Chapter 2

Basic Concepts

The aim of this chapter is to give some definitions and properties of the gamma functions, distributions, and partial differential equations which will be used in the later chapters.

2.1 The Gamma Function

In this section, we shall present the definition of the gamma function given by Euler. In addition, we shall give some properties of the gamma function.

Definition 2.1.1 *(Euler) The gamma function is denoted by* Γ *and is defined by*

$$\Gamma(z) = \int_0^\infty e^t t^{z-1} dt, \tag{2.1}$$

where z *is a complex number with* $Re\, z > 0$.

A result that yields an immediate analytic continuation from the left haft plane is the following properties.

Proposition 2.1.2 *Let* z *be a complex number. Then*

$$(1)\ \Gamma(z) = \frac{\Gamma(z+1)}{z}, z \neq 0, -1, -2, \ldots, \tag{2.2}$$

$$(2)\ \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, z \neq 0, -1, -2, \ldots, z \neq 0, \pm 1, \pm 2, \ldots. \tag{2.3}$$

Proof. (1) See [6, p63] and (2) see [6, p65]. □

Proposition 2.1.3 *(Legendre's duplication formula) Let* z *be a complex number. Then*

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), z \neq 0, -1, -2, \ldots. \tag{2.4}$$

Proof. See [6, p74]. □

Proposition 2.1.4 *The residue of* $\Gamma(\lambda)$ *at* $\lambda = -k$ *is*

$$\underset{\lambda=-k}{res}\,\Gamma(\lambda) = \frac{(-1)^k}{k!}, k = 0, 1, 2, \ldots. \tag{2.5}$$

Proof. See [4, p2]. □

2.2 Distributions

In this section, we shall use the standard notation $\mathcal{D}$ the space of testing functions, which consists of all real or complex functions with continuous derivative of all orders and with compact support. Every element in $\mathcal{D}$ is called a *testing function.*

Definition 2.2.1 *A sequence of testing function* $\{\varphi_i(x)\}_{i=1}^{\infty}$ *is said to converge to* $\varphi(x)$ *in* $\mathcal{D}$ *if all* $\varphi_i(x)$ *are zero outside a certain region in* $\mathbb{R}^n$ *and if for every nonnegative integers* $m_1, m_2, \ldots, m_n$*, the sequence* $\{\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi_i(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}\}_{i=1}^{\infty}$ *converges uniformly to* $\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}$ *on* $\mathbb{R}^n$.

Proposition 2.2.2 $\mathcal{D}$ *is closed under convergence, that is, the limit of every sequence that converge in* $\mathcal{D}$ *is also in* $\mathcal{D}$.

Proof. See [1, p161]. □

Definition 2.2.3 *A functional on a linear space* E *is a mapping* $f : E \to \mathbb{C}$ *which assigns to each member* φ *of* E *a certain complex number; the image of* $\varphi \in E$ *under* f *is usually written as* $f(\varphi)$ *or* $< f, \varphi >$.

Definition 2.2.4 *A distribution(or generalized function) is a mapping* $f : \mathcal{D} \to \mathbb{C}$ *which satisfies the following conditions :*

(1) *for any* $\varphi_1, \varphi_2 \in \mathcal{D}$ *and any scalars* a_1, a_2*,*

$$< f, a_1\varphi_1 + a_2\varphi_2 >= a_1 < f, \varphi_1 > + a_2 < f, \varphi_2 >,$$

(2) *for any sequence of testing functions* $\{\varphi_i(x)\}_{i=1}^{\infty}$ *that converges in* $\mathcal{D}$ *to* φ*, the sequence* $\{< f, \varphi_i >\}_{i=1}^{\infty}$ *converges to* $< f, \varphi >$ *in the ordinary sense.*

One way to generate distributions is as follows. Let $f(x)$ be a locally integrable function, that is, a function that is integrable in the Lebesgue sense over every compact subset of $\mathbb{R}^n$. Corresponding to $f(x)$, we can define a distribution through the convergence integral

$$< f, \varphi >= \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx. \tag{2.6}$$

Then by [5, p3], $< f, \varphi >$ is a distribution.

Definition 2.2.5 *Distributions that can be generated through (2.6) from locally integrable functions shall be called regular, and all others will be called singular.*

An important singular distribution is the so-called *Dirac-delta function* δ, which is defined by

$$< \delta, \varphi >= \varphi(0). \tag{2.7}$$

It is to note that the Dirac-delta function is a singular distribution see [5, p4].

