CHAPTER 3

Main Result

3.1 The Dynamical Systems

As in introduction, we will study the following dynamical system

$$\dot{x} = -\mu x + y(z+\alpha) - bxz, \quad \dot{y} = -\mu y + x(z-\alpha) - byz, \quad \dot{z} = 1 - xy. \qquad (1.1)$$

where x and y are the currents

μ and b are positive constants representing dissipative effects and $b \in (0,1)$

α is a constant of the motion.

The equilibrium points of the system (1.1) are

$$E_1 = (\beta_1,\ \beta_2,\ \gamma) \text{ and } E_2 = (-\beta_1,\ -\beta_2,\ \gamma), \qquad (1.2)$$

where $\beta_1 = \sqrt{(\gamma+\alpha)/(\mu+b\gamma)}$,

$\beta_2 = \sqrt{(\gamma-\alpha)/(\mu+b\gamma)}$,

$$\gamma = \frac{\mu b + \sqrt{\mu^2 + \alpha^2(1-b^2)}}{(1-b^2)}.$$

We assume that $\gamma - \alpha > 0$. The Jacobian of the linearized system about the first E_1 is

$$J = \begin{bmatrix} -\mu - b\gamma & \gamma + \alpha & \beta_2 - b\beta_1 \\ \gamma - \alpha & -\mu - b\gamma & \beta_1 - b\beta_2 \\ -\beta_2 & -\beta_1 & 0 \end{bmatrix}.$$

The characteristic equation of the Jacobian J has the form

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0\,,$$

that is $c_1 = 2b\gamma + 2\mu > 0\,,$

$$c_2 = \beta_1^2 + \beta_2^2 - 2b\beta_1\beta_2 = (\beta_1 - \beta_2)^2 + 2\beta_1\beta_2(1-b) > 0\,,$$

$$c_3 = 2\beta_1\beta_2\gamma + \alpha b(\beta_1^2 - \beta_2^2) + \mu(\beta_1^2 + \beta_2^2) - 2b\beta_1\beta_2(\mu + b\gamma)$$

$$= 2\gamma\left(1 - b^2\right) + \frac{2\gamma\alpha b}{\mu + b\gamma} + \mu c_2 > 0\,,$$

and $\quad c_1c_2 - c_3 = \dfrac{-2\gamma b\alpha}{\mu + b\gamma} < 0\,.$

We see that $c_1c_2 - c_3 < 0$ is not satisfied the Routh-Hurwitz conditions, that is the equilibrium points are not unstable. Consequently, we cannot conclude anything about the behavior of the dynamical system (1.1) by means of linearization. However, a numerical integration of these equations is quite revealing. It shows that the equilibria are actually unstable and that the orbits encircle one of them a number of times before switching suddenly to encircle the other equilibrium where it oscillates for awhile before again switching rapidly back about the origin point. This orbits are never captured by either equilibrium point, and the limiting behavior is apparently chaotic since the number of oscillations in the neighborhood of each equilibrium point is unpredictable. Fig. 1 shows a three-dimensional image of the chaotic trajectory at $\mu = 0.75$, $\alpha = 1.80$ and $b = 0.10$.

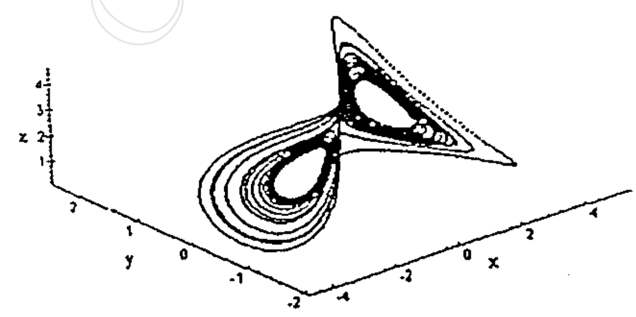


Fig. 1. Three-dimensional image of the chaotic system (1.1) at $\mu = 0.75$, $\alpha = 1.80$ and $b = 0.10$.

3.2 Controlling chaos to equilibrium points

In this section, the chaos of the dynamical system (1.1) is controlled to one of the two equilibria of the system. Two feedback methods are applied; linear feedback and bounded feedback to achieve this goal.

3.2.1 Feedback control method

The linear feedback control is applied to system (1.1). Our goal is to guide the chaotic trajectories of the system to one of the two unstable equilibria (E_1 or E_2). For the purpose of controlling chaos by feedback control approach, let us assume that the equations of the controlled system are given by

$$\begin{aligned} \dot{x} &= -\mu x + y(z+\alpha) - bxz + u_1, \\ \dot{y} &= -\mu y + x(z-\alpha) - byz + u_2, \\ \dot{z} &= 1 - xy + u_3, \end{aligned} \tag{2.1}$$

where u_1, u_2 and u_3 are external control inputs.

It will be suitably designed to drive the trajectory of the system, specified by (x, y, z) to any of the two equilibrium points of the uncontrolled (i.e., $u_1 = u_2 = u_3 = 0$) system by only a single state variable feedback. For practical applications, a simple feedback controller is more desirable so the control law is

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{bmatrix} x-\overline{x} \\ y-\overline{y} \\ z-\overline{z} \end{bmatrix},$$

where $(\overline{x}, \overline{y}, \overline{z})$ is the desired unstable equilibrium of the chaotic system (1.1), and k_{11}, k_{22} and k_{33}, positive feedback gains, are needed to be chosen such that the trajectory of the controlled system is stabilized to any of the two equilibrium points of the uncontrolled system.

$$J = \begin{bmatrix} -\mu - b\gamma & \gamma + \alpha & \beta_2 - b\beta_1 \\ \gamma - \alpha & -\mu - b\gamma & \beta_1 - b\beta_2 \\ -\beta_2 & -\beta_1 & -k_{33} \end{bmatrix},$$

where $k_{11} = k_{22} = 0$.

The characteristic equation of the Jacobian J has the form

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

that is $c_1 = A + k_{33}$,

$$c_2 = B + 2k_{33}\mu + 2k_{33}b\gamma,$$

$$c_3 = C,$$

where $A = 2b\gamma + 2\mu$,

$$B = \beta_1^2 + \beta_2^2 - 2b\beta_1\beta_2,$$

$$C = 2\beta_1\beta_2\gamma + \alpha b(\beta_1^2 - \beta_2^2) + \mu(\beta_1^2 + \beta_2^2) - 2b\beta_1\beta_2(\mu + b\gamma).$$

Since A, B, $C > 0$, $\mu, \gamma, b > 0$ and $k_{33} > 0$, hence $c_1 > 0, c_2 > 0$ and $c_3 > 0$.

$$c_1c_2 - c_3 = 4k_{33}(\mu^2 + 2\mu b\gamma + b^2\gamma^2) + k_{33}\left[(\beta_1 - \beta_2)^2 + 2\beta_1\beta_2(1-b)\right] + 2k_{33}^2(\mu + b\gamma) > 0.$$

We see that c_1, c_2 and c_3 are satisfied the Routh-Hurwitz conditions (i) and (ii) of (2.3) and the asymptotic stability of $E_1 = (\beta_1, \beta_2, \gamma)$ of the controlled system (2.4) is established.

