0534241583715065
1	0.05	1.1401679966888393	22.6562320619280984
2	0.10	2.2716085600433971	22.6213970653774877
3	0.15	3.4005807762515446	22.4855682037600673
4	0.20	4.5117893734521522	21.8514737759736440
5	0.25	5.5722109326762490	20.4131538747729660
6	0.30	6.5366033050965797	17.9934387280360228
7	0.35	7.3546363837452189	14.5675812398743733
8	0.40	7.9784980048853267	10.2575161002460330
9	0.45	8.3694969185456680	5.2993427585959290
10	0.50	8.5026953594982496	-0.0000000000086268
11	0.55	8.3694403285430175	-5.3018108668591312
12	0.60	7.9781417114573551	-10.2672350831409436
13	0.65	7.3535089742021712	-14.5889027485536655
14	0.70	6.5340310301442479	-18.0301255570414196
15	0.75	5.5673452178238870	-20.4684896348792641
16	0.80	4.5036188480044519	-21.9287906431535535
17	0.85	3.3879051132606579	-22.5892711160692743
18	0.90	2.2529490221677306	-22.7585051074665657
19	0.95	1.1135862812907824	-22.8385612737623269
20	1.00	-0.0372179049769395	-23.3009249357420766

Table 3.5: Third numerical solution of the BVP (3.12)

i	x(i)	y(x)	y′(x)
0	0.00	0.0000000000000000	73.7951528102731729
1	0.05	3.6834322698622325	73.5894657482668082
2	0.10	7.3280691608219056	71.6319883360008790
3	0.15	10.7755865723967543	65.5436071512241385
4	0.20	13.8181610225647223	55.6826751002012662
5	0.25	16.3155345820381524	44.1168851978731227
6	0.30	18.2356606672366997	32.8534173115936524
7	0.35	19.6233343077121003	22.9021997992896931
8	0.40	20.5498285308909401	14.3746709204378293
9	0.45	21.0785794195629738	6.9133497365448877
10	0.50	21.2502481077481563	0.0000000000849718
11	0.55	21.0785049716919877	-6.9159311774462994
12	0.60	20.5494172599755712	-14.3851977056242417
13	0.65	19.6220451383154801	-22.9265278041151158
14	0.70	18.2326563092525452	-32.8976487001175670
15	0.75	16.3097046490244860	-44.1858103896611030
16	0.80	13.8082500059789492	-55.7767768047350970
17	0.85	10.7604461513192919	-65.6575069310302784
18	0.90	7.3068636253577398	-71.7596081538684494
19	0.95	3.6554059200738877	-73.7363972915171072
20	1.00	-0.0364214856913980	-73.9921369576122400

Table 3.6: Fourth numerical solution of the BVP (3.12)

i	x(i)	y(x)	y′(x)
0	0.00	0.0000000000000000	95.3011617740843861
1	0.05	4.7583979156269380	94.9616211941741420
2	0.10	9.4363663384360167	91.2661091843286499
3	0.15	13.7906626915740554	82.1530453775351050
4	0.20	17.6105650815883782	70.6322683199884437
5	0.25	20.8894023200845919	61.1517175292952170
6	0.30	23.8030446451737256	56.1913086418456092
7	0.35	26.5739479162068307	54.9719325334906776
8	0.40	29.2748825691403803	51.9423330402312398
9	0.45	31.5402059613903927	35.6030901411083503
10	0.50	32.4768509774475420	0.0000000000662947
11	0.55	31.5282183924661581	-35.5102661875351842
12	0.60	29.2671797989037665	-51.7614473148958590
13	0.65	26.5753202714266651	-54.7867826251468857
14	0.70	23.8134403336095849	-56.0153951575965097
15	0.75	20.9085964452331216	-60.9733834553444104
16	0.80	17.6386808944853566	-70.4544733517294481
17	0.85	13.8270549152220159	-82.0073575645227538
18	0.90	9.4782852813199256	-91.1973006679501217
19	0.95	4.8014060770403993	-94.9832406598057274
20	1.00	0.0404180929923909	-95.3725885499345888

