CHAPTER 3

MAIN RESULTS

In this chapter, the analysis of the solving methods as mentioned in chapter 1, and the numerical results will be presented.

3.1 The direct solution method combined with the series solution method

3.1.1 Analysis of the method

Consider the population growth model characterized by the nonlinear Volterra-Fredholm integro-differential equation

$$\kappa\frac{du}{dt} = u(t) - u^2(t) - u(t)\int_0^t u(x)dx - Cu(t)\int_{t_1}^{t_2} u(x)dx, \qquad u(0) = u_0. \quad (3.1)$$

Assume that

$$\alpha = \int_{t_1}^{t_2} u(x)dx, \quad (3.2)$$

where α is a constant.

This assumption carries the Eq.(3.1) into

$$\frac{du}{dt} = \frac{1}{\kappa}\{(1-\alpha C)u(t) - u^2(t) - u(t)\int_0^t u(x)dx\}, \qquad u(0) = u_0. \quad (3.3)$$

Assume that the solution $u(t)$ is represented as a power series of the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^n. \quad (3.4)$$

The main concern is to formally determine the coefficients $a_n, n \geq 0$. To achieve this goal we substitute Eq.(3.4) into both sides of Eq.(3.3)

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \frac{1}{\kappa}\{(1-\alpha C)(\sum_{n=0}^{\infty} a_n t^n) - (\sum_{n=0}^{\infty} a_n t^n)^2 - (\sum_{n=0}^{\infty} a_n t^n)\int_0^t (\sum_{n=0}^{\infty} a_n x^n) dx\} \tag{3.5}$$

by this substitution, the difficult integral in the right-hand side of Eq.(3.3) will be transformed into a more readily solvable integral. Evaluating the integral at the right-hand side, equating the coefficients of similar powers of t from both sides, and using the initial condition lead to

$$a_n = \sum_{j=0}^{n} b_{jn}\alpha^j, \tag{3.6}$$

where b_{jn} is the coefficients of α^j which depend on a_0, κ, or C.
Substitute (3.6) into (3.4), we obtain

$$u(t) = \sum_{n=0}^{\infty}[\sum_{j=0}^{n} b_{jn}\alpha^j]t^n. \tag{3.7}$$

Substituting $u(t)$ of (3.7) into the integral of (3.2). With this substitution, the unsolvable integral (3.2) will be transformed into

$$\alpha = \int_{t_1}^{t_2} \sum_{n=0}^{\infty}[\sum_{j=0}^{n} b_{jn}\alpha^j]x^n dx, \tag{3.8}$$

a more readily solvable integral that formally determines a numerical value for the constant α. Having established α, the approximation solution $u(t)$ follows immediately upon substituting the value of α obtained from (3.8) into (3.7). This completes the technique.

3.1.2 Numerical results

In this section, the nonlinear Volterra-Fredholm integro-differential equation will be tested by using the proposed algorithm discussed above.

Consider the population growth model characterized by the nonlinear Volterra-Fredholm integro-differential equation

$$\frac{du}{dt} = 10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx - u(t)\int_{0.9}^{1.0} u(x)dx, \qquad u(0) = 0.1, \tag{3.9}$$

where the initial condition $u(0) = 0.1$ and the nondimensional parameter $\kappa = 0.1$, $C = 0.1$, and the interval time $t_1 = 0.9$, $t_2 = 1.0$.

Let

$$\alpha = \int_{0.9}^{1.0} u(x)dx, \tag{3.10}$$

and substitute into (3.9) gives

$$\frac{du}{dt} = (10-\alpha)u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx. \tag{3.11}$$

Assume that the solution $u(t)$ is represented as a power series of the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{3.12}$$

To determine the coefficients $a_n, n \geq 0$, we substitute Eq.(3.12) into both sides of Eq.(3.11) to find

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = (10-\alpha)(\sum_{n=0}^{\infty} a_n t^n) - 10(\sum_{n=0}^{\infty} a_n t^n)^2 - 10(\sum_{n=0}^{\infty} a_n t^n)\int_0^t (\sum_{n=0}^{\infty} a_n x^n)dx. \tag{3.13}$$

Expanding the power series, evaluating the integral at the right-hand side lead to

$$
\begin{aligned}
& a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + 5a_5t^4 + 6a_6t^5 + 7a_7t^6 + 8a_8t^7 + \cdots \\
&\quad = (10-\alpha)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots) \\
&\qquad - 10(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots)^2 \\
&\qquad - 10(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots) \times (a_0t + \frac{a_1}{2}t^2 + \frac{a_2}{3}t^3 + \frac{a_3}{4}t^4 + \cdots) \\
&\quad = ((10-\alpha)a_0 - 10a_0^2)) + ((10-\alpha)a_1 - 20a_0a_1 - 10a_0^2)t \\
&\qquad ((10-\alpha)a_2 - 20a_0a_2 - 10a_1^2 - 15a_0a_1)t^2 \\
&\qquad + ((10-\alpha)a_3 - 20a_2a_1 - 20a_0a_3 - 5a_1^2 - 13.33333333a_0a_2)t^3 \\
&\qquad + ((10-\alpha)a_4 - 20a_0a_4 - 20a_3a_1 - 10a_2^2 - 8.33333333a_2a_1 - 12.5a_0a_3)t^4 \\
&\qquad + \cdots
\end{aligned}
\tag{3.14}
$$

Equating the coefficients of similar powers of t from both sides, and using the initial condition leads to

$$
\begin{aligned}
a_0 &= 0.1 \\
a_1 &= ((10-\alpha)a_0 - 10a_0^2) \\
&= ((10-\alpha)(0.1) - 10(0.1)^2) \\
&= 0.9 - 0.1\alpha, \\
a_2 &= ((10-\alpha)a_1 - 20a_0a_1 - 10a_0^2)/2 \\
&= ((10-\alpha)(0.9-\alpha) - 20(0.1)(0.9-\alpha) - 10(0.1)^2)/2 \\
&= 3.55 - 0.85\alpha + 0.05\alpha^2, \\
a_3 &= ((10-\alpha)a_2 - 20a_0a_2 - 10a_1^2 - 15a_0a_1)/3 \\
&= 6.316666667 - 2.8\alpha + 0.3833333333\alpha^2 - 0.01666666667\alpha^3, \\
a_4 &= ((10-\alpha)a_3 - 20a_2a_1 - 20a_0a_3 - 5a_1^2 - 13.33333333a_0a_2)/4, \\
&= -5.537499997 - 1.070833334\alpha + 0.7875\alpha^2 - 0.1041666667\alpha^3 \\
&\quad + 0.004166666667\alpha^4,
\end{aligned}
$$

$$a_5 = ((10-\alpha)a_4 - 20a_0a_4 - 20a_3a_1 - 10a_2^2 - 8.33333333a_2a_1 - 12.5a_0a_3)/5,$$
$$= -63.70916666 + 26.6375\alpha - 3.493333333\alpha^2 + 0.07166666663\alpha^3$$
$$+ 0.01583333333\alpha^4 - 0.000833333333\alpha^5,$$
$$a_6 = ((10-\alpha)a_5 - 20a_0a_5 - 20a_1a_4 - 20a_2a_3 - 12a_0a_4 - 7.5a_1a_3 - 3.333333333a_2^2)/6,$$
$$= -156.0804167 + 106.0386111\alpha - 26.87638889\alpha^2 + 3.1375\alpha^3$$
$$- 0.1534722222\alpha^4 - 0.000416666666\alpha^5 + 0.0001388888889\alpha^6,$$
$$a_7 = ((10-\alpha)a_6 - 20a_0a_6 - 20a_2a_4 - 20a_1a_5 - 10a_3^2 - 7a_1a_4 - 5.833333333a_2a_3$$
$$- 11.66666667a_0a_5)/7,$$
$$= -18.47323422 + 113.4596231\alpha - 60.80011906\alpha^2 + 13.33035714\alpha^3$$
$$- 1.439424603\alpha^4 + 0.07357142857\alpha^5 - 0.001130952381\alpha^6 - 0.00001984126984\alpha^7,$$
$$a_8 = ((10-\alpha)a_7 - 20a_0a_7 - 20a_3a_4 - 20a_2a_5 - 20a_1a_6 - 2.5a_3^2 - 6.666666667a_1a_5$$
$$- 5.333333333a_2a_4 - 11.42857143a_0a_6)/8,$$
$$= 1056.288569 - 585.4652901\alpha + 95.37555799\alpha^2 + 4.387817468\alpha^3$$
$$- 3.276068949\alpha^4 + 0.4197941469\alpha^5 - 0.02293402778\alpha^6 + 0.0004340277778\alpha^7$$
$$+ 0.2480158730 \times 10^{-5}\alpha^8 \qquad (3.15)$$