3.2.1.2 Stabilizing the equilibrium $E_2 = (-\beta_1, -\beta_2, \gamma)$

In order to control chaos of system (1.1) to the unstable equilibrium $E_2 = (-\beta_1, -\beta_2, \gamma)$, linear feedback control is proposed to obtain the controlled system

$$\begin{aligned} \dot{x} &= -\mu x + y(z + \alpha) - bxz - k_{11}(x + \beta_1), \\ \dot{y} &= -\mu y + x(z - \alpha) - byz - k_{22}(y + \beta_2), \\ \dot{z} &= 1 - xy - k_{33}(z - \gamma). \end{aligned} \tag{2.5}$$

Theorem 2. The equilibrium point $E_2 = (-\beta_1, -\beta_2, \gamma)$ of controlled system (2.5) is asymptotically stable provide that $k_{11} = 0,\ k_{22} = 0$ and $k_{33} > 0$.

Proof. The Jacobian matrix of the linearized system of at the equilibrium is given by

$$J = \begin{bmatrix} -\mu - b\gamma & \gamma + \alpha & -\beta_2 + b\beta_1 \\ \gamma - \alpha & -\mu - b\gamma & -\beta_1 + b\beta_2 \\ \beta_2 & \beta_1 & -k_{33} \end{bmatrix},$$

where $k_{11} = k_{22} = 0$.

The characteristic equation of the Jacobian J has the form

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

that is $c_1 = A + k_{33}$,

$c_2 = B + 2k_{33}\mu + 2k_{33}b\gamma$,

$c_3 = C$,

where $A = 2b\gamma + 2\mu$,

$B = \beta_1^2 + \beta_2^2 - 2b\beta_1\beta_2$,

$C = 2\beta_1\beta_2\gamma + \alpha b(\beta_1^2 - \beta_2^2) + \mu(\beta_1^2 + \beta_2^2) - 2b\beta_1\beta_2(\mu + b\gamma)$.

Since $A, B, C > 0$, $\mu, \gamma, b > 0$ and $k_{33} > 0$, thus $c_1 > 0, c_2 > 0$ และ $c_3 > 0$.

Consider

$$c_1c_2 - c_3 = 4k_{33}(\mu^2 + 2\mu b\gamma + b^2\gamma^2) + k_{33}\left[(\beta_1 - \beta_2)^2 + 2\beta_1\beta_2(1-b)\right] + 2k_{33}^2(\mu + b\gamma) > 0.$$

We see that c_1, c_2 and c_3 are satisfied the Routh-Hurwitz in equation (2.3), hence $E_2 = (-\beta_1, -\beta_2, \gamma)$ of the controlled system (2.5) is asymptotic stability.

3.2.1.3 Numerical simulations

The chaotic and controlled systems of the ordinary differential equations (1.1) and (2.1), respectively, are integrated by using fourth-order Runge-Kutta method with time step size 0.01. The parameters α, μ and b are taken $\alpha = 1.8$, $\mu = 0.75$ and $b = 0.1$ to ensure chaotic behavior in the absence of the control. The initial conditions $x = 0.3$, $y = 0.4$ and $z = 0.5$ are chosen in all simulations. The gain matrix K is chosen to be $k_{11} = 0$, $k_{22} = 0$ and $k_{33} = 0.2$. Fig. 2 shows the convergence of the trajectory of the controlled system (2.4) to the equilibrium point of $E_1 = (2.00578,\ 0.49854,\ 2.03704)$ of system (1.1) in three-dimensional image. Fig. 3 shows in three-dimensional image the stabilization of the equilibrium point $E_2 = (-2.00578,\ -0.49854,\ 2.03704)$, where $k_{11} = 0$, $k_{22} = 0$ and $k_{33} = 0.5$.

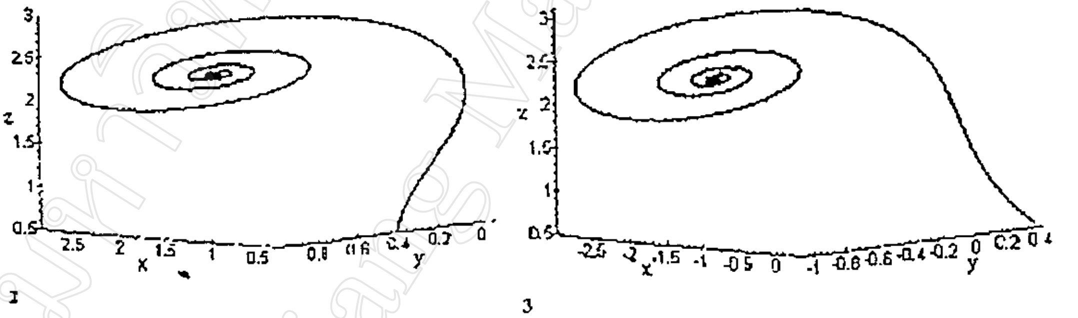


Fig. 2. Three-dimensional image of stabilizing the positive equilibrium $E_1 = (2.00578,\ 0.49854,\ 2.03704)$ of controlled system (2.2) by using linear feedback at $\alpha = 1.8$, $\mu = 0.75$, $b = 0.1$, $k_{11} = 0$, $k_{22} = 0$ and $k_{33} = 0.2$.

Fig. 3. Three-dimensional image of the stabilizing of equilibrium point $E_2 = (-2.00578,\ -0.49854,\ 2.03704)$ of controlled system (2.5), where $\alpha = 1.8$, $\mu = 0.75$, $b = 0.1$, $k_{11} = k_{33} = 0$ and $k_{22} = 0.5$.

3.2.2 Bounded feedback control

The controlling chaotic systems of practical problems, we need bounded control. In order to ensure small values of the control perturbations, the controller $u(t)$ is restricted in the following manner:

$$u(t)=\begin{cases} -u_0, & u(t)<u_0 \\ u(t), & -u_0<u(t)<u_0 \\ u_0, & u_0<u(t) \end{cases},$$

where u_0 is a small saturating positive value of the control.

3.2.2.1 Stabilizing the equilibrium $E_1=(\beta_1, \beta_2, \gamma)$

In order to stabilize the unstable equilibrium $E_1=(\beta_1, \beta_2, \gamma)$ by bounded feedback control, the proposed control is designed for system (1.1) as follows:

$$\begin{aligned} \dot{x} &= -\mu x + y(z+\alpha) - bxz, \\ \dot{y} &= -\mu y + x(z-\alpha) - byz, \\ \dot{z} &= 1 - xy + u(t), \end{aligned} \tag{2.6}$$

where $u(t)=-k\left[bxz+y(z+\alpha)\right]$, $k>0$.