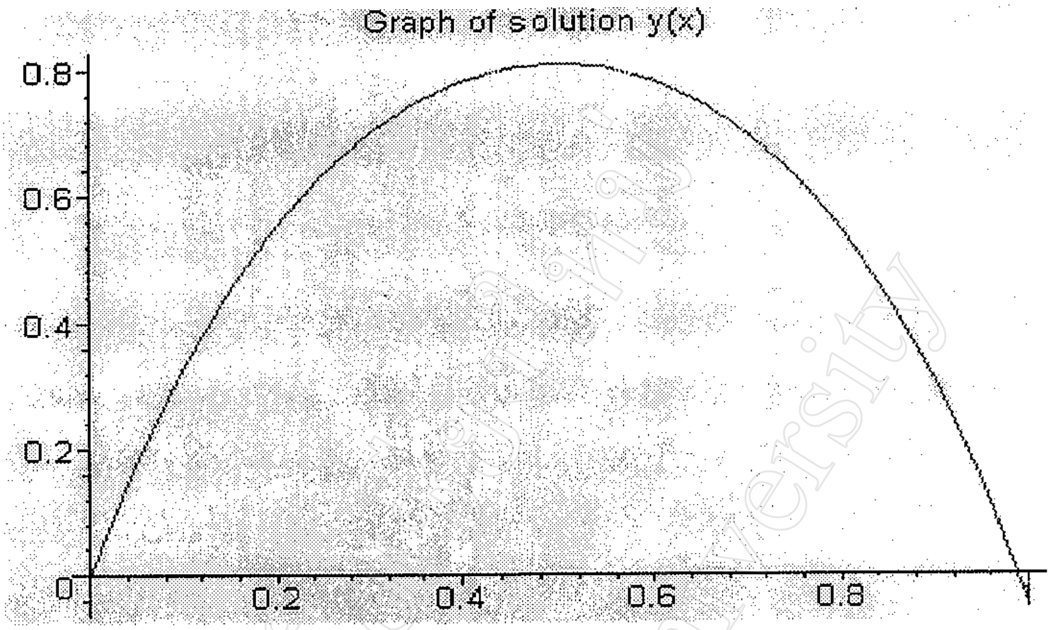

Figure 3.4: First numerical solution of the BVP (3.12)

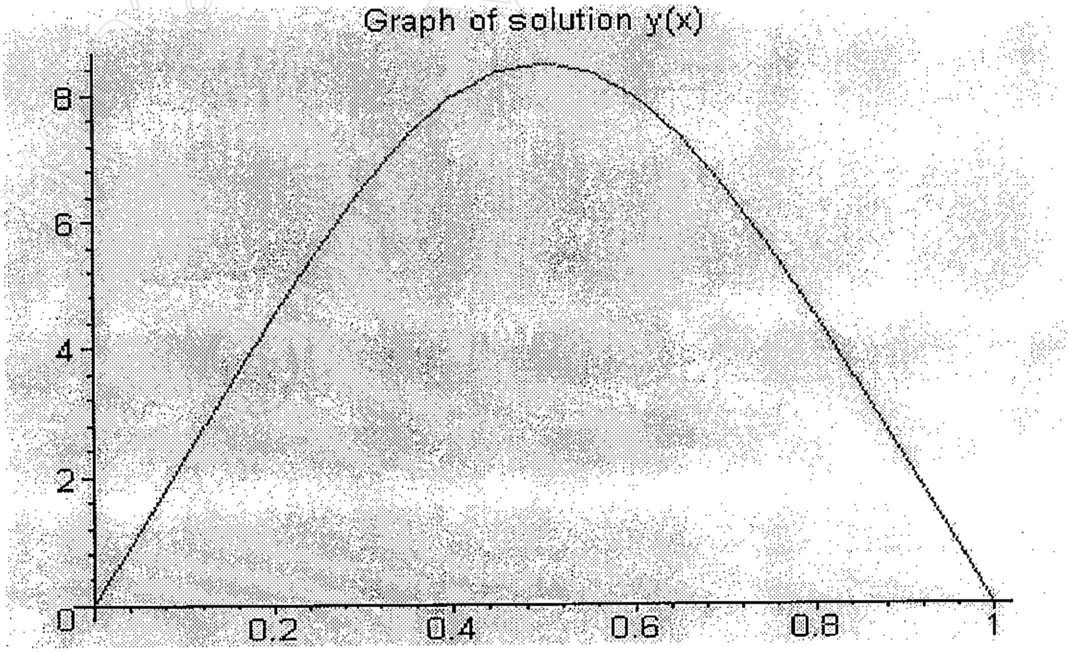

Figure 3.5: Second numerical solution of the BVP (3.12)