Proposition 2.2.6 *Let x be an n-dimensional real variable and y an m-dimensional real variable. Also, let $\phi(x, y)$ be a testing function in $\mathcal{D}$ define over $\mathbb{R}^{n+m}$. If $f(x)$ is a distribution defined over $\mathbb{R}^n$, then $\theta(y) =< f(x), \phi(x, y) >$ is a testing function of y in $\mathcal{D}$.*

Proof. See [13, p74]. □

Proposition 2.2.7 *Let f be any distribution (in one dimension), then the functional g defined by*

$$< g, \varphi >=< f, -\varphi' >$$

is also a distribution.

Proof. See [5, p19]. □

Definition 2.2.8 *The distribution g in proposition 2.2.7 is called the derivative of f and is denoted by f' or $\frac{df}{dx}$, that is,*

$$< f', \varphi >=< f, -\varphi' > . \tag{2.8}$$

Similarly, in the case of several variable, the partial derivative of a distribution f with respect to each of the variables can be defined as

$$< \frac{\partial f}{\partial x_i}, \varphi >=< f, -\frac{\partial \varphi}{\partial x_i} >, \tag{2.9}$$

for $i = 1, \ldots, n$.

Proposition 2.2.9 *Given P is a hyper-surface, then $P\delta^{(k)}(P) + k\delta^{(k-1)}(P) = 0$ where $\delta^{(k)}$ is the Dirac-delta function with k derivatives.*

Proof. See [5, p233]. □

Proposition 2.2.10 *Let f be a distribution in m dimensions and g be a distribution in n dimensions. Then the functional h defined by*

$$< h(x,y), \varphi(x,y) >=< f(x), < g(y), \varphi(x,y) >>$$

is a distribution in $m+n$ dimensions.

Proof. See [5, p100]. □

Definition 2.2.11 *The distribution h in proposition 2.2.10 is called the tensor (or direct) product of $f(x)$ and $g(y)$ and is denoted by $h(x,y) = f(x) \times g(y)$, that is,*

$$< f(x) \times g(y), \varphi(x,y) >=< f(x), < g(y), \varphi(x,y) >> . \tag{2.10}$$

Definition 2.2.12 *The support of a distribution f is defined as the complement of the largest open set on which f is zero.*

Proposition 2.2.13 *Let f and g be distributions in n dimensions. Then the function h defined by*

$$< h, \varphi >=< f(x) \times g(y), \varphi(x+y) > \tag{2.11}$$

is a distribution provided that it satisfies either of the following conditions :

(1) *Either f or g has bounded support, or*

(2) *In one dimension the supports of both f and g are bounded on the same side (for instance, $f = 0$ for $x < a$, and $g = 0$ for $y < b$).*

Proof. See [5, p104]. □

Definition 2.2.14 *The distribution h in proposition 2.2.13 is called the convolution of f and g and is denoted by $h = f * g$, that is,*

$$< f * g, \varphi >=< f(x) \times g(y), \varphi(x+y) > . \tag{2.12}$$

Now we shall give some helpful properties of convolutions.

Proposition 2.2.15 *Let f, g and h be distributions.*

(1) *For δ is the Dirac-delta function, we have*

$$f * \delta = f. \tag{2.13}$$

(2) *If f and g satisfy at least one of the (1) and (2) of proposition 2.2.13, then*

$$f * g = g * f. \tag{2.14}$$

(3) *If $P(D)$ is linear partial differential operator with constant coefficients and f and g satisfy at least one of the (1) and (2) of proposition 2.2.13, then*

$$P(D)f * g = P(D)(f * g) = f * P(D)g. \tag{2.15}$$

Proof. (1) and (2) see [5, p104].

(3) see [13, p49]. □

2.3 Introduction to Partial Differential Equations

A *partial differential equation* is an equation containing a partial derivative which is to be taken of an unknown function of more than one variable. A partial differential is called *linear* if it can be written in the form

$$\sum_{m_1+\ldots+m_n \leq m} a_{m_1,\ldots,m_n}(x_1,\ldots,x_n)\frac{\partial^{m_1+\ldots+m_n}u(x_1,\ldots,x_n)}{\partial x_1^{m_1}\ldots\partial x_n^{m_n}} = f(x_1,\ldots,x_n)$$

where summation is taken over all nonnegative integers $m_1, \ldots, m_n$, the a's and f are given functions, u is an unknown function, and m is a nonnegative integer. A partial differential equation is called *nonlinear* if it is not linear.

Now we shall give some examples of partial differential equations.

Example 2.3.1 *The wave equation in n dimensions for $u = u(x_1, \ldots, x_n, t)$ is*

$$u_{tt} = c^2 \Delta u, \tag{2.16}$$

where c is a positive constant and Δ is defined by (1.3).

It represents vibration of springs or propagation of sound wave in tube for $n = 1$, wave on the surface of shallow water for $n = 2$, and a acoustic or light waves for $n = 3$.