The controller $u(t)$ tends to zero if the solutions converge to one of the two unstable equilibria. The original and controlled systems have the same equilibrium points E_1 and E_2.

Theorem 3. The equilibrium solution $E_1=(\beta_1,\beta_2,\gamma)$ of the controlled system (2.6) is asymptotically stable if the constant $k>0$.

Proof. Let the controller $u(t)=-k\left[bxz+y(z+\alpha)\right]$, $k>0$ be designed to the chaotic system (1.1). The Jacobian matrix J of the linearized system of (2.6) at $E_1=(\beta_1,\beta_2,\gamma)$ is

$$J=\begin{bmatrix} -\mu-b\gamma & \gamma+\alpha & \beta_2-b\beta_1 \\ \gamma-\alpha & -\mu-b\gamma & \beta_1-b\beta_2 \\ -\beta_2-kb\gamma & -\beta_1-k(\gamma+\alpha) & -k(b\beta_1+\beta_2) \end{bmatrix}. \tag{2.7}$$

The characteristic equation of the Jacobian matrix (2.7) is

$$\lambda^3+c_1\lambda^2+c_2\lambda+c_3=0.$$

That is $c_1=A+k(b\beta_1+\beta_2)$,

$$\begin{aligned} c_2 &= B+k(\beta_1\gamma+\beta_1\alpha+2\beta_2\mu+b^2\gamma\beta_1+2b\gamma\beta_2+2\mu b\beta_1-\beta_2 b\alpha) \\ &= B+k(\beta_1\gamma+\beta_1\alpha+2\beta_2\mu+b^2\gamma\beta_1+2\mu b\beta_1+b\beta_2(2\gamma-\alpha)), \\ c_3 &= C+k\big(\mu^2\beta_2+\mu\beta_1\alpha+\mu\beta_1\gamma-2b^2\gamma\beta_2\alpha+2b\gamma\beta_1\alpha+2\alpha^2 b\beta_1+2\mu b\gamma\beta_2 \\ &\quad +\mu b^2\gamma\beta_1+\mu^2 b\beta_1-\mu b\beta_2\alpha\big) \\ &= C+k\left(\mu^2\beta_2+\mu\beta_1\gamma+2b\gamma\alpha\left(\frac{r(1-b^2)+\alpha(1+b^2)}{\sqrt{\mu+b\gamma}\left(\sqrt{r+\alpha}+b\sqrt{r-\alpha}\right)}\right)+2\alpha^2 b\beta_1\right. \\ &\quad \left.+2\mu b\gamma\beta_2+\mu b^2\gamma\beta_1+\mu\alpha\left(\frac{r(1-b^2)+\alpha(1+b^2)}{\sqrt{\mu+b\gamma}\left(\sqrt{r+\alpha}+b\sqrt{r-\alpha}\right)}\right)+\mu^2 b\beta_1\right). \end{aligned}$$

where $A=2b\gamma+2\mu$,

$$B=\beta_1^2+\beta_2^2-2b\beta_1\beta_2,$$

$$C=2\beta_1\beta_2\gamma+\alpha b(\beta_1^2-\beta_2^2)+\mu(\beta_1^2+\beta_2^2)-2b\beta_1\beta_2(\mu+b\gamma).$$

Since $A,B,C>0$, $\mu,\gamma,b>0$ and $k>0$. Hence $c_1>0, c_2>0$ and $c_3>0$.

$$
\begin{aligned}
c_1c_2 - c_3 = k\Big(&3\mu^2\beta_2 + \mu\alpha\beta_1 + \mu\gamma\beta_1 + 5b^2\gamma\mu\beta_1 + 3\mu^2\beta_1 b + 4b^2\gamma^2\beta_2 \\
&+ \mu b\beta_2(6\gamma-\alpha) + 2b\beta_1(\mu+b\gamma)^2 + 2b^3\gamma^2\beta_1\Big) \\
&+ (kb\beta_1 + k\beta_2)\big((\beta_1-\beta_2)^2 + 2\beta_1\beta_2(1-b)\big) \div k^2\big(2\mu b\beta_1\beta_2 \\
&+ \alpha\beta_1\beta_2 + b^2\gamma\beta_1\beta_2 + 2\mu\beta_2^2 + b\beta_2^2(2\gamma-\alpha) \div \gamma\beta_1\beta_2 + 2\mu b^2\beta_1^2 \\
&+ b\alpha\beta_1^2 + b^2\beta_1\beta_2(2\gamma-\alpha) + b^3\gamma\beta_1^2 + 2b\mu\beta_1\beta_2 + b\gamma\beta_1^2\big)\cdot
\end{aligned}
$$

Since $b \in (0,1)$ and $\gamma > \alpha$. Hence $c_1c_2 - c_3 > 0$.

We see that c_1, c_2 and c_3 are satisfied conditions of Routh – Hurwitz. Hence equilibrium $E_1 = (\beta_1, \beta_2, \gamma)$ of controlled system (2.6) is asymptotically stable.

3.2.2.2 Stabilizing the equilibrium $E_2 = (-\beta_1, -\beta_2, \gamma)$

For stabilizing the unstable equilibrium $E_2 = (-\beta_1, -\beta_2, \gamma)$ of (1.1) by bounded feedback control, the proposed control is designed for system (1.1) as follows:

$$
\begin{aligned}
\dot{x} &= -\mu x + y(z+\alpha) - bxz + u(t), \\
\dot{y} &= -\mu y + x(z-\alpha) - byz, \\
\dot{z} &= 1 - xy,
\end{aligned}
\tag{2.8}
$$

where $u(t) = -k\big(\mu x - y(z+\alpha) - bxz\big), \quad k > 0.$

The controller $u(t)$ converges to zero as chaos suppressed to the equilibrium E_2. In order to study the stability of the controlled system (2.8) in the neighborhood of $E_2 = (-\beta_1, -\beta_2, \gamma)$, lineariztion about the equilibrium $E_2 = (-\beta_1, -\beta_2, \gamma)$ gives the Jacobian matrix

$$J=\begin{bmatrix} -\mu-b\gamma & \gamma+\alpha & -\beta_2+b\beta_1 \\ \gamma-\alpha & -\mu-b\gamma & -\beta_1+b\beta_2 \\ \beta_2-k(\mu-b\gamma) & \beta_1+k(\gamma+\alpha) & -k(\beta_2+b\beta_1) \end{bmatrix}. \tag{2.9}$$