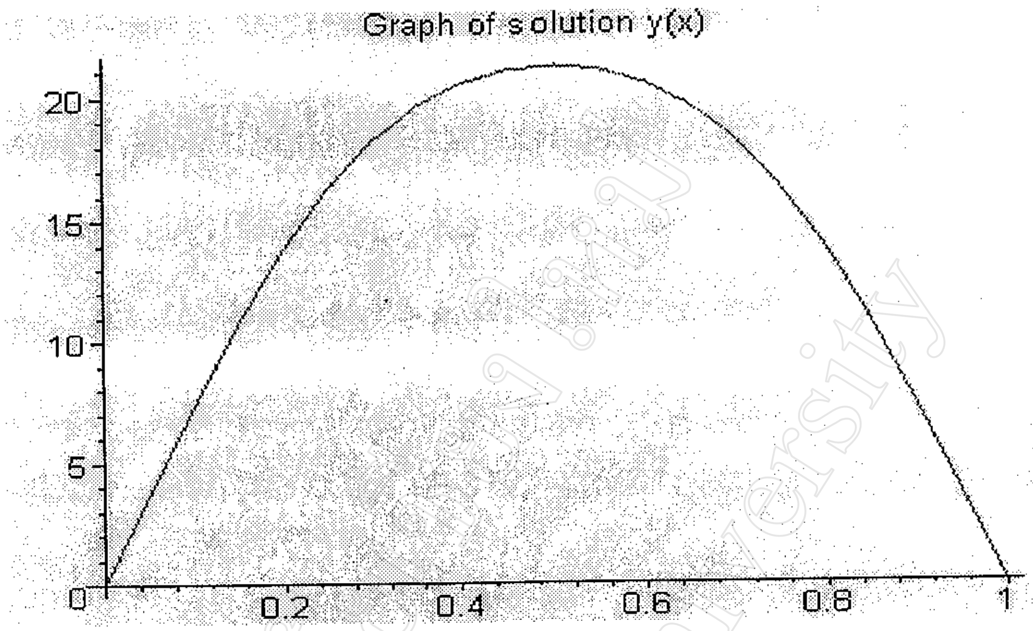

Figure 3.6: Third numerical solution of the BVP (3.12)

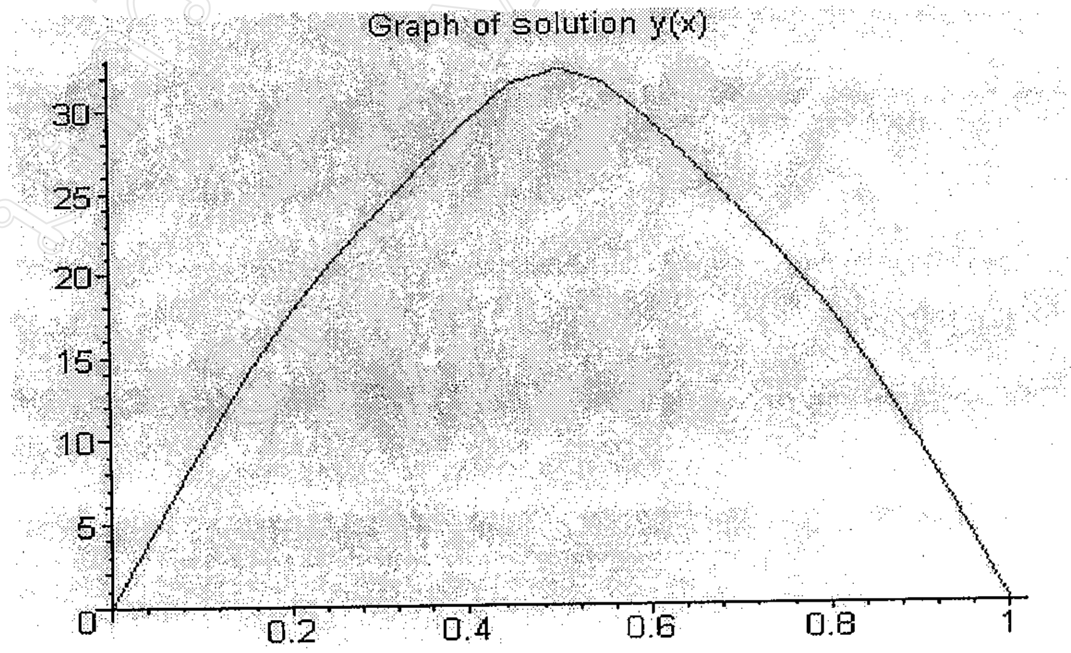

Figure 3.7: Fourth numerical solution of the BVP (3.12)