Example 2.3.2 *Elastic waves are described classically by the linear system*

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \mu\Delta u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}(\nabla u) \tag{2.17}$$

(i=1,2,3,...), where the $u_i(x_1, x_2, x_3, t)$ *are the components of the displacement vector* u*, and* ρ *is the density and* λ, μ *the Lame constants of the elastic material. Each* u_i *satisfies the fourth-order equation*

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\lambda + 2\mu}{\rho}\triangle\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\mu}{\rho}\triangle\right) u_i = 0, \tag{2.18}$$

formed from to different wave operators.

Proposition 2.3.3 *Let* $c > 0$*. Given the equation*

$$\triangle_c u(x) = f(x, u(x)), \tag{2.19}$$

where

$$\triangle_c = \frac{1}{c^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}$$

and f *defined and having continuous first derivatives for all* $x \in \Omega \cup \partial\Omega$*, where* Ω *is an open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *denotes the boundary of* Ω*. Assume that* f *is bounded on* Ω *,that is, there exists an* $N > 0$ *such that* $|f(x, u(x))| \leq N$ *for all* $x \in \Omega$*. Then we obtain a continuous function* $u(x)$ *as a unique solution of (2.19) with the boundary condition* $u(x) = 0$ *for* $x \in \partial\Omega$*.*

Proof. See[2, p369]. □

Definition 2.3.4 *Consider the linear partial differential equation*

$$P(D)u = f, \tag{2.20}$$

where f *is a distribution,* u *an unknown function, and* $P(D)$ *a linear partial differential operator with constant coefficients. A function* $E(x)$ *is called elementary solution of equation (2.20) if* $P(D)E = \delta$*, where* δ *is the Dirac-delta function.*

Definition 2.3.5 *Let* $x = (x_1, x_2, ..., x_n)$ *be a point of* $\mathbb{R}^n$*. We shall write* $x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2 = U$*, where* $p + q = n$*. By* Γ_+ *we designate the interior of the forward cone :* $\{x \in \mathbb{R}^n : x_1 > 0$ *and* $U > 0\}$*, and by* $\overline{\Gamma}_+$ *designates of its closure. Similarly,* Γ_- *designates the domain* $\{x \in \mathbb{R}^n : x_1 < 0$ *and* $U > 0\}$ *and* $\overline{\Gamma}_-$ *designates of its closure.*

Let $F(\lambda)$ *be a function of the scalar variable* λ*, and let* $\Phi(x)$ *be a function endowed with the following properties :*

(1) $\Phi(x) = F(U)$,

(2) $supp\,\Phi(x) \subset \overline{\Gamma}_+$,

(3) $e^{<x,y>}\Phi(x) \in L_1$ *if* $y \in V_-$, *where*

$$V_- = \{y \in \mathbb{R}^n : y_1 > 0, y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \cdots - y_{p+q}^2 > 0\}.$$

We call $\mathcal{R}$ the family of functions $\Phi(x)$ which satisfies condition (1)-(3).

Similarly, we call $\mathcal{A}$ the family of functions which satisfies the following conditions :

(1) $\Phi(x) = F(U)$,

(2) $supp\,\Phi(x) \subset \overline{\Gamma}_-$,

(3) $e^{<x,y>}\Phi(x) \in L_1$ *if* $y \in V_+$, *where*

$$V_+ = \{y \in \mathbb{R}^n : y_1 < 0, y_1^2 + y_2^2 + \ldots + y_p^2 - y_{p+1}^2 - y_{p+2}^2 - \cdots - y_{p+q}^2 > 0\}.$$

We shall consider the following functions of the family $\mathcal{R}$ introduced by Y. Nozaki [9, p72] :

$$R_\alpha^H(x) = \begin{cases} \frac{U^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+. \end{cases} \qquad (2.21)$$

where α is a complex parameter, n is the dimension of the space, the constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{\alpha+2-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}. \qquad (2.22)$$

and p is the number of positive terms of

$$U = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2, \quad p+q=n. \qquad (2.23)$$

It is well known that $R_\alpha^H(x)$ is an ordinary function if $Re\,\alpha \geq n$ and it is a distribution of α if $Re\,\alpha < n$. Let $supp\,R_\alpha^H(x)$ denote the support of $R_\alpha^H(x)$. By putting $p = 1$ in (2.21) and (2.22) and by (2.4), then (2.21) reduces to

$$M_\alpha(s) = \begin{cases} \frac{s^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+. \end{cases} \qquad (2.24)$$

Here $s = x_1^2 - x_2^2 - \ldots - x_n^2$ and

$$H_n(\alpha) = \pi^{\frac{n-1}{2}} 2^{\alpha-1}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{\alpha}{2}),$$