The characteristic equation of the Jacobian matrix (2.9) is

$$\lambda^3+c_1\lambda^2+c_2\lambda+c_3=0.$$

That is $c_1=A+k(b\beta_1+\beta_2)$,

$$\begin{aligned} c_2 &= B+k(\gamma\beta_1+\mu\beta_2+b^2\gamma\beta_1+2b\gamma\beta_2-\alpha b\beta_2+\alpha\beta_1+3\mu b\beta_1) \\ &= B+k(\gamma\beta_1+\mu\beta_2+b^2\gamma\beta_1+b\beta_2(2\gamma-\alpha)+\alpha\beta_1+3\mu b\beta_1), \end{aligned}$$

$$\begin{aligned} c_3 &= C+k(2\mu b^2\gamma\beta_1+2\alpha^2 b\beta_1+2\mu b\gamma\beta_2+2b\gamma\beta_1\alpha+2\mu^2 b\beta_1-2b^2\gamma\beta_2\alpha) \\ &= C+k\Big[2\mu b^2\gamma\beta_1+2\alpha^2 b\beta_1+2\mu b\gamma\beta_2+2\mu^2 b\beta_1 \\ &\quad +2b\gamma\alpha\left(\frac{\gamma(1-b^2)+\alpha(1+b^2)}{\sqrt{\mu+b\gamma}\left(\sqrt{\gamma+\alpha}+b\sqrt{\gamma-\alpha}\right)}\right)\Big], \end{aligned}$$

where $A=2b\gamma+2\mu$,

$B=\beta_1^2+\beta_2^2-2b\beta_1\beta_2$,

$C=2\beta_1\beta_2\gamma+\alpha b(\beta_1^2-\beta_2^2)+\mu(\beta_1^2+\beta_2^2)-2b\beta_1\beta_2(\mu+b\gamma)$.

Since $A, B, C>0$, $\mu,\gamma,b>0$, $\gamma>\alpha$, $b\in(0,1)$ and $k>0$. Hence $c_1>0$, $c_2>0$ and $c_3>0$.

Consider

$$\begin{aligned} c_1c_2-c_3 &= k\Big(2b\beta_1(\mu+b\gamma)^2+4\mu b\gamma\beta_2+2\mu\alpha\left(\frac{\gamma(1-b^2)+\alpha(1+b^2)}{\sqrt{\mu+b\gamma}\left(\sqrt{\gamma+\alpha}+b\sqrt{\gamma-\alpha}\right)}\right) \\ &\quad +4\mu^2\beta_1 b+2\mu\gamma\beta_1+2\mu^2\beta_2+2b^3\gamma^2\beta_1+6b^2\gamma\mu\beta_1+4b^2\gamma^2\beta_2\Big) \\ &\quad +(k\beta_2+kb\beta_1)\left((\beta_1-\beta_2)^2+2\beta_1\beta_2(1-b)\right)+k^2\Big(\alpha\beta_1\beta_2(1-b^2) \\ &\quad +3b^2\gamma\beta_1\beta_2+\gamma\beta_1\beta_2-b\alpha\beta_2^2+3\mu b\beta_1\beta_2+2b\gamma\beta_2^2+\mu\beta_2^2+b\alpha\beta_1^2+b^3\gamma\beta_1^2 \\ &\quad +b\gamma\beta_1^2+3\mu b^2\beta_1^2+b\mu\beta_1\beta_2\Big) \end{aligned}$$

Since $b \in (0,1)$ and $\gamma > \alpha$. Hence $c_1c_2 - c_3 > 0$.

Applying Routh-Hurwitz theorem of classical stability theory, we get the necessary and sufficient conditions(2.3) are satisfied and asymptotic stability for the equilibrium point $E_2 = (-\beta_1, -\beta_2, \gamma)$ is established and this completes the proof of the following lemma.

Theorem 4. The equilibrium solution $E_2 = (-\beta_1, -\beta_2, \gamma)$ of the controlled system (2.8) is asymptotically stable for $k > 0$.

3.2.2.3 Numerical simulations

Numerical experiments are carried out to integrate the controlled systems (2.6) and (2.8). System (2.6) and (2.8) is numerically integrated by using fourth-order Runge-Kutta method with time step 0.01. The parameters α, μ and b are chosen $\alpha = 1.8$, $\mu = 0.75$ and $b = 0.1$ to ensure the existence of chaos in the absence of the control. The initial conditions x = 0.3, y = 0.4 and z = 0.5 are chosen in all simulations. The equilibrium point $E_1 = (\beta_1, \beta_2, \gamma) = (2.00578,\ 0.49854,\ 2.03704)$ is stabilized for $k = 0.1$ and $u_0 = 0.5$. Fig. 4(a)-(c) shows the behavior of the states x, y and z of system(2.6) and the controller $u(t)$ with time. The control is started at t = 100. The equilibrium point $E_2 = (-\beta_1, -\beta_2, \gamma) = (-2.00578,\ -0.49854,\ 2.03704)$ of system(2.8) is stabilized for $k = 0.1$ and $u_0 = 0.5$. Fig. 5(a)-(c) shows the behavior of the states x, y and z of system(2.8) and the controller $u(t)$ with time. The control is activated at t = 100.

