CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some basic concepts and some methods which will be used in our research.

2.1 The Logistic population model

Let $N(t)$ denote the number of the population at time t. Consider $N(t)$ as a continuous function of t. We shall make the following basic assumptions:

Assumption (A): The derivative of $N(t)$ is a continuous function of t for all $t > 0$. (The derivative of $N(t)$ describes the rate of change of the population.)

Assumption (B): The rate of change of the population N is of the form $(b - sN)N$ when b and s are positive constants.

Assumption (C): If α denotes the population at $t = 0$, then $b - s\alpha > 0$.

The mathematical model resulting from Assumptions (A), (B), and (C) for the population growth is the following initial value problem:

$$\frac{dN}{dt} = (b - sN)N, \qquad N(0) = \alpha \qquad (0 < t < \infty). \tag{2.1}$$

The Eq.(2.1) is known as the *logistic equation*, and the corresponding graph of this equation is called the *logistic curve*.

2.2 Integral equations

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics, where they are not only useful but indispensable even for numerical computation.

2.2.1 Volterra integral equations

Let $K(x,y)$ be a complex valued continuous function defined on a domain $a \leq x, y \leq b$; $f(x)$ be a complex valued continuous function defined on interval $a \leq x \leq b$ and λ be an arbitrary complex number; then the following integral equations

$$\phi(x) - \lambda \int_a^x K(x,y)\phi(y)dy = f(x) \qquad (a \leq x \leq b) \tag{2.2}$$

and

$$\int_a^x K(x,y)\phi(y)dy = f(x) \qquad (a \leq x \leq b). \tag{2.3}$$

are called *Volterra integral equation* of the second and first kind, respectively, for function $\phi(x)$.

2.2.2 Fredholm integral equations

Let $K(x,y)$ be a complex valued continuous function defined on a domain $a \leq x, y \leq b$; $f(x)$ be a complex valued continuous function defined on interval $a \leq x \leq b$ and λ be an arbitrary complex number; then the general linear *Fredholm integral equation of the second kind* for function $\phi(x)$ is an equation of the type

$$\phi(x) - \lambda \int_a^b K(x,y)\phi(y)dy = f(x) \qquad (a \leq x \leq b) \tag{2.4}$$

while the linear *Fredholm integral equation of the first kind* is given by

$$\int_a^b K(x,y)\phi(y)dy = f(x) \qquad (a \leq x \leq b). \tag{2.5}$$

2.3 The Volterra model

Consider the Volterra model for population growth of a species within a closed system

$$\frac{dp}{d\tilde{t}} = ap - bp^2 - cp\int_0^{\tilde{t}} p(x)dx, \qquad p(0) = p_0, \tag{2.6}$$

where $a > 0$ is the birth rate coefficient,

$b > 0$ is the intraspecies competition coefficient,

$c > 0$ is the toxicity coefficient,

p_0 is the initial population,

and $p = p(\tilde{t})$ denotes the population at time $\tilde{t}$.

The term $cp\int pdx$ represents the effect of toxin accumulation on the species.

Several time scales and population scales may be employed (see, e.g., [2]). However, we shall scale time and population by introducing the nondimensional variables

$$t = \frac{\tilde{t}}{b/c}, \qquad u = \frac{p}{a/b} \tag{2.7}$$

to produce the nondimensional problem

$$\kappa\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx, \qquad u(0) = u_0, \tag{2.8}$$

where $\kappa \equiv c/ab$ is a nondimensional parameter.

2.4 Series solutions of Volterra integral equations of the second kind

Consider the nonhomogeneous Volterra integral equations of the second kind, with variable limits of integration of the form

$$u(x) = f(x) + \int_a^x K(x,t)u(t)dt, \tag{2.9}$$

where $K(x,t)$ is the kernel of the integral equation.

The strategy consists first of representing $u(x)$ as a power series expansion given by

$$u(x) = \sum_{k=0}^{\infty} a_k x^k \tag{2.10}$$

Next, we use the Taylor series expansions for $f(x)$ and $K(x,t)$. Assuming that the solution exists, it remains to formally determine the coefficients $a_k, k \geq 0$. To achieve this goal we insert (2.10) and the above mentioned Taylor expansions into both sides of (2.9). With this substitution, the difficult integral in the right-hand side of (2.9) will be transformed into a more readily solvable integral, which can be solved more easily than the original. Integrate the result with classical integral term by term and equating the coefficients of similar powers of x from both sides lead to the complete determination of the coefficients $a_k, k \geq 0$. Having established these coefficients, the solution of (2.9) is readily obtained in a power series form.