By using the first few terms of each expansion in Eq.(3.12), we obtain

$$
\begin{aligned}
u(t) = {} & 0.1 + (0.9 - 0.1\alpha)t + (3.55 - 0.85\alpha + 0.05\alpha^2)t^2 \\
& + (6.316666667 - 2.8\alpha + 0.3833333333\alpha^2 - 0.01666666667\alpha^3)t^3 \\
& + (-5.537499997 - 1.070833334\alpha + 0.7875\alpha^2 - 0.1041666667\alpha^3 \\
& + 0.004166666667\alpha^4)t^4 + (-63.70916666 + 26.6375\alpha - 3.493333333\alpha^2 \\
& + 0.07166666663\alpha^3 + 0.01583333333\alpha^4 - 0.000833333333\alpha^5)t^5 \\
& + (-156.0804167 + 106.0386111\alpha - 26.87638889\alpha^2 + 3.1375\alpha^3 \\
& - 0.1534722222\alpha^4 - 0.000416666666\alpha^5 + 0.0001388888889\alpha^6)t^6 \\
& + (-18.47323422 + 113.4596231\alpha - 60.80011906\alpha^2 + 13.33035714\alpha^3 \\
& - 1.439424603\alpha^4 + 0.07357142857\alpha^5 - 0.001130952381\alpha^6 \\
& - 0.00001984126984\alpha^7)t^7 + (1056.288569 - 585.4652901\alpha \\
& + 95.37555799\alpha^2 + 4.387817468\alpha^3 - 3.276068949\alpha^4 + 0.4197941469\alpha^5 \\
& - 0.02293402778\alpha^6 + 0.0004340277778\alpha^7 + 0.2480158730 \times 10^{-5}\alpha^8)t^8 \\
& + O(t^9). \qquad (3.16)
\end{aligned}
$$

Substituting (3.16) into (3.10) yields the integrable integral

$$
\begin{aligned}
\alpha = \int_{0.9}^{1.0} & [0.1 + (0.9 - 0.1\alpha)x + (3.55 - 0.85\alpha + 0.05\alpha^2)x^2 \\
& + (6.316666667 - 2.8\alpha + 0.3833333333\alpha^2 - 0.01666666667\alpha^3)x^3 \\
& + (-5.537499997 - 1.070833334\alpha + 0.7875\alpha^2 - 0.1041666667\alpha^3 \\
& + 0.004166666667\alpha^4)x^4 + \cdots + O(x^9)]dx. \qquad (3.17)
\end{aligned}
$$

Solving for positive real α, we obtain

$$\alpha = 3.693452238, 14.16054255 \qquad (3.18)$$

With $\alpha = 3.693452238$,the approximation of $u(t)$ given by

$$
\begin{aligned}
u(t) = {} & 0.1 + 0.5306547762t + 1.092645070t^2 + 0.364533692t^3 - 3.222824317t^4 \\
& - 6.99485762t^5 - 0.9081106895t^6 + 22.46202251t^7 + 40.86978039t^8 \\
& + O(t^9). \qquad (3.19)
\end{aligned}
$$

It is interest to point out that eight steps were employed to determine the approximation (3.19)with α in (3.18), and further terms can be obtained to increase the degree of accuracy.

Since we are interested in the small perturbation in the system, then we use only small α.

3.2 The direct solution method combined with the decomposition method

In this section, the direct solution method and the decomposition method will be implemented to the model of Eq.(3.1).

3.2.1 Analysis of the method

Recall the population growth model in section 3.1

$$\kappa\frac{du}{dt} = u(t) - u^2(t) - u(t)\int_0^t u(x)dx - Cu(t)\int_{t_1}^{t_2} u(x)dx, \qquad u(0) = u_0. \quad (3.20)$$

The assumption

$$\alpha = \int_{t_1}^{t_2} u(x)dx, \quad (3.21)$$

carries the Eq.(3.20) into

$$\frac{du}{dt} = \frac{1}{\kappa}\{(1-\alpha C)u(t) - u^2(t) - u(t)\int_0^t u(x)dx\}, \qquad u(0) = u_0. \quad (3.22)$$

Rewrite the Volterra-Fredholm's population model (3.22) in an operator form

$$Lu(t) = \frac{1}{\kappa}\{(1-\alpha C)u(t) - u^2(t) - u(t)\int_0^t u(x)dx\}, \qquad u(0) = u_0, \quad (3.23)$$

where the differential operator L is defined by

$$L = \frac{d}{dt}. \quad (3.24)$$

It is clear that L is invertible so that the integral operator is defined by

$$L^{-1}(.) = \int_0^t (.)dt. \quad (3.25)$$

Applying L^{-1} to both sides of Eq.(3.23) and using the initial condition lead to

$$u(t) = u_0 + L^{-1}(\frac{1}{\kappa}\{(1-\alpha C)u(t) - u^2(t) - u(t)\int_0^t u(x)dx\}). \tag{3.26}$$

We usually represent $u(t)$ in Eq.(3.26) by the decomposition series

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \tag{3.27}$$

Accordingly, the main concern here is to formally determine the components $u_n(t), n \geq 0$. To do, we substitute Eq.(3.27) into both sides of Eq.(3.26) to obtain

$$\sum_{n=0}^{\infty} u_n(t) = u_0 + L^{-1}(\frac{1}{\kappa}\{(1-\alpha C)\sum_{n=0}^{\infty} u_n(t) - \sum_{n=0}^{\infty} A_n(t) - \int_0^t \sum_{n=0}^{\infty} B_n(x,t)dx\}), \tag{3.28}$$

where the nonlinear term $u^2(t)$ and $u(x)u(t)$ are represented by the so-called Adomian polynomials $A_n(t)$ and $B_n(x,t)$, respectively. In other words,we set

$$u^2(t) = \sum_{n=0}^{\infty} A_n(t), \tag{3.29}$$

$$u(x)u(t) = \sum_{n=0}^{\infty} B_n(x,t). \tag{3.30}$$

The Adomian polynomials $A_n(t)$ and $B_n(x,t)$ are generated specifically according to algorithms in the Preliminaries. To determine the components $u_0, u_1, u_2, \ldots$ of $u(t)$ we follow the recursive relationship

$$u_0(t) = u_0, \tag{3.31}$$

$$u_{k+1}(t) = L^{-1}(\frac{1}{\kappa}\{(1-\alpha C)u_k(t) - A_k(t) - \int_0^t B_k(x,t)dx\}), k \geq 0. \tag{3.32}$$

After we obtained the components $u_n(t)$ of $u(t)$, we find that

$$\sum_{n=0}^{\infty} u_n(t) = \sum_{n=0}^{\infty}[\sum_{j=0}^{n} b_{jn}\alpha^j]t^n, \tag{3.33}$$

where b_{jn} is the coefficients of α^j which depend on a_0, κ, or C.

Then

$$u(t) = \sum_{n=0}^{\infty}[\sum_{j=0}^{n} b_{jn}\alpha^j]t^n. \tag{3.34}$$

Substituting $u(t)$ of (3.34) into the integral of (3.21). With this substitution, the unsolvable integral (3.21) will be transformed into

$$\alpha = \int_{t_1}^{t_2} \sum_{n=0}^{\infty} [\sum_{j=0}^{n} b_{jn} \alpha^j] x^n dx, \tag{3.35}$$

a more readily solvable integral that formally determines a numerical value for the constant α. Having established α, the approximation solution $u(t)$ follows immediately upon substituting the value of α obtained from (3.35) into (3.34). This completes the technique.