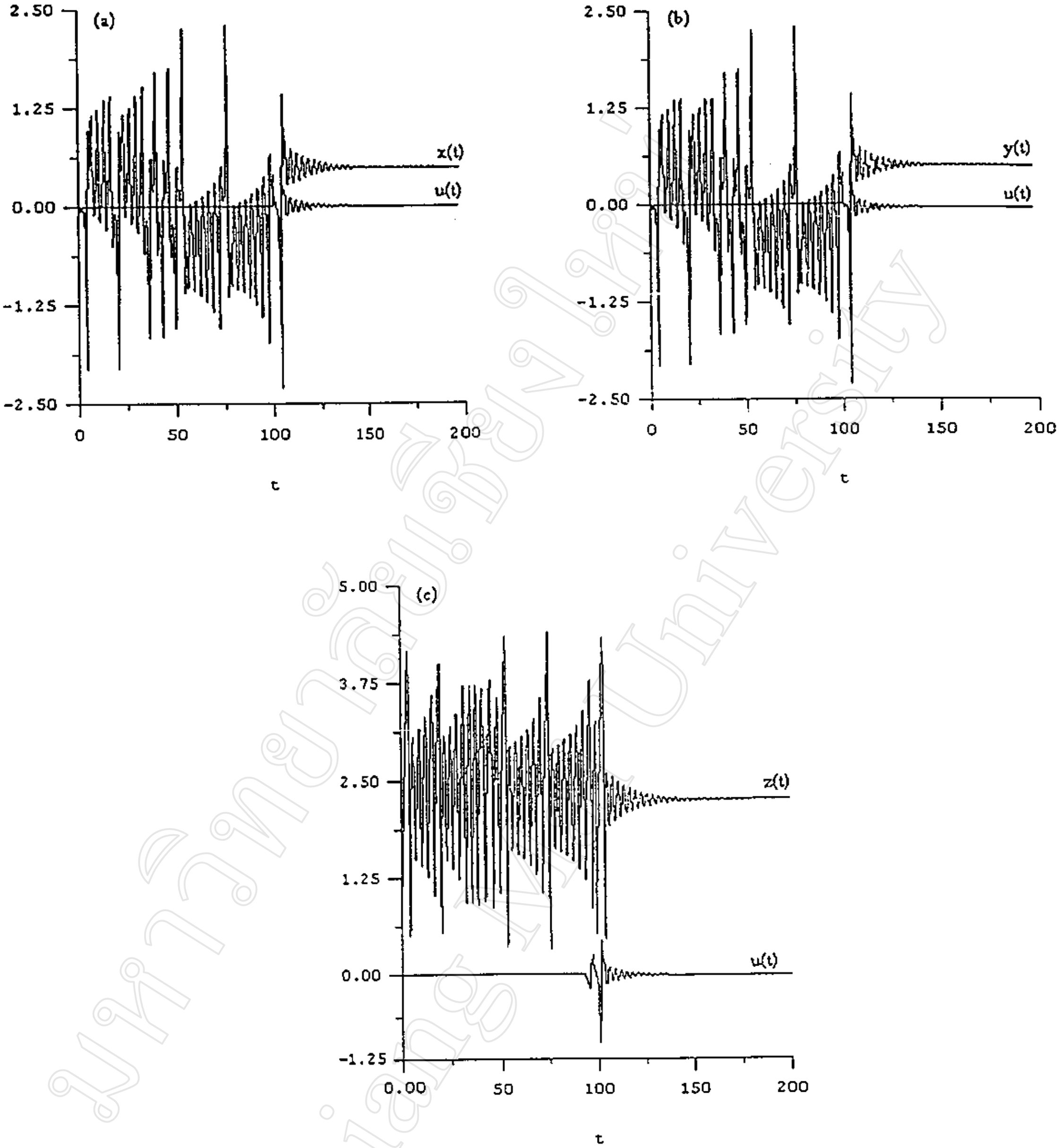


Fig. 4. The states of the controlled system (2.6) and the control $u(t)$ response with time before and after control activation. (a) The response of the state x, (b) the response of the state y, and (c) the response of the state z. The control is activated at t=100, $\alpha=1.8$, $\mu=0.75$, $b=0.1$, $k=0.1$ and $u_0=0.5$.

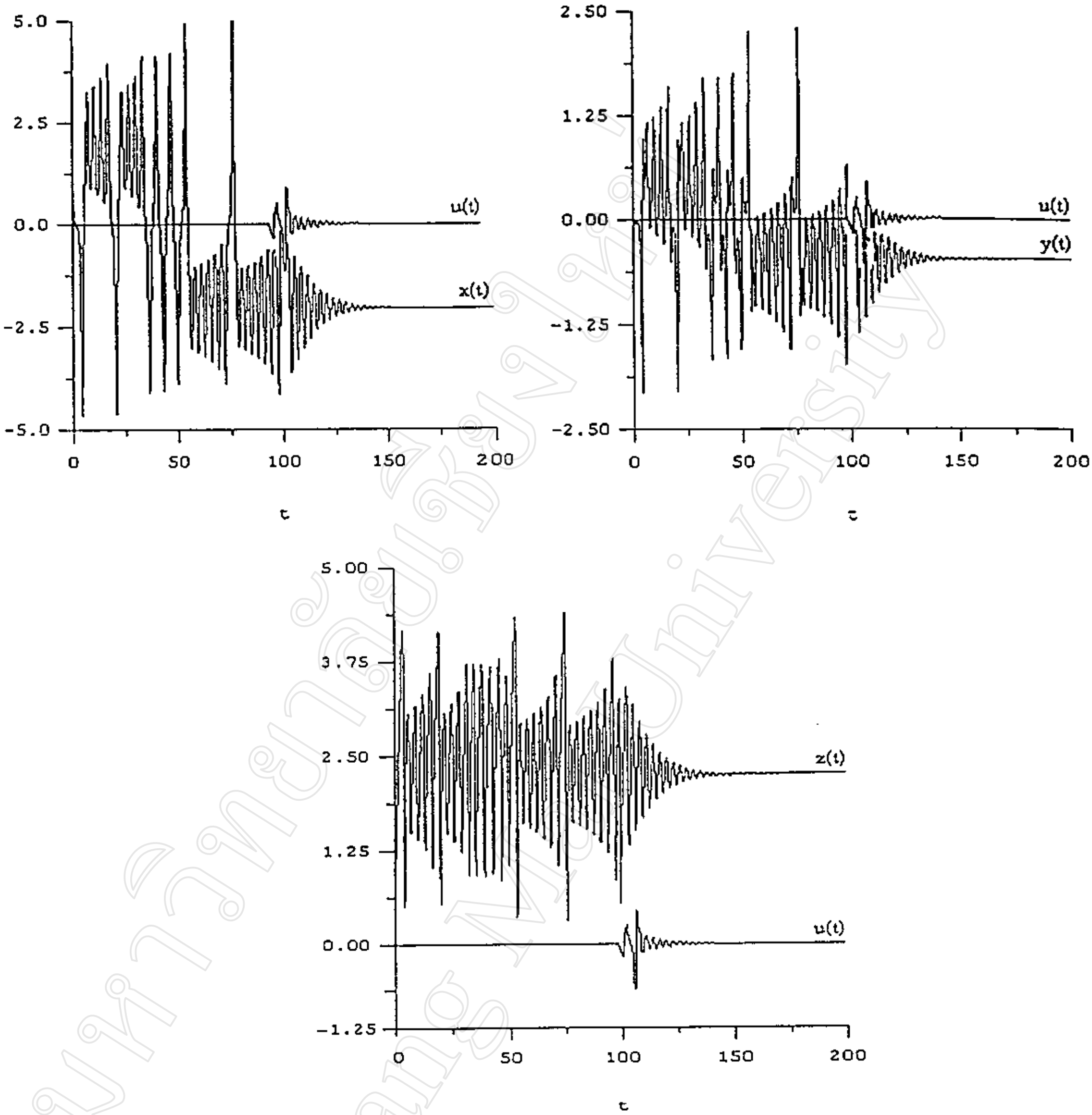


Fig. 5. The states of the controlled system (2.8) and the control $u(t)$ response with time before and after control activation. (a) The response of the state x, (b) the response of the state y, and (c) the response of the state z. The control is activated at t=100, $\alpha = 1.8$, $\mu = 0.75$, $b = 0.1$, $k = 0.1$ and $u_0 = 0.5$.

3.3 Controlling chaos to limit cycles

The strange attractor of system (1.1) contains infinite number of unstable periodic orbits (UPOs) embedded within the attractor. In order to suppress the chaotic behavior of system(1.1) to one of these UPOs, We apply two methods of control in this section. The nonfeedback and a derived method based on the delay feedback control are applied to system (1.1) to guide the chaotic trajectory to a limit cycle.