2.5 Direct solutions of Fredholm integral equations of the second kind

Consider the nonhomogeneous Fredholm integral equations of the second kind, those with fixed limits of integration, given by

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt, \tag{2.11}$$

where $K(x,t)$ is the kernel of the integral equation. We may assume that kernel $K(x,t)$ is a separable kernel given by

$$K(x,t) = h(x)g(t). \tag{2.12}$$

Substituting (2.12) into (2.11) we get

$$u(x) = f(x) + h(x)\int_a^b g(t)u(t)dt. \tag{2.13}$$

We can easily observe that the integral in (2.13) is a definite integral, hence it will be reasonable to assume that

$$\int_a^b g(t)u(t)dt = \alpha, \tag{2.14}$$

where α is a constant. Clearly this assumption carries the Eq.(2.13) into

$$u(x) = f(x) + \alpha h(x). \tag{2.15}$$

It remains to evaluate the constant α to establish the exact solution. This can be done easily by inserting $u(x)$ of (2.15) into the integral of (2.14). With this substitution, the unsolvable integral (2.14) will be transformed into

$$\alpha = \int_a^b g(t)[f(t) + \alpha h(t)]dt, \tag{2.16}$$

a more readily solvable integral that formally determines a numerical value for the constant α. Having established α, the exact solution $u(x)$ follows immediately upon substituting the value of α obtained from (2.16) into (2.15). This completes the technique.

2.6 The decomposition method

The Adomian decomposition method enables the accurate and efficient analytic solution of nonlinear ordinary or partial differential equations without the need to resort to linearization or perturbation approaches. It unifies the treatment of linear and nonlinear, ordinary or partial differential equations, or systems of such equations, into a single basis method, which is applicable to both initial and boundary-value problems.

2.6.1 Description of the technique

Consider the differential equation

$$Lu + Ru + Nu = g, \tag{2.17}$$

where L is the highest order derivative which is assumed to be easily invertible, Ru is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of Eq.(2.17), and using the given conditions we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \tag{2.18}$$

where the function f represents the terms arising by integrating the source term g and from using the given conditions, all are assumed to be prescribed.

The standard Adomian method defines the solution $u(x)$ by the series

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.19}$$

where the components $u_0, u_1, u_2, ...$ are usually determined recursively by:

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= -L^{-1}(Ru_k) - L^{-1}(Nu_k), \qquad k \geq 0. \end{aligned} \tag{2.20}$$

It is important to note that the decomposition method suggests that the zeroth component u_0 usually defined by the function f described above. Having determined the components $u_0, u_1, u_2, ...$, the solution u in a series form defined by Eq.(2.19) follows immediately.

2.6.2 Algorithm for calculating Adomian polynomials for nonlinear operators

The Adomian decomposition method decomposes the linear term $u(x,t)$ into an infinite sum of components $u_n(x,t)$ defined by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.21}$$

Moreover, the decomposition method identifies the nonlinear term $F(u(x,t))$ by the decomposition series

$$F(u(x,t)) = \sum_{n=0}^{\infty} A_n \tag{2.22}$$

where A_n are the so-called *Adomian polynomials.*

To calculate Adomian polynomials, given a nonlinear operator $F(u(x,t))$, then the first few polynomials are given by

$$
\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1F'(u_0),\\
A_2 &= u_2F'(u_0) + \frac{1}{2!}u_1^2F''(u_0),\\
A_3 &= u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^3F'''(u_0),\\
A_4 &= u_4F'(u_0) + (\frac{1}{2!}u_2^2 + u_1u_3)F''(u_0) + \frac{1}{2!}u_1^2u_2F'''(u_0) + \frac{1}{4!}u_1^4F^{(iv)}(u_0).\\
&\vdots
\end{aligned} \tag{2.23}
$$

Other polynomials can be generated in a similar manner.