3.2.2 Numerical results

Consider the population growth model characterized by the nonlinear Volterra-Fredholm integro-differential equation

$$\frac{du}{dt} = 10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx - u(t)\int_{0.9}^{1.0} u(x)dx, \qquad u(0) = 0.1, \tag{3.36}$$

where the initial condition $u(0) = 0.1$ and the nondimensional parameter $\kappa = 0.1$, $C = 0.1$, and the interval time $t_1 = 0.9$, $t_2 = 1.0$.

We set

$$\alpha = \int_{0.9}^{1.0} u(x)dx, \tag{3.37}$$

where α is a constant. Using (3.37) into (3.36) gives

$$\frac{du}{dt} = (10-\alpha)u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx, \qquad u(0) = 0.1. \tag{3.38}$$

Rewrite the Volterra-Fredholm's population model (3.38) in an operator form

$$Lu(t) = (10-\alpha)u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx, \qquad u(0) = 0.1, \tag{3.39}$$

Applying L^{-1} to both sides of Eq.(3.39) and using the initial condition lead to

$$u(t) = 0.1 + L^{-1}((10-\alpha)u(t) - 10u^2(t) - 10\int_0^t u(t)u(x)dx). \tag{3.40}$$

Substitute the decomposition series

$$u(t) = \sum_{n=0}^{\infty} u_n(t). \tag{3.41}$$

into both sides of Eq.(3.40) to obtain

$$\sum_{n=0}^{\infty} u_n(t) = 0.1 + L^{-1}((10-\alpha)\sum_{n=0}^{\infty} u_n(t) - 10\sum_{n=0}^{\infty} A_n(t) - 10\int_0^t \sum_{n=0}^{\infty} B_n(x,t)dx), \tag{3.42}$$

To evaluate the Adomian polynomials $A_n(t)$ and $B_n(x,t)$, we follow the procedure in the Preliminaries. The expressions for $A_n(t)$ and $B_n(x,t), n = 0, 1, 2, \ldots$ are as follows

For $A_n(t)$ **:** Let $F(u) = u^2(t)$, then we obtain

$$\begin{aligned}
A_0 &= u_0^2(t), \\
A_1 &= 2u_0(t)u_1(t), \\
A_2 &= 2u_0(t)u_2(t) + u_1^2(t), \\
A_3 &= 2u_0(t)u_3(t) + 2u_1(t)u_2(t), \\
A_4 &= 2u_0(t)u_4(t) + 2u_1(t)u_3(t) + u_2^2(t), \\
A_5 &= 2u_0(t)u_5(t) + 2u_1(t)u_4(t) + 2u_2(t)u_3(t), \\
A_6 &= 2u_0(t)u_6(t) + 2u_1(t)u_5(t) + 2u_2(t)u_4(t) + u_3^2(t), \\
A_7 &= 2u_0(t)u_7(t) + 2u_1(t)u_6(t) + 2u_2(t)u_5(t) + u_3(t)u_4(t), \ldots
\end{aligned} \tag{3.43}$$

For $B_n(x,t)$ **:** Let $F(u) = u(x)u(t)$, then we obtain

$$\begin{aligned}
B_0(x,t) &= u_0(x)u_0(t), \\
B_1(x,t) &= u_0(x)u_1(t) + u_1(x)u_0(t), \\
B_2(x,t) &= u_0(x)u_2(t) + u_1(x)u_1(t) + u_2(x)u_0(t), \\
B_3(x,t) &= u_0(x)u_3(t) + u_1(x)u_2(t) + u_2(x)u_1(t) + u_3(x)u_0(t), \\
B_4(x,t) &= u_0(x)u_4(t) + u_1(x)u_3(t) + u_2(x)u_2(t) + u_3(x)u_1(t) + u_4(x)u_0(t), \\
B_5(x,t) &= u_0(x)u_5(t) + u_1(x)u_4(t) + u_2(x)u_3(t) + u_3(x)u_2(t) + u_4(x)u_1(t) \\
&\quad + u_5(x)u_0(t), \\
B_6(x,t) &= u_0(x)u_6(t) + u_1(x)u_5(t) + u_2(x)u_4(t) + u_3(x)u_3(t) + u_4(x)u_2(t) \\
&\quad + u_5(x)u_1(t) + u_6(x)u_0(t), \\
B_7(x,t) &= u_0(x)u_7(t) + u_1(x)u_6(t) + u_2(x)u_5(t) + u_3(x)u_4(t) + u_4(x)u_3(t) \\
&\quad + u_5(x)u_2(t) + u_6(x)u_1(t) + u_7(x)u_0(t), \ldots
\end{aligned} \tag{3.44}$$

To determine the components $u_0, u_1, u_2, \ldots$ of $u(t)$, we follow the recursive relationship

$$u_0(t) = 0.1, \tag{3.45}$$
$$u_{k+1}(t) = L^{-1}((10-\alpha)u_k(t) - 10A_k(t) - 10\int_0^t B_k(x,t)dx), \ k \geq 0. \tag{3.46}$$

With $u_0(t)$ defined as shown above, we can evaluate the other components as follows

$$\begin{aligned}
u_0(t) &= 0.1,\\
u_1(t) &= L^{-1}((10-\alpha)u_0(t) - 10A_0(t) - 10\int_0^t B_0(x,t)dx)\\
&= L^{-1}((10-\alpha)u_0(t) - 10u_0^2(t) - 10\int_0^t u_0(x)u_0(t)dx)\\
&= \int_0^t ((10-\alpha)(0.1) - 10(0.1)^2 - 10\int_0^t (0.1)(0.1))dx\\
&= (0.9 - 0.1\alpha)t - 0.05t^2,\\
u_2(t) &= L^{-1}((10-\alpha)u_1(t) - 10A_1(t) - 10\int_0^t B_1(x,t)dx)\\
&= (3.6 - 0.85\alpha + 0.05\alpha^2)t^2 + (-0.583333333 + 0.066666667\alpha)t^3\\
&\quad + 0.016666667,\\
u_3(t) &= L^{-1}((10-\alpha)u_2(t) - 10A_2(t) - 10\int_0^t B_2(x,t)dx)\\
&= (6.9 - 2.86666667\alpha + 0.383333333\alpha^2 - 0.016666667\alpha^3)t^3 +\\
&\quad (-3.154166667 + 0.7625\alpha - 0.045833333\alpha^2)t^4\\
&\quad + (0.2425 - 0.028333333\alpha)t^5 - 0.004722222t^6,\\
u_4(t) &= L^{-1}((10-\alpha)u_3(t) - 10A_3(t) - 10\int_0^t B_3(x,t)dx)\\
&= (-2.4 - 1.833333333\alpha + 0.833333333\alpha^2 - 0.104166667\alpha^3 + 0.004166667\alpha^4)t^4\\
&\quad + (-9.351666667 + 3.799166667\alpha - 0.501666667\alpha^2 + 0.021666667\alpha^3)t^5\\
&\quad + (1.663194444 - 0.409166667\alpha + 0.025\alpha^2)t^6(-0.082738095 + 0.00984127\alpha)t^7\\
&\quad + 0.001230159t^8,
\end{aligned}$$