3.3.1 Nonfeedback control method

The essential advantages of nonfeedback technique lie in their speed flexibility, which are suited for practical applications. The nonfeedback methods include weak periodic force use to controlling the chaotic behavior of system (1.1). The proposed nonfeedback control of system (1.1) consists of a constant and an external periodic force function in the form

$$u(t) = f_1 + f_2 \cos(\omega t), \tag{3.1}$$

where f_1 is a constant bias,

f_2 is the amplitude, and

ω is the frequency of the external periodic force signal.

The nonfeedback method with the controller (3.1) is applied in two cases.

Case 1. In this case, the controller (3.1) is added to the second equation of(1.1). The controlled system has the following equations:

$$\dot{x} = -\mu x + y(z+\alpha) - bxz,$$
$$\dot{y} = -\mu y + x(z-\alpha) - byz + f_1 + f_2\cos(\omega t), \qquad (3.2)$$
$$\dot{z} = 1 - xy.$$

Chaos is suppressed to a periodic solution within the chaotic attractor for certain values of f_1, f_2 and ω.

Case 2. In this case, the controller (3.1) is added to the third equation of (1.1). The resulting controlled system has the form

$$\dot{x} = -\mu x + y(z+\alpha) - bxz,$$
$$\dot{y} = -\mu y + x(z-\alpha) - byz, \qquad (3.3)$$
$$\dot{z} = 1 - xy + f_1 + f_2\cos(\omega t).$$

Also chaos is suppressed to another periodic solution within the chaotic attractor for certain values of f_1, f_2 and ω.

3.3.1.1 Numerical simulations

In this parts, a number of numerical simulations is carried out by using fourth-order Runge-Kutta method with time step size 0.01. The initial conditions are chosen in all these simulations to be x = 0.3, y = 0.4 and z = 0.5. The parameters α, μ and b are chosen $\alpha = 1.8, \mu = 0.75$ and $b = 0.1$ so that system (1.1) will exhibit chaotic behavior if no controls are applied.

Fig. 6 shows the projection of the chaotic attractor in x-y phase plane

Many numerical experiments are carried out to guide chaos to a periodic solution around the positive equilibrium E_1. The controlled system (3.2), for the values $f_1 = 0.452, f_2 = 1.499$ and $\omega = 2.712$, chaos is suppressed to a limit cycle as shown in Fig. 7(a). The orbits of the second controlled system (3.3), with the same value $f_1 = 0.452, f_2 = 1.499$ and $\omega = 2.712$ are converged to another limit cycle that surrounds the two equilibria Fig. 7(b).

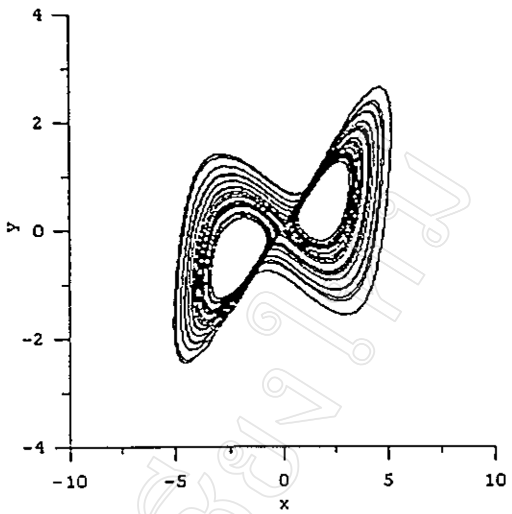


Fig. 6. The chaotic attractor of system (1.1) in the x-y phase plane at $\alpha = 1.8$, $\mu = 0.75$ and $b = 0.1$.

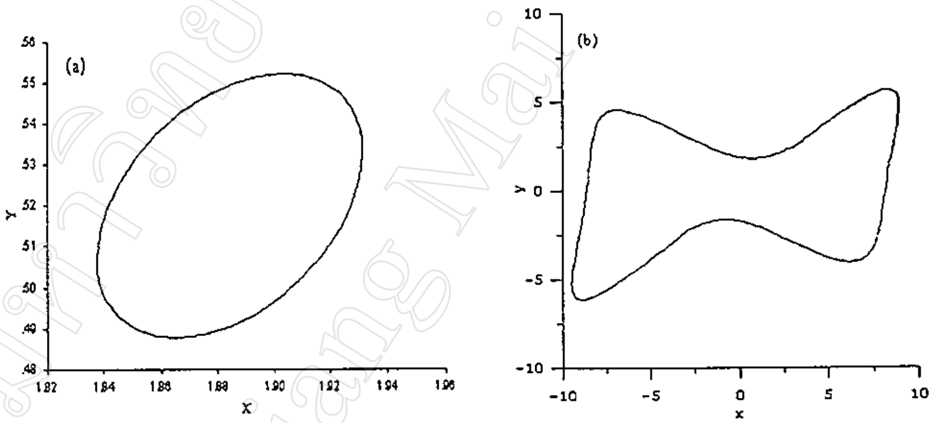


Fig. 7. (a) The limit cycle of the controlled system (3.2) in x-y plane for the value $f_1 = 0.452$, $f_2 = 1.499$ and $\omega = 2.712$. (b) The limit cycle of the controlled system (3.3), for the value $f_1 = 0.452, f_2 = 1.499$ and $\omega = 2.712$, is plotted in the x-y plane.

3.3.2 Delayed feedback control

The DFC is proposed by Pyragas [9] to control chaos in continuous dynamical systems. The DFC method is based on a feedback of the difference between the current state and the delayed state. The delay time is set to correspond to the desired period of the periodic orbit, therefore the feedback vanishes after stabilizing UPO. This method does not require finding UPO, but it needs its period only.

The DFC method is applied to system (1.1) to stabilize a UPO of period τ. The controlled system is obtained by adding a delayed feedback term to (1.1) as follows:

$$\begin{aligned} \dot{x} &= -\mu x + y(z+\alpha) - bxz - k_{11}\left[x(t) - x(t-\tau)\right], \\ \dot{y} &= -\mu y + x(z-\alpha) - bxy - k_{22}\left[y(t) - y(t-\tau)\right], \\ \dot{z} &= 1 - xy - k_{33}\left[z(t) - z(t-\tau)\right]. \end{aligned} \tag{4.1}$$

where K is 3×3 feedback gain matrix, and

τ is delay time which is the period of the target UPO.

The controller in system (4.1) converges to zero when the trajectories converge to periodic orbit with period orbit with period τ. The controlled and uncontrolled systems have the same fixed points E_1, E_2.