In another way, we can calculate Adomian polynomials A_n by using a new algorithm. We will show just only case which is used in our research.

Case 1. $F(u) = u^2$

We first set

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.24}$$

Substituting (2.24) into $F(u) = u^2$ gives

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots)^2. \tag{2.25}$$

Expanding the expression at the right-hand side gives

$$F(u) = u_0^2 + 2u_0u_1 + 2u_0u_2 + u_1^2 + 2u_0u_3 + 2u_1u_2 + \cdots . \tag{2.26}$$

The expansion in (2.26) can be rearranged by grouping all terms with the sum of the subscripts of the components of u_n is the same. This means that we can rewrite (2.26) as

$$
\begin{aligned}
F(u) = {} & \underbrace{u_0^2}_{A_0} + \underbrace{2u_0u_1}_{A_1} + \underbrace{2u_0u_2 + u_1^2}_{A_2} + \underbrace{2u_0u_3 + 2u_1u_2}_{A_3}\\
& + \underbrace{2u_0u_4 + 2u_1u_3 + u_2^2}_{A_4} + \underbrace{2u_0u_5 + 2u_1u_4 + 2u_2u_3}_{A_5} + \cdots .
\end{aligned} \tag{2.27}
$$

This gives Adomian polynomials for $F(u) = u^2$ by

$$
\begin{aligned}
A_0 &= u_0^2, \\
A_1 &= 2u_0u_1, \\
A_2 &= 2u_0u_2 + u_1^2, \\
A_3 &= 2u_0u_3 + 2u_1u_2, \\
A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \\
A_5 &= 2u_0u_5 + 2u_1u_4 + 2u_2u_3. \\
&\vdots
\end{aligned} \tag{2.28}
$$

Case 2. $F(u) = uu_x$.

We first set

$$
\begin{aligned}
u &= \sum_{n=0}^{\infty} u_n, \\
u_x &= \sum_{n=0}^{\infty} u_{n_x}.
\end{aligned} \tag{2.29}
$$

Substituting (2.29) into $F(u) = uu_x$ yields

$$F(u) = (u_0 + u_1 + u_2 + u_3 + u_4 + \cdots) \times (u_{0_x} + u_{1_x} + u_{2_x} + u_{3_x} + u_{4_x} + \cdots). \tag{2.30}$$

Multiplying the two factors gives

$$
\begin{aligned}
F(u) = {} & u_0u_{0_x} + u_{0_x}u_1 + u_0u_{1_x} + u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0 + u_{0_x}u_3 \\
& + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0 + u_{0_x}u_4 + u_0u_{4_x} + u_{1_x}u_3 \\
& + u_1u_{3_x} + u_2u_{2_x} \cdots .
\end{aligned} \tag{2.31}
$$

Collecting all terms with the same sum of subscripts of the components u_n we can rewritten Eq.(2.31) in the form

$$
\begin{aligned}
F(u) = {} & \underbrace{u_{0_x}u_0}_{A_0} + \underbrace{u_{0_x}u_1 + u_0u_{1_x}}_{A_1} + \underbrace{u_{0_x}u_2 + u_{1_x}u_1 + u_{2_x}u_0}_{A_2} \\
& + \underbrace{u_{0_x}u_3 + u_{1_x}u_2 + u_{2_x}u_1 + u_{3_x}u_0}_{A_3} \\
& + \underbrace{u_{0_x}u_4 + u_0u_{4_x} + u_{1_x}u_3 + u_1u_{3_x} + u_2u_{2_x}}_{A_4} + \cdots .
\end{aligned} \tag{2.32}
$$

Consequently, the Adomian polynomials are given by

$$
\begin{aligned}
A_0 &= u_{0_x} u_0, \\
A_1 &= u_{0_x} u_1 + u_0 u_{1_x}, \\
A_2 &= u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0, \\
A_3 &= u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{3_x} u_0, \\
A_4 &= u_{0_x} u_4 + u_0 u_{4_x} + u_{1_x} u_3 + u_1 u_{3_x} + u_2 u_{2_x}. \\
&\vdots
\end{aligned}
$$