$$
\begin{aligned}
u_5(t) &= L^{-1}((10-\alpha)u_4(t) - 10A_4(t) - 10\int_0^t B_4(x,t)dx) \\
&= (-54.6 + 22.866666667\alpha - 2.991666667\alpha^2 + 0.05\alpha^3 + 0.015833333\alpha^4 \\
&\quad - 0.000833333\alpha^5)t^5 + (-9.338888889 + 8.208888889\alpha - 2.118055556\alpha^2 \\
&\quad + 0.218055556\alpha^3 - 0.007916667\alpha^4)t^6 + (6.603948413 - 2.670059524\alpha \\
&\quad + 0.352261905\alpha^2 - 0.015238095\alpha^3)t^7 + (-0.68844494 + 0.171760913\alpha \\
&\quad - 0.010634921\alpha^2)t^8 + O(t^9), \\
u_6(t) &= L^{-1}((10-\alpha)u_5(t) - 10A_5(t) - 10\int_0^t B_5(x,t)dx) \\
&= (-148.4 + 98.238888889\alpha - 24.783333333\alpha^2 + 2.919444444\alpha^3 \\
&\quad - 0.14555556\alpha^4 + 0.000416667\alpha^5 + 0.000138889\alpha^6)t^6 \\
&\quad + (47.534126984 - 12.496031746\alpha - 1.042857143\alpha^2 + 0.626547619\alpha^3 \\
&\quad - 0.06827381\alpha^4 + 0.002380952\alpha^5)t^7 + (12.976478175 - 8.969293155\alpha \\
&\quad + 2.093980655\alpha^2 - 0.204553571\alpha^3 + 0.007202381\alpha^4)t^8 + O(t^9), \\
u_7(t) &= L^{-1}((10-\alpha)u_6(t) - 10A_1(t) - 10\int_0^t B_6(x,t)dx) \\
&= (-72.528571429 + 128.615873016\alpha - 60.10952381\alpha^2 + 12.719047619\alpha^3 \\
&\quad - 1.371150794\alpha^4 + 0.071190476\alpha^5 - 0.001130952\alpha^6 - 0.000019841\alpha^7)t^7 \\
&\quad + (249.827876984 - 148.812202381\alpha + 32.028720238\alpha^2 - 2.636200397\alpha^3 \\
&\quad - 0.02390625\alpha^4 + 0.015230655\alpha^5 - 0.000612599\alpha^6)t^8 + O(t^9), \\
u_8(t) &= L^{-1}((10-\alpha)u_7(t) - 10A_1(t) - 10\int_0^t B_7(x,t)dx) \\
&= (794.171428571 - 427.855555556\alpha + 61.263492063\alpha^2 + 7.228571429\alpha^3 \\
&\quad - 3.259365079\alpha^4 + 0.404563492\alpha^5 - 0.022321429\alpha^6 + 0.000434028\alpha^7 \\
&\quad + (2.48015873 \times 10^{-6})\alpha^8)t^8 + O(t^9) \qquad (3.47)
\end{aligned}
$$

substitute (3.47) into (3.41) we obtain

$$
\begin{aligned}
u(t) = {} & 0.1 + (0.9 - 0.1\alpha)t + (3.55 - 0.85\alpha + 0.05\alpha^2)t^2 \\
& + (6.316666667 - 2.8\alpha + 0.3833333333\alpha^2 - 0.01666666667\alpha^3)t^3 \\
& + (-5.537499997 - 1.070833334\alpha + 0.7875\alpha^2 - 0.1041666667\alpha^3 \\
& + 0.004166666667\alpha^4)t^4 + (-63.70916666 + 26.6375\alpha - 3.493333333\alpha^2 \\
& + 0.07166666663\alpha^3 + 0.01583333333\alpha^4 - 0.000833333333\alpha^5)t^5 \\
& + (-156.0804167 + 106.0386111\alpha - 26.87638889\alpha^2 + 3.1375\alpha^3 \\
& - 0.1534722222\alpha^4 - 0.000416666666\alpha^5 + 0.0001388888889\alpha^6)t^6 \\
& + (-18.47323422 + 113.4596231\alpha - 60.80011906\alpha^2 + 13.33035714\alpha^3 \\
& - 1.439424603\alpha^4 + 0.07357142857\alpha^5 - 0.001130952381\alpha^6 \\
& - 0.00001984126984\alpha^7)t^7 + (1056.288569 - 585.4652901\alpha \\
& + 95.37555799\alpha^2 + 4.387817468\alpha^3 - 3.276068949\alpha^4 + 0.4197941469\alpha^5 \\
& - 0.02293402778\alpha^6 + 0.0004340277778\alpha^7 + 0.2480158730 \times 10^{-5}\alpha^8)t^8 \\
& + O(t^9). \qquad (3.48)
\end{aligned}
$$

Substituting (3.48) into (3.37) yields the integrable integral

$$
\begin{aligned}
\alpha = \int_{0.9}^{1.0} & [0.1 + (0.9 - 0.1\alpha)x + (3.55 - 0.85\alpha + 0.05\alpha^2)x^2 \\
& + (6.316666667 - 2.8\alpha + 0.3833333333\alpha^2 - 0.01666666667\alpha^3)x^3 \\
& + (-5.537499997 - 1.070833334\alpha + 0.7875\alpha^2 - 0.1041666667\alpha^3 \\
& + 0.004166666667\alpha^4)x^4 + \cdots + O(x^9)]dx. \qquad (3.49)
\end{aligned}
$$

Solving (3.49) for positive real α, gives

$$\alpha = 3.693452238, 14.16054255 \qquad (3.50)$$

For $\alpha = 3.693452238$, the approximation of $u(t)$ is given by

$$
\begin{aligned}
u(t) = {} & 0.1 + 0.5306547762t + 1.092645070t^2 + 0.364533692t^3 - 3.222824317t^4 \\
& - 6.99485762t^5 - 0.9081106895t^6 + 22.46202251t^7 + 40.86978039t^8 \\
& + O(t^9). \qquad (3.51)
\end{aligned}
$$

The approximation (3.51) is in full agreement with the approximation (3.19) obtained before by using the direct solution method with the series solution method. As the series solution method, eight iterations only were used for the derivation of Eq.(3.51).

3.3 The direct solution method combined with the method of converting the model to a nonlinear ODE

In this section, it will be useful to convert the Volterra-Fredholm's population model to an equivalent nonlinear ODE.

3.3.1 Analysis of the method

Consider the population growth model

$$\kappa\frac{du}{dt} = u(t) - u^2(t) - u(t)\int_0^t u(x)dx - Cu(t)\int_{t_1}^{t_2} u(x)dx, \qquad u(0) = u_0. \quad (3.52)$$

with

$$\alpha = \int_{t_1}^{t_2} u(x)dx, \quad (3.53)$$

Eq.(3.53) becomes

$$\frac{du}{dt} = \frac{1}{\kappa}\{(1-\alpha C)u(t) - u^2(t) - u(t)\int_0^t u(x)dx\}, \qquad u(0) = u_0. \quad (3.54)$$

In order to convert (3.54) to an ODE, we set

$$y(t) = \int_0^t u(x)dx. \quad (3.55)$$

This transformation readily leads to

$$y'(t) = u(t), \quad (3.56)$$

$$y''(t) = u'(t). \quad (3.57)$$

Substituting Eqs.(3.55)-(3.57) into Eq.(3.54) yields the nonlinear differential equation

$$y''(t) = \frac{1}{\kappa}\{(1-\alpha C)y'(t) - (y'(t))^2 - y(t)y'(t)\}, \quad (3.58)$$

with initial conditions

$$y(0) = 0, \tag{3.59}$$

$$y'(0) = u_0, \tag{3.60}$$

obtained by using Eqs.(3.55) and (3.56), respectively.

We now seek to solve the initial value problem Eqs.(3.58)-(3.60). This can be done by using the series solution method or the Adomian decomposition method so defined in Section 3.1 and 3.2, respectively. However, the Adomian decomposition method will be implemented here. We rewrite the initial value problem in an operator form by

$$L_t y(t) = \frac{1}{\kappa}\{(1-\alpha C)y'(t) - (y'(t))^2 - y(t)y'(t)\},\ y(0)=0,\ y'(0)=u_0, \tag{3.61}$$

where the differential operator L_t is defined by

$$L_t = \frac{d^2}{dt^2}, \tag{3.62}$$

so that the two fold integral operator L_t^{-1} is defined by

$$L_t^{-1} = \int_0^t \int_0^t (.)dtdt. \tag{3.63}$$

Operating with L_t^{-1} on both sides of Eq.(3.61) we obtain

$$y(t) = u_0(t) + L_t^{-1}(\frac{1}{\kappa}\{(1-\alpha C)y'(t) - (y'(t))^2 - y(t)y'(t)\}). \tag{3.64}$$