3.3.2.1 Approximated delay feedback control method

Our aim is to stabilize the chaotic system (1.1) to a limit cycle. Instead of applying DFC method, we approximate the controlled system (4.1) by applying Taylor theorem, for small τ we have

$$x(t-\tau) = x(t) - \tau\dot{x}(t) + O(\tau^2).$$

Hence, we get

$$\begin{aligned} x(t)-x(t-\tau) &= \tau\dot{x}(t)=\tau\left[y(z+\alpha)-\mu x-bxz\right], \\ y(t)-y(t-\tau) &= \tau\dot{y}(t)=\tau\left[x(z-\alpha)-\mu y-byz\right], \\ z(t)-z(t-\tau) &= \tau\dot{z}(t)=\tau\left(1-xy\right). \end{aligned}$$

This method can be applied for small value of τ and the chaotic orbit $(x(t),\ y(t),\ z(t))$ of (1.1) is very close to the periodic orbit with the period τ.

Thus, system(4.1) can be approximated by the following controlled system:

$$\begin{aligned} \dot{x} &= (1-k_{11}\tau)\left[y(z+\alpha)-\mu x-bxz\right], \\ \dot{y} &= (1-k_{22}\tau)\left[x(z-\alpha)-\mu y-byz\right], \\ \dot{z} &= (1-k_{33}\tau)\left(1-xy\right). \end{aligned} \qquad (4.2)$$

The controlled system (4.2) has the same equilibrium points E_1, E_2 as of the original system(1.1). The Jacobian matrix J of the linearized system (4.2) at $E_1=(\beta_1,\ \beta_2,\ \gamma)$ is

$$J=\begin{bmatrix} -d\mu-db\gamma & d\gamma+d\alpha & d\beta_2-db\beta_1 \\ e\gamma-e\alpha & -e\mu-eb\gamma & e\beta_1-eb\beta_2 \\ -f\beta_2 & -f\beta_1 & 0 \end{bmatrix},$$

Let $1-k_{11}\tau=d$, $1-k_{22}\tau=e$ and $1-k_{33}\tau=f$.

The characteristic equation of the Jacobian J has the form

$$\lambda^3+c_1\lambda^2+c_2\lambda+c_3=0,$$

where $c_1=\mu e+b\gamma d+b\gamma e+\mu d$,

$$\begin{aligned} c_2 &= f\left(e\beta_1^2+d\beta_2^2-db\beta_1\beta_2-eb\beta_1\beta_2\right) \\ &= f\left(e(\beta_1^2-b)+d(\beta_2^2-b)\right), \\ c_3 &= f\left(-2\mu deb\,\beta_1\beta_2+e\alpha db\left(\beta_1^2-\beta_2^2\right)+\mu ed\left(\beta_1^2+\beta_2^2\right)-2b^2\gamma de\beta_1\beta_2\right. \\ &\quad \left.+ed\gamma\beta_1\beta_2\right) \\ &= f\left[ed\gamma\beta_1\beta_2(1-2b^2)+\mu ed\left((\beta_1-\beta_2)^2+2\beta_1\beta_2(1-b)\right)\right. \\ &\quad \left.+e\alpha db\left(\beta_1^2-\beta_2^2\right)\right]. \end{aligned}$$

Since $A, B, C > 0$, $\mu, \gamma, \alpha, b > 0$ and $e, d, f \in (0,1)$, hence $c_1 > 0$. The Routh-Hurwitz in condition (i) of equation (2.3) are not satisfied that is $c_2, c_3 \le 0$, for some values of parameter. For example, if we let $b = 0.7$, $\alpha = 1.8$, $\mu = 0.75$ and if we choose $k_{11} = 0.001$, $k_{22} = 9.5$, $k_{33} = 1.5$ and $\tau = 0.1$. We have $c_1 > 0$ and $c_2, c_3 \le 0$, thus the controlled system (4.2) is not satisfied the Routh-Hurwitz. Consequently, there are for some values of parameter which are satisfied the Routh-Hurwitz in condition (i) of equation (2.3), that is $c_2, c_3 > 0$. The case of $k_{11}, k_{22} \ne 0$ and $k_{33} = 0$ or $k_{11}, k_{33} \ne 0$ and $k_{22} = 0$ or $k_{22}, k_{33} \ne 0$ and $k_{11} = 0$ or $k_{22} = k_{33} = 0$ and $k_{11} \ne 0$ or $k_{11} = k_{33} = 0$ and $k_{22} \ne 0$ have the similar results as in the case $k_{11}, k_{22}, k_{33} \ne 0$, that is $c_1 > 0$ and $c_2, c_3 \le 0$ for some values of parameter. Case $k_{11} = k_{22} = 0$ and $k_{33} \ne 0$ is a borderline stability case, we will show later. If $b = 0$, conditions (i) and (ii) of equation (2.3) are true for $0 < k_{11}\tau < 1$, $0 < k_{22}\tau < 1$ and $0 < k_{33}\tau < 1$. If $b \ne 0$, we can choose b which is very small in the interval $(0,1)$ and can choose some point or some subinterval of $0 < k_{11}\tau < 1$, $0 < k_{22}\tau < 1$ and $0 < k_{33}\tau < 1$ such that chaotic behavior suppresses to a limit cycle and c_1, c_2, c_3 are satisfied the Routh-Hurwitz. For example, if we let $b = 0.1$ and if we choose $k_{11} = k_{22} = 8.3$, $k_{33} = 6.5$ or $k_{11} = k_{22} = 6.5$, $k_{33} = 0$ and $\tau = 0.1$, the chaotic orbit converges to a limit cycle, see Fig. 9(a). If we choose $k_{11} = k_{22} = 8.5$, $k_{33} = 3.5$ or $k_{11} = k_{22} = 8.1$, $k_{33} = 0$ and $\tau = 0.1$, then chaos suppresses to a limit cycle, see Fig. 9(b).

Fix $b = 0.01$, choose $k_{11}, k_{22} \in [8,9]$, $k_{33} \in [0,3.5]$ and $\tau = 0.1$, chaotic behavior suppresses to a limit cycle which have the same Fig. 9(b).

In the remark, we show a borderline stability case:

Remark Let $k_{11} = k_{22} = 0$ and $0 < k_{33}\tau < 1$, then the Jacobian matrix J of the linearized system of (4.2) at $E_1 = (\beta_1, \beta_2, \gamma)$ is

$$J = \begin{bmatrix} -\mu - b\gamma & \gamma + \alpha & \beta_2 - b\beta_1 \\ \gamma - \alpha & -\mu - b\gamma & \beta_1 - b\beta_2 \\ -f\beta_2 & -f\beta_1 & 0 \end{bmatrix},$$

where $k_{11}\tau = k_{22}\tau = 0$ and let $1 - k_{33}\tau = f$.

The characteristic equation of the Jacobian J has the form

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

where $c_1 = A$,

$c_2 = fB$,

$c_3 = fC$,

where $A = 2b\gamma + 2\mu$,

$B = \beta_1^2 + \beta_2^2 - 2b\beta_1\beta_2$,

$C = 2\beta_1\beta_2\gamma + \alpha b(\beta_1^2 - \beta_2^2) + \mu(\beta_1^2 + \beta_2^2) - 2b\beta_1\beta_2(\mu + b\gamma)$.