2.7 Padé approximation

Suppose that we are given a Power series $\sum_{i=0}^{\infty} c_i x^i$, representing a function $f(x)$, so that

$$f(x) = \sum_{i=0}^{\infty} c_i x^i. \tag{2.33}$$

This expansion is the fundamental starting point of any analysis using Padé approximants. Throughout this work we reserve the notation $c_i = 0, 1, 2, \cdots$ for the given set of coefficients, and $f(x)$ is the associated function. A Padé approximant is a rational fraction

$$[L/M] = \frac{a_0 + a_1 x + \cdots + a_L x^L}{b_0 + b_1 x + \cdots + b_M x^M}, \qquad a_L \neq 0, b_M \neq 0 \tag{2.34}$$

which has a Maclaurin expansion which agrees with (2.33) as far as possible. The most useful of the Padé approximations are those with the degree of the numerator equal to, or one more than the degree of the denominator. Notice that in (2.34) there are $L+1$ numerator coefficients and $M+1$ denominator coefficients. There is a more or less irrelevant common factor between them, and for definiteness we take $b_0 = 1$. This choice turns out to be an essential part of the precise definition, and (2.34) is our conventional notation with this choice for b_0. So there are $L+1$ independent numerator coefficients and M independent denominator coefficients, making $L+M+1$ unknown coefficients in all. This number suggests that normally the $[L/M]$ ought to fit the power series (2.33) through the orders $1, x, x^2, \ldots, x^{L+M}$

in the notation of formal power series,

$$\sum_{i=0}^{\infty} c_i x^i = \frac{a_0 + a_1 x + \cdots + a_L x^L}{b_0 + b_1 x + \cdots + b_M x^M} + Q(x^{L+M+1}). \qquad (2.35)$$

Returning to (2.35) and cross-multiplying, we find that

$$(b_0 + b_1 x + \cdots + b_M x^M)(c_0 + c_1 x + \cdots) = a_0 + a_1 x + \cdots + a_L x^L + \cdots \qquad (2.36)$$

Equating the coefficients of $x^{L+1}, x^{L+2}, \ldots, x^{L+M}$ we find

$$\begin{aligned} b_M c_{L-M+1} + b_{M-1} c_{L-M+2} + \cdots + b_0 c_{L+1} &= 0, \\ b_M c_{L-M+2} + b_{M-1} c_{L-M+3} + \cdots + b_0 c_{L+2} &= 0, \\ &\vdots \\ b_M c_L + b_{M-1} c_{L+1} + \cdots + b_0 c_{L+M} &= 0. \end{aligned} \qquad (2.37)$$

If $j < 0$, we define $c_j = 0$ for consistency. Since $b_0 = 1$, Eqs.(2.37) become a set of M linear equations for M unknown denominator coefficients:

$$\begin{bmatrix} c_{L-M+1} & c_{L-M+2} & c_{L-M+3} & \cdots & c_L \\ c_{L-M+2} & c_{L-M+3} & c_{L-M+4} & \cdots & c_{L+1} \\ c_{L-M+3} & c_{L-M+4} & c_{L-M+5} & \cdots & c_{L+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_L & c_{L-M+1} & c_{L-M+2} & \cdots & c_{L+M+1} \end{bmatrix} \begin{bmatrix} b_M \\ b_{M-1} \\ b_{M-2} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} c_{L+1} \\ c_{L+2} \\ c_{L+3} \\ \vdots \\ c_{L+M} \end{bmatrix} \qquad (2.38)$$

from which the b_i may be found. The numerator coefficients $a_0, a_1, \ldots, a_L$ follow immediately from (2.36) by equating the coefficients $1, x, x^2, \ldots, x^L$:

$$\begin{aligned} a_0 &= c_0, \\ a_1 &= c_1 + b_1 c_0, \\ a_2 &= c_2 + b_1 c_1 + b_2 c_0, \\ &\vdots \\ a_L &= c_L + \sum_{i=1}^{\min(L,M)} b_i c_{L-i}. \end{aligned} \qquad (2.39)$$

Thus (2.38) and (2.39) normally determine the Padé numerator and denominator and called the Padé equations; we have constructed a $[L/M]$ Padé approximant, which agrees with $\sum_{i=0}^{\infty} c_i x^i$ through order x^{L+M}.