We next represent $y(t)$ by the decomposition series

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{3.65}$$

so that it remains to formally determine the components $y_n(t),\ n \geq 0$. Substituting Eq.(3.65) into both side of Eq.(3.64) yields

$$\sum_{n=0}^{\infty} y_n(t) = u_0(t) + L_t^{-1}(\frac{1}{\kappa}\{(1-\alpha C)\sum_{n=0}^{\infty} y_n'(t) - \sum_{n=0}^{\infty}\tilde{A}_n(t) - \sum_{n=0}^{\infty}\tilde{B}_n(t)\}), \tag{3.66}$$

where $\tilde{A}_n(t)$ and $\tilde{B}_n(t)$ are the so-called Adomian polynomials that represent the nonlinear terms $(y'(t))^2$ and $y(t)y'(t)$, respectively. In other words, we set

$$(y'(t))^2 = \sum_{n=0}^{\infty} \tilde{A}_n(t), \qquad (3.67)$$

$$y(t)y'(t) = \sum_{n=0}^{\infty} \tilde{B}_n(t) \qquad (3.68)$$

To determine the components $y_0, y_1, y_2, \ldots$ of $y(t)$ we follow the recursive relationship

$$y_0(t) = u_0(t), \qquad (3.69)$$

$$y_{k+1}(t) = L_t^{-1}(\frac{1}{\kappa}\{(1-\alpha C)y_k'(t) - \tilde{A}_k(t) - \tilde{B}_k(t)\}),\ k \geq 0. \qquad (3.70)$$

After we obtained the components $y_n(t)$ of $y(t)$, we find that

$$\sum_{n=0}^{\infty} y_n(t) = \sum_{n=0}^{\infty}[\sum_{j=0}^{n} b_{jn}\alpha^j]t^n, \qquad (3.71)$$

where b_{jn} is the coefficients of α^j which depend on a_0, κ, or C.

Then

$$y(t) = \sum_{n=0}^{\infty}[\sum_{j=0}^{n} b_{jn}\alpha^j]t^n, \qquad (3.72)$$

and

$$y'(t) = \sum_{n=1}^{\infty} n[\sum_{j=0}^{n} b_{jn}\alpha^j]t^{n-1}. \qquad (3.73)$$

Recall that

$$u(t) = y'(t), \qquad (3.74)$$

then we obtain

$$u(t) = \sum_{n=1}^{\infty} n[\sum_{j=0}^{n} b_{jn}\alpha^j]t^{n-1}. \qquad (3.75)$$

Substituting $u(t)$ of (3.75) into the integral of (3.53). With this substitution, the unsolvable integral (3.53) will be transformed into

$$\alpha = \int_{t_1}^{t_2} \sum_{n=1}^{\infty} n[\sum_{j=0}^{n} b_{jn}\alpha^j]x^{n-1}dx, \qquad (3.76)$$

a more readily solvable integral that formally determines a numerical value for the constant α. Having established α, the approximation solution $u(t)$ follows immediately upon substituting the value of α obtained from (3.76) into (3.75). This completes the technique.

3.3.2 Numerical results

Consider the population growth model characterized by the nonlinear Volterra-Fredholm integro-differential equation

$$\frac{du}{dt} = 10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx - u(t)\int_{0.9}^{1.0} u(x)dx, \qquad u(0) = 0.1, \tag{3.77}$$

where the initial condition $u(0) = 0.1$ and the nondimensional parameter $\kappa = 0.1$, $C = 0.1$, and the interval time $t_1 = 0.9$, $t_2 = 1.0$.

We set

$$\alpha = \int_{0.9}^{1.0} u(x)dx, \tag{3.78}$$

where α is a constant. Using (3.37) into (3.36) gives

$$\frac{du}{dt} = (10-\alpha)u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx, \qquad u(0) = 0.1. \tag{3.79}$$

We set

$$y(t) = \int_0^t u(x)dx. \tag{3.80}$$

This transformation readily leads to

$$y'(t) = u(t), \tag{3.81}$$

$$y''(t) = u'(t). \tag{3.82}$$

Substituting Eqs.(3.80)-(3.82) into Eq.(3.79) yields the nonlinear differential equation

$$y''(t) = (10-\alpha)y'(t) - 10(y'(t))^2 - 10y(t)y'(t), \tag{3.83}$$

with initial conditions

$$y(0) = 0, \tag{3.84}$$

$$y'(0) = 0.1, \tag{3.85}$$

obtained by using Eqs.(3.80) and (3.81), respectively.

Rewrite the initial value problem in an operator form

$$L_t y(t) = (10-\alpha)y'(t) - 10(y'(t))^2 - 10y(t)y'(t),\ y(0)=0,\ y'(0)=0.1, \quad (3.86)$$

Operating with L_t^{-1} on both sides of Eq.(3.86) we obtain

$$y(t) = 0.1t + L_t^{-1}((10-\alpha)y'(t) - 10(y'(t))^2 - 10y(t)y'(t)). \quad (3.87)$$

Representing $y(t)$ by the decomposition series

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \quad (3.88)$$

To determine the components $y_n(t),\ n \geq 0$ by substituting Eq.(3.88) into both side of Eq.(3.87) we obtain

$$\sum_{n=0}^{\infty} y_n(t) = 0.1t + L_t^{-1}((10-\alpha)\sum_{n=0}^{\infty} y'_n(t) - 10\sum_{n=0}^{\infty} \tilde{A}_n(t) - 10\sum_{n=0}^{\infty} \tilde{B}_n(t)), \quad (3.89)$$

where

$$\sum_{n=0}^{\infty} \tilde{A}_n(t) = (y'(t))^2, \quad (3.90)$$

$$\sum_{n=0}^{\infty} \tilde{B}_n(t) = y(t)y'(t). \quad (3.91)$$

Following the algorithms set by Adomian for generating these polynomials, it readily follows that

For $\tilde{A}_n(t)$: Let $F(y) = (y'(t))^2$, we find

$$\begin{aligned}
\tilde{A}_0(t) &= (y'_0(t))^2,\\
\tilde{A}_1(t) &= 2y'_0(t)y'_1(t),\\
\tilde{A}_2(t) &= 2y'_0(t)y'_2(t) + (y'_1(t))^2,\\
\tilde{A}_3(t) &= 2y'_0(t)y'_3(t) + 2y'_1(t)y'_2(t),\\
\tilde{A}_4(t) &= 2y'_0(t)y'_4(t) + 2y'_1(t)y'_3(t) + (y'_2(t))^2,\\
\tilde{A}_5(t) &= 2y'_0(t)y'_5(t) + 2y'_1(t)y'_4(t) + 2y'_2(t)y'_3(t),\\
\tilde{A}_6(t) &= 2y'_0(t)y'_6(t) + 2y'_1(t)y'_5(t) + 2y'_2(t)y'_4(t) + (y'_3(t))^2,\\
\tilde{A}_7(t) &= 2y'_0(t)y'_7(t) + 2y'_1(t)y'_6(t) + 2y'_2(t)y'_5(t) + 2y'_3(t)y'_4(t), \ldots \quad (3.92)
\end{aligned}$$

For $\tilde{B}_n(t)$: Let $F(y) = y(t)y'(t)$, we find

$$
\begin{aligned}
\tilde{B}_0(t) &= y_0(t)y_0'(t),\\
\tilde{B}_1(t) &= y_0(t)y_1'(t) + y_1(t)y_0'(t),\\
\tilde{B}_2(t) &= y_0(t)y_2'(t) + y_1(t)y_1'(t) + y_2(t)y_0'(t),\\
\tilde{B}_3(t) &= y_0(t)y_3'(t) + y_1(t)y_2'(t) + y_2(t)y_1'(t) + y_3(t)y_0'(t),\\
\tilde{B}_4(t) &= y_0(t)y_4'(t) + y_1(t)y_3'(t) + y_2(t)y_2'(t) + y_3(t)y_1'(t) + y_4(t)y_0'(t),\\
\tilde{B}_5(t) &= y_0(t)y_5'(t) + y_1(t)y_4'(t) + y_2(t)y_3'(t) + y_3(t)y_2'(t) + y_4(t)y_4'(t) + y_5(t)y_0'(t),\\
\tilde{B}_6(t) &= y_0(t)y_6'(t) + y_1(t)y_5'(t) + y_2(t)y_4'(t) + y_3(t)y_3'(t) + y_4(t)y_2'(t) + y_5(t)y_1'(t)\\
&\quad + y_6(t)y_0'(t),\\
\tilde{B}_7(t) &= y_0(t)y_7'(t) + y_1(t)y_6'(t) + y_2(t)y_5'(t) + y_3(t)y_4'(t) + y_4(t)y_3'(t) + y_5(t)y_2'(t)\\
&\quad + y_6(t)y_1'(t) + y_7(t)y_0'(t), \ldots
\end{aligned}
\tag{3.93}
$$