Since $A, B, C > 0$, $\mu, \gamma, b > 0$ and $f > 0$, hence $c_1 > 0, c_2 > 0$ and $c_3 > 0$.

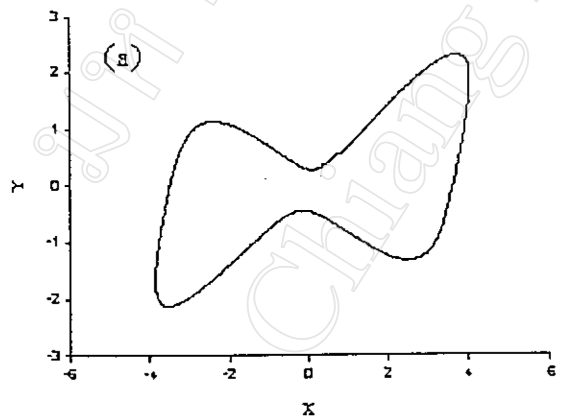
Now, consider

$$c_1c_2 - c_3 = f\Big[\mu\left(\beta_1^2 + \beta_2^2\right) - 2\mu b\beta_1\beta_2 - 2b^2\gamma\beta_1\beta_2 + 2b\gamma\left(\beta_1^2 + \beta_2^2\right) - 2\beta_1\beta_2\gamma - \alpha b\left(\beta_1^2 - \beta_2^2\right)\Big] = 0.$$

We see that the condition $c_1c_2 - c_3 = 0$ results in a borderline stability case. So, we cannot conclude the system of (4.2) at $E_1 = (\beta_1, \beta_2, \gamma)$ is asymptotically stable. However, we can show numerically that there exist some point of $k_{11}\tau$, $k_{22}\tau$ and $k_{33}\tau$ in the interval $(0,1)$ which suppress the chaotic system (1.1) to a limit cycle in the next section. The same result is true at the equilibrium E_2. We see in Fig. 10(a) and 10(b).

3.3.2.2 Numerical simulations

The controlled system of ordinary differential equations (4.2) is integrated by using fourth-order Runge-Kutta method with time step 0.01, $\alpha = 1.8$, $\mu = 0.75$ and $b = 0.1$. The initial values are taken at x = 0.3, y = 0.4 and z = 0.5. The time delay τ is chosen to be $\tau = 0.1$. Fig. 9(a) shows the stabilization of the limit cycle in the xy-phase plane when the gain matrix has the values $k_{11} = k_{22} = 8.3$, $k_{33} = 6.5$ or $k_{11} = k_{22} = 6.5$, $k_{33} = 0$ and $\tau = 0.1$. Fig. 9(b) shows the stabilization of the limit cycle in the xy-phase plane when the gain matrix has the values $k_{11} = k_{22} = 8.5$, $k_{33} = 3.5$ or $k_{11} = k_{22} = 8.1$, $k_{33} = 0$ and $\tau = 0.1$. When the gain matrix has the values $k_{11} = k_{22} = 0$ and $k_{33} = 4.7$, the chaotic orbit converges to a periodic orbit with three period, see Fig. 10(a). Fig. 10(b) shows the stabilization of the limit cycle in the xy-phase plane when the gain matrix has the values $k_{11} = k_{22} = 0$, $k_{33} = 6.5$ and $\tau = 0.1$.



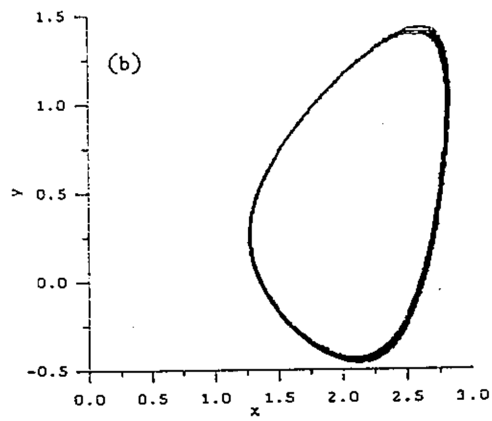


Fig. 9(a) The limit cycle of the controlled system (4.2), for the values $k_{11} = k_{22} = 8.3$, $k_{33} = 6.5$ or $k_{11} = k_{22} = 6.5$, $k_{33} = 0$ and $\tau = 0.1$ is plotted in the x-y phase plane. (b) Periodic orbit is stabilized in the x-y phase plane of the controlled system (4.2) at the values $k_{11} = k_{22} = 8.5$, $k_{33} = 3.5$ or $k_{11} = k_{22} = 8.1$, $k_{33} = 0$ and $\tau = 0.1$.

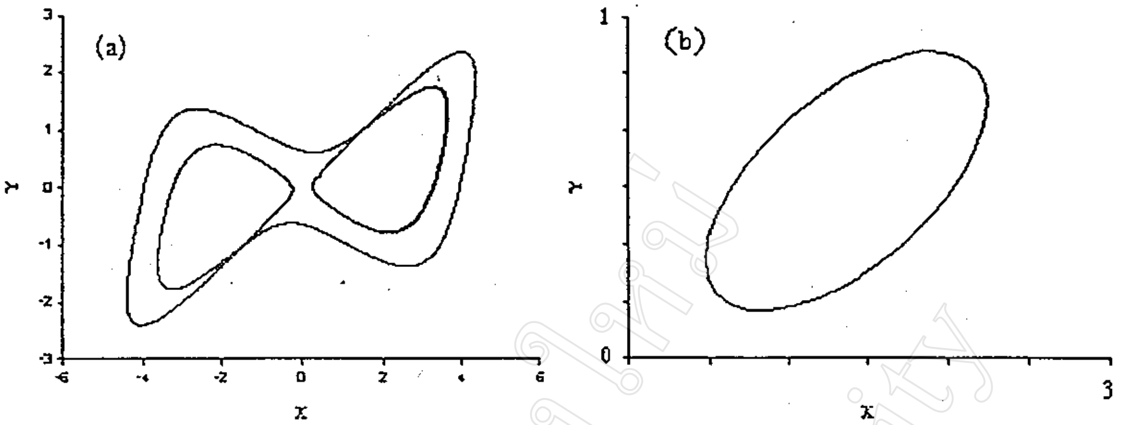


Fig. 10. (a) The periodic trajectory of period three of the controlled system (4.2), for the values $k_{11} = k_{22} = 0$, $k_{33} = 4.7$ and $\tau = 0.1$ is plotted in the x-y phase plane. (b) Periodic orbit is stabilized in the x-y phase plane of the controlled system (4.2) at the values $k_{11} = k_{22} = 0$, $k_{33} = 6.5$ and $\tau = 0.1$.