To determine the components $y_0, y_1, y_2, \ldots$ of $y(t)$, we follow the recursive relationship

$$y_0(t) = 0.1t, \tag{3.94}$$

$$y_{k+1}(t) = L_t^{-1}((10-\alpha)y_k'(t) - 10\tilde{A}_k(t) - 10\tilde{B}_k(t)),\ \ k \geq 0. \tag{3.95}$$

Proceeding as before, we set

$$y_0(t) = 0.1t, \tag{3.96}$$

so that the successive components may be determined as

$$
\begin{aligned}
y_1(t) &= L_t^{-1}((10-\alpha)y_0'(t) - 10\tilde{A}_0(t) - 10\tilde{B}_0(t))\\
&= L_t^{-1}((10-\alpha)y_0'(t) - 10(y_0'(t))^2 - 10y_0(t)y_0'(t))\\
&= L_t^{-1}((10-\alpha)(0.1) - 10(0.1)^2 - 10(0.1)(0.1))\\
&= (0.45 - 0.5\alpha)t^2 - 0.016666667t^3,\\
y_2(t) &= L_t^{-1}((10-\alpha)y_1'(t) - 10\tilde{A}_1(t) - 10\tilde{B}_1(t))\\
&= (1.2 - 2.833333333\alpha + 1.666666667\alpha^2)t^3 + (-0.145833333\\
&\quad + 0.166666667\alpha)t^4 + 0.003333333t^5,
\end{aligned}
$$

$$
\begin{aligned}
y_3(t) &= L_t^{-1}((10-\alpha)y_2'(t) - 10\tilde{A}_2(t) - 10\tilde{B}_2(t)) \\
&= (1.725 - 7.166666667\alpha + 9.583333333\alpha^2 - 4.166666667\alpha^3)t^4 \\
&\quad + (-0.630833333 + 1.525\alpha - 0.916666667\alpha^2)t^5 + (0.040416667 \\
&\quad - 0.047222222\alpha)t^6 - 0.000674603t^7, \\
y_4(t) &= L_t^{-1}((10-\alpha)y_3'(t) - 10\tilde{A}_3(t) - 10\tilde{B}_3(t)) \\
&= (-0.48 - 3.666666667\alpha + 16.666666667\alpha^2 - 20.833333333\alpha^3 \\
&\quad + 8.333333333\alpha^4)t^5 + (-1.558611111 + 6.331944444\alpha - 8.361111111\alpha^2 \\
&\quad + 3.611111111\alpha^3)t^6 + (0.237599206 - 0.58452381\alpha + 0.357142857\alpha^2)t^7 \\
&\quad + (-0.010342262 + 0.012301587\alpha)t^8 + 0.000136684t^9, \\
y_5(t) &= L_t^{-1}((10-\alpha)y_4'(t) - 10\tilde{A}_4(t) - 10\tilde{B}_4(t)) \\
&= (-9.1 + 38.111111111\alpha - 49.861111111\alpha^2 + 8.333333333\alpha^3 \\
&\quad + 26.388888889\alpha^4 - 13.888888889\alpha^5)t^6 + (-1.334126984 \\
&\quad + 11.726984127\alpha - 30.257936508\alpha^2 + 31.150793651\alpha^3 \\
&\quad - 11.30952381\alpha^4)t^7 + (0.825493552 - 3.337574405\alpha \\
&\quad + 4.40327381\alpha^2 - 1.904761905\alpha^3)t^8 + (-0.076493882 \\
&\quad + 0.190845459\alpha - 0.118165785\alpha^2)t^9 + O(t^{10}), \\
y_6(t) &= L_t^{-1}((10-\alpha)y_5'(t) - 10\tilde{A}_5(t) - 10\tilde{B}_5(t)) \\
&= (-21.2 + 140.341269841\alpha - 354.047619048\alpha^2 + 417.063492063\alpha^3 \\
&\quad - 207.936507937\alpha^4 + 5.952380952\alpha^5 + 19.841269841\alpha^6)t^7 \\
&\quad + (5.941765873 - 15.620039683\alpha - 13.035714286\alpha^2 + 78.318452381\alpha^3 \\
&\quad - 85.342261905\alpha^4 + 29.761904762\alpha^5)t^8 + (1.441830908 - 9.965881283\alpha \\
&\quad + 23.26645172\alpha^2 - 22.728174603\alpha^3 + 8.002645503\alpha^4)t^9 + O(t^{10})
\end{aligned}
$$

$$
\begin{aligned}
y_7(t) &= L_t^{-1}((10-\alpha)y_6'(t) - 10\tilde{A}_6(t) - 10\tilde{B}_6(t)) \\
&= (-9.066071429 + 160.76984127\alpha - 751.369047619\alpha^2 + 1589.880952381\alpha^3 \\
&\quad - 1713.938492063\alpha^4 + 889.880952381\alpha^5 - 141.369047619\alpha^6 \\
&\quad - 24.801587302\alpha^7)t^8 + (27.758652998 - 165.346891534\alpha \\
&\quad + 355.874669312\alpha^2 - 292.911155203\alpha^3 - 26.5625\alpha^4 + 169.229497354\alpha^5 \\
&\quad - 68.066578483\alpha^6)t^9 + O(t^{10}) \\
y_8(t) &= L_t^{-1}((10-\alpha)y_7'(t) - 10\tilde{A}_7(t) - 10\tilde{B}_7(t)) \\
&= (88.241269841 - 475.395061728\alpha + 680.705467372\alpha^2 + 803.174603175\alpha^3 \\
&\quad - 3621.51675485\alpha^4 + 4495.149911817\alpha^5 - 2480.158730159\alpha^6 \\
&\quad + 482.25308642\alpha^7 + 27.557319224\alpha^8)t^9 + O(t^{10}) \qquad (3.97)
\end{aligned}
$$

substitute Eqs.(3.97) into Eq.(3.88) we obtain

$$
\begin{aligned}
y(t) &= 0.1t + (0.45 - 0.5\alpha)t^2 + (1.183333333 - 2.833333333\alpha + 1.666666667\alpha^2)t^3 \\
&\quad + (1.579166667 - 7\alpha + 9.583333333\alpha^2 - 4.166666667\alpha^3)t^4 \\
&\quad + (-1.1075 - 2.141666667\alpha + 15.75\alpha^2 - 20.833333333\alpha^3 + 8.333333333\alpha^4)t^5 \\
&\quad + (-10.618194444 + 44.395833333\alpha - 58.222222222\alpha^2 + 11.944444444\alpha^3 \\
&\quad + 26.388888889\alpha^4 - 13.888888889\alpha^5)t^6 + (-22.297202381 + 151.483730159\alpha \\
&\quad - 383.948412698\alpha^2 + 448.214285714\alpha^3 - 219.246031746\alpha^4 + 5.952380952\alpha^5 \\
&\quad + 19.841269841\alpha^6)t^7 + (-2.309154266 + 141.82452877\alpha - 760.001488095\alpha^2 \\
&\quad + 1666.294642857\alpha^3 - 1799.280753968\alpha^4 + 919.642857143\alpha^5 \\
&\quad - 141.369047619\alpha^6 - 24.801587302\alpha^7)t^8 + (117.36539655 \\
&\quad - 650.516989087\alpha + 1059.728422619\alpha^2 + 487.535273369\alpha^3 \\
&\quad - 3640.076609347\alpha^4 + 4664.379409171\alpha^5 - 2548.225308642\alpha^6 \\
&\quad + 482.25308642\alpha^7 + 27.557319224\alpha^8)t^9 + O(t^{10}). \qquad (3.98)
\end{aligned}
$$

Thus

$$
\begin{aligned}
y'(t) = {} & 0.1 + (0.9 - 0.1\alpha)t + (3.55 - 0.85\alpha + 0.05\alpha^2)t^2 \\
& + (6.316666667 - 2.8\alpha + 0.3833333333\alpha^2 - 0.01666666667\alpha^3)t^3 \\
& + (-5.537499997 - 1.070833334\alpha + 0.7875\alpha^2 - 0.1041666667\alpha^3 \\
& + 0.004166666667\alpha^4)t^4 + (-63.70916666 + 26.6375\alpha - 3.493333333\alpha^2 \\
& + 0.07166666663\alpha^3 + 0.01583333333\alpha^4 - 0.000833333333\alpha^5)t^5 \\
& + (-156.0804167 + 106.0386111\alpha - 26.87638889\alpha^2 + 3.1375\alpha^3 \\
& - 0.1534722222\alpha^4 - 0.000416666666\alpha^5 + 0.0001388888889\alpha^6)t^6 \\
& + (-18.47323422 + 113.4596231\alpha - 60.80011906\alpha^2 + 13.33035714\alpha^3 \\
& - 1.439424603\alpha^4 + 0.07357142857\alpha^5 - 0.001130952381\alpha^6 \\
& - 0.00001984126984\alpha^7)t^7 + (1056.288569 - 585.4652901\alpha \\
& + 95.37555799\alpha^2 + 4.387817468\alpha^3 - 3.276068949\alpha^4 + 0.4197941469\alpha^5 \\
& - 0.02293402778\alpha^6 + 0.0004340277778\alpha^7 + 0.2480158730 \times 10^{-5}\alpha^8)t^8 \\
& + O(t^9).
\end{aligned}
\tag{3.99}
$$

Since $u(t) = y'(t)$, then substituting $u(t)$ into (3.78)and solving for real positive α yields

$$
\alpha = 3.693452238, 14.16054255 \tag{3.100}
$$

Combining (3.99) and (3.100) yields the approximation of $u(t)$ given by

$$
\begin{aligned}
u(t) = {} & 0.1 + 0.5306547762t + 1.092645070t^2 + 0.364533692t^3 - 3.222824317t^4 \\
& - 6.99485762t^5 - 0.9081106895t^6 + 22.46202251t^7 + 40.86978039t^8 \\
& + O(t^9)
\end{aligned}
\tag{3.101}
$$

in a complete agreement with the results previously obtained in the previous sections.

3.4 Analysis

The results of the present work can be employed to examine the mathematical structure of $u(t)$.

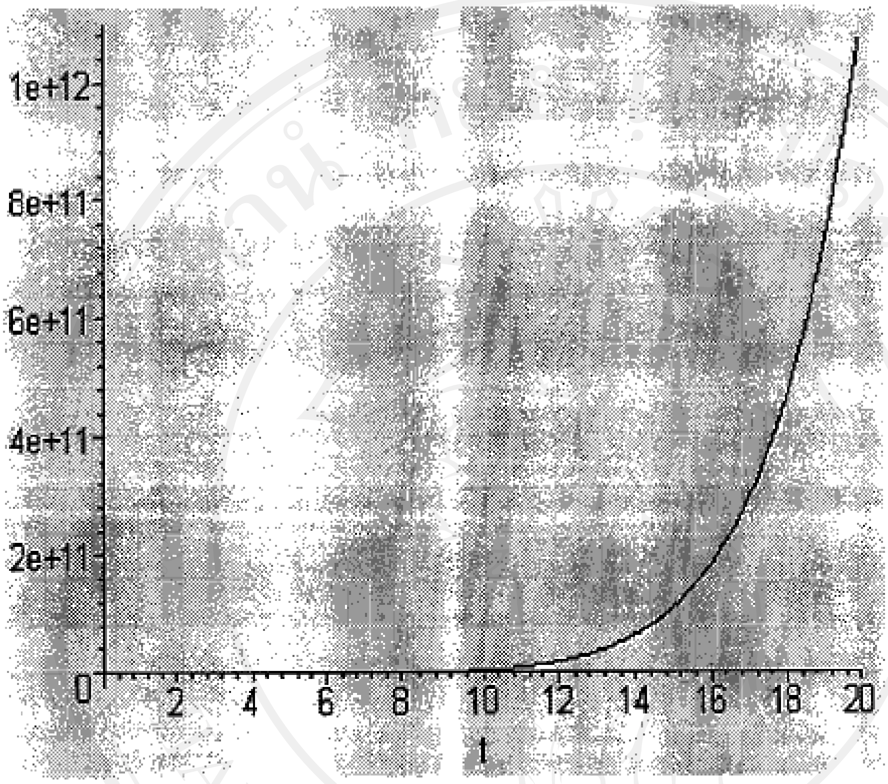

Figure 3.1: Relation between $u(t)$ and t for $u(0) = 0.1, \kappa = 0.1, C = 0.1, t_1 = 0.9, t_2 = 1.0$

From Figure 3.1, we could not predict the behavior of the population $u(t)$, so the Padé approximations will be used. The way of studying the mathematical structure of $u(t)$ is to define Padé approximants which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(t)$. It is of interest to note that Padé approximants give results with no greater error bounds [8] than approximation by polynomials.

By using Maple 7 software, the Padé approximation obtained for $u(t)$ in Eq.(3.19) is

$$[4/4] = \frac{0.1 + 0.2790310687t + 0.3209568115t^2 + 0.1816855009t^3 + 0.04332238664t^4}{1 - 2.516237074t + 5.635649635t^2 - 4.240785533t^3 + 2.760282005t^4}. \quad (3.102)$$

To analyze the behavior of $u(t)$ we will consider two time intervals.

First interval : For $0 \leq t \leq t_1$ we will consider the solution of the equation

$$\kappa\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx. \tag{3.103}$$

Second interval : For $t \geq t_1$ which is the time interval that the perturbation had occurred in the system, we will consider the solution of the equation

$$\kappa\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx - Cu\int_{t_1}^{t_2} u(x)dx \tag{3.104}$$

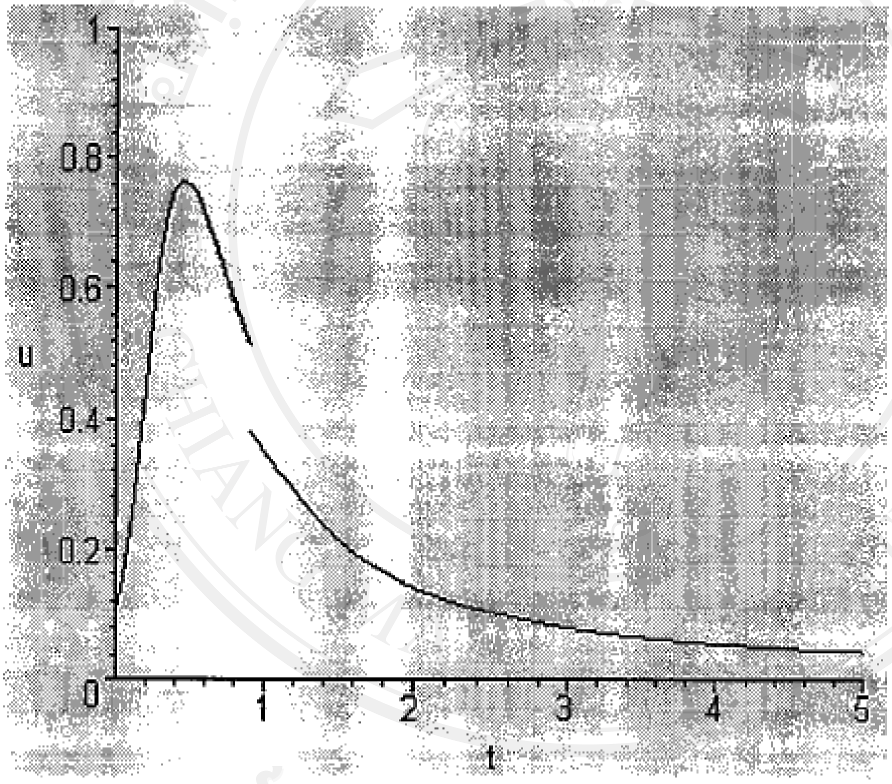

Figure 3.2: Relation between Padé's approximants [4/4] of $u(t)$ and t for $u(0) = 0.1, \kappa = 0.1, C = 0.1, t_1 = 0.9, t_2 = 1.0$

From Fig 3.2, we can observe that at the time from $t = 0$ to $t = t_1$ the rapid rise occurs along the logistic curve followed by the slow exponential decay after reaching the maximum point, when the perturbation occurs at $t = t_1$ to $t = t_2$ we will see that the population $u(t)$ rapid decrease at $t = t_1$ and exponentially decay where $t \rightarrow \infty$.

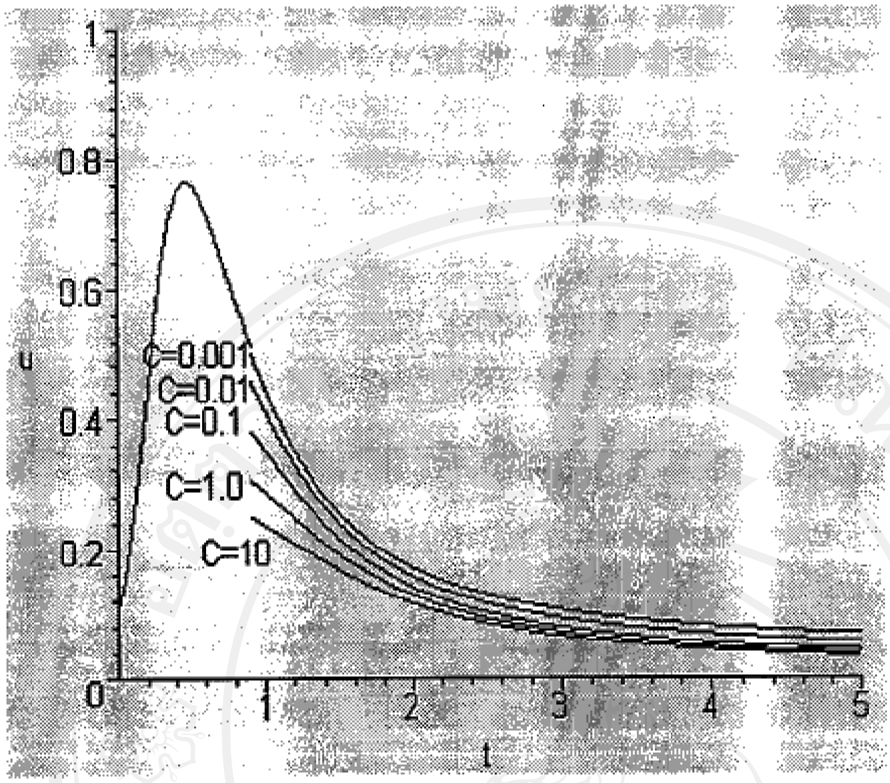

Figure 3.3: Relation between Padé's approximants [4/4] of $u(t)$ and t for $u(0) = 0.1, \kappa = 0.1, C = 0.001, 0.01, 0.1, 1.0$, and $10, t_1 = 0.9, t_2 = 1.0$

From Fig 3.3, for fixed κ, t_1, and t_2 we can easily observe that as C increases, the population $u(t)$ decreases at the instant of time, whereas the exponential decay increases.

Table 1

Error from the direct solution method combined with the series solution method is defined by

$$Error = \mid \kappa\frac{du}{dt} - (u(t) - u^2(t) - u(t)\int_0^t u(x)dx - Cu(t)\int_{t_1}^{t_2} u(x)dx) \mid$$

for $u(0) = 0.1, \kappa = 0.1, C = 0.1, t_1 = 0.9, t_2 = 1.0$

t	Error	t	Error	t	Error	t	Error
0.0	$0.200000 * 10^{-9}$	2.3	0.3856760772	4.6	0.0946442400	6.9	0.0362050898
0.1	$0.161520 * 10^{-6}$	2.4	0.3543798969	4.7	0.0904656982	7.0	0.0347033788
0.2	$0.118274 * 10^{-3}$	2.5	0.3265947244	4.8	0.0865282355	7.1	0.0332487971
0.3	0.0046701249	2.6	0.3018558472	4.9	0.0828115953	7.2	0.0318387505
0.4	0.0395501843	2.7	0.2797622157	5.0	0.0792976075	7.3	0.0304708261
0.5	0.1364925872	2.8	0.2599689998	5.1	0.0759699398	7.4	0.0291427775
0.6	0.2786209824	2.9	0.2421802666	5.2	0.0728138805	7.5	0.0278525108
0.7	0.4138720330	3.0	0.2261421944	5.3	0.0698161504	7.6	0.0265980720
0.8	0.5031515152	3.1	0.2116369922	5.4	0.0669647416	7.7	0.0253776355
0.9	1.4685771510	3.2	0.1984775837	5.5	0.0642487733	7.8	0.0241894957
1.0	1.3635508440	3.3	0.1865030186	5.6	0.0616583683	7.9	0.0230320551
1.1	1.2521249750	3.4	0.1755745492	5.7	0.0591845447	8.0	0.0219038179
1.2	1.1404276220	3.5	0.1655723012	5.8	0.0568191203	8.1	0.0208033818
1.3	1.0328172340	3.6	0.1563924549	5.9	0.0545546288	8.2	0.0197294305
1.4	0.9321047231	3.7	0.1479448676	6.0	0.0523842454	8.3	0.0186807273
1.5	0.8398193424	3.8	0.1401510659	6.1	0.0503017235	8.4	0.0176561105
1.6	0.7565238875	3.9	0.1329425540	6.2	0.0483013340	8.5	0.0166544866
1.7	0.6821220692	4.0	0.1262593848	6.3	0.0463778187	8.6	0.0156748259
1.8	0.6161148998	4.1	0.1200489542	6.4	0.0445263396	8.7	0.0147161587
1.9	0.5577912866	4.2	0.1142649793	6.5	0.0427424435	8.8	0.0137775695
2.0	0.5063573217	4.3	0.1088666330	6.6	0.0410220224	8.9	0.0128581956
2.1	0.4610170448	4.4	0.1038178087	6.7	0.0393612840	9.0	0.0119572210
2.2	0.4210183992	4.5	0.0990864946	6.8	0.0377567221	9.1	0.0110738749

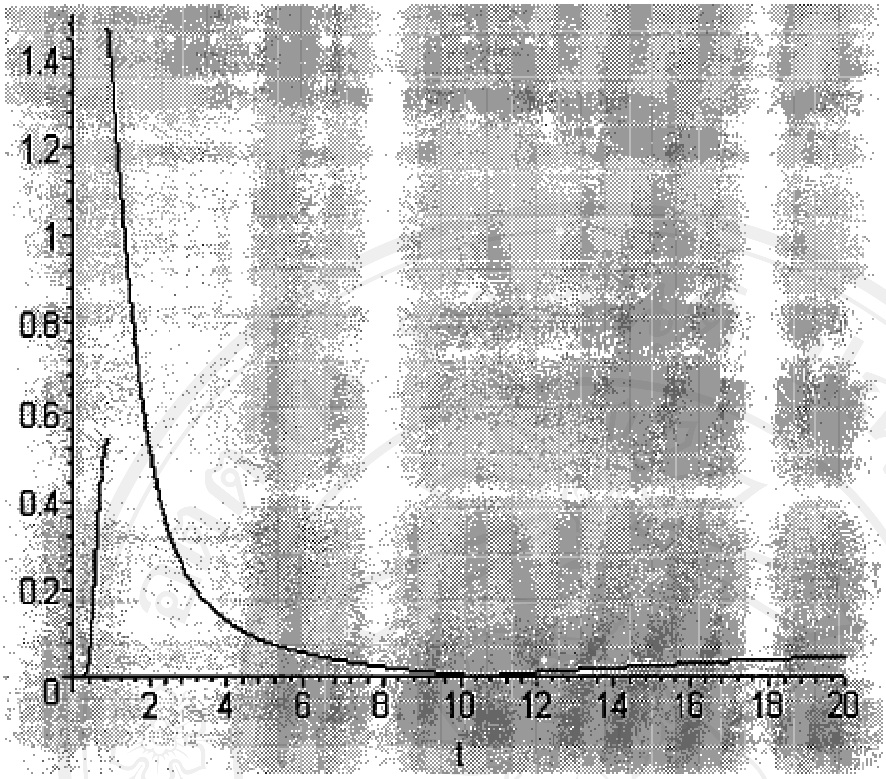

Figure 3.4: Relation between error from the direct solution method combined with the series solution method and t for $u(0) = 0.1, \kappa = 0.1, C = 0.1, t_1 = 0.9, t_2 = 1.0$