$M_\alpha(s)$ is precisely, *the hyperbolic kernel of Marcel Riesz.*

Definition 2.3.6 *Let $x = (x_1, x_2, ..., x_n)$ be a point in $\mathbb{R}^n$ and the function $R^\ell_\beta(x)$ defined by*

$$R^\ell_\beta(x) = \frac{|x|^{\beta-n}}{W_n(\beta)}, \tag{2.25}$$

where β is a complex parameter, $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$ and

$$W_n(\beta) = \frac{\pi^{\frac{n}{2}} 2^\beta \Gamma(\frac{\beta}{2})}{\Gamma(\frac{n-\beta}{2})}. \tag{2.26}$$

Definition 2.3.7 *Let α and β be complex parameters and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. The family $K_{\alpha,\beta}$, the kernel of Marcel Riesz, is defined by*

$$K_{\alpha,\beta}(x) = R^H_\alpha(x) * R^\ell_\beta(x), \tag{2.27}$$

where $R^H_\alpha(x)$ and $R^\ell_\beta(x)$ are defined by (2.21) and (2.25) respectively. In the special case $\alpha = \beta = 2k$, the kernel $K_{2k,2k}$ is called the Diamond kernel of Marcel Riesz.

Proposition 2.3.8 *Given the equation*

$$\Box^k u(x) = 0, \tag{2.28}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated k-times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \ldots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{2.29}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then we obtain $u(x) = \left(R^H_{2(k-1)}(x)\right)^{(m)}$ as a solution of (2.28) with $m = (n-4)/2$, $n \geq 4$ and n is even. The function $\left(R^H_{2(k-1)}(x)\right)^{(m)}$ is defined by (2.21) with m derivatives and $\alpha = 2(k-1)$.

Proof. See [7, Lemma 2.3]. □

Proposition 2.3.9 *Given the equation*

$$\Diamond^k u(x) = f(x), \tag{2.30}$$

where $\Diamond^k$ is the Diamond operator iterated k-times defined by (1.1), $f(x)$ is a distribution, $u(x)$ is an unknown distribution and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ the n-dimensional Euclidean space and n is even, then (2.30) has the general solution

$$u(x) = (-1)^k R^\ell_{2k}(x) * \left(R^H_{2(k-1)}(x)\right)^{(m)} + (-1)^k K_{2k,2k}(x) * f(x)$$

where $\left(R^H_{2(k-1)}(x)\right)^{(m)}$ is a function with m derivatives defined by (2.21) and $K_{2k,2k}(x)$ is defined by (2.27) with $\alpha = \beta = 2k$.

Proof. See [7, Theorem 3.1]. □

Let us note that in the proof of Proposition 2.3.9, $u(x) = (-1)^k K_{2k,2k}(x) * f(x)$ is a particular solution of the equation (2.30).

Proposition 2.3.10 *$R_0^H(x) = \delta$, where δ is a Dirac-delta function and $R_0^H(x)$ is given in (2.21) with $\alpha = 0$.*

Proof. See [12, p10]. □

Definition 2.3.11 *Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional space $\mathbb{R}^n$,*

$$V = c_1^2(x_1^2 + x_2^2 + \ldots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2 \tag{2.31}$$

and

$$W = c_2^2(x_1^2 + x_2^2 + \ldots + x_p^2) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.32}$$

where $p + q = n$. The interior of forward cone defined by $\Gamma_+(A) = \{x \in \mathbb{R}^n : x_1 > 0, A > 0\}$ for $A = V, W$. For any complex numbers α and β, define

$$S_\alpha^H(x) = \begin{cases} \frac{V^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{for } x \in \Gamma_+(V), \\ 0 & \text{for } x \notin \Gamma_+(V), \end{cases} \tag{2.33}$$

and

$$T_\beta^H(x) = \begin{cases} \frac{W^{\frac{\beta-n}{2}}}{K_n(\beta)} & \text{for } x \in \Gamma_+(W), \\ 0 & \text{for } x \notin \Gamma_+(W), \end{cases} \tag{2.34}$$

where

$$K_n(a) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{a+2-n}{2})\Gamma(\frac{1-a}{2})\Gamma(a)}{\Gamma(\frac{a-p+2}{2})\Gamma(\frac{p-a}{2})} \quad \text{for } a = \alpha, \beta. \tag{2.35}$$

The function $S_\alpha^H(x)$ and $T_\beta^H(x)$ are introduced by Y. Nozaki [9, p72]. It is well known that such functions are ordinary functions if $Re(\alpha) \geq n$ and $Re(\beta) \geq n$ and are distributions of α and β if $Re(\alpha) < n$ and $Re(\beta) < n$ respectively.

By putting $p = 1$ in (2.31), (2.32) and (2.35) and using the Legendre's duplication of $\Gamma(z)$: $\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$ then (2.33) and (2.34) reduce to

$$M_\alpha^H(x) = \begin{cases} \frac{V^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{for } x \in \Gamma_+(V), \\ 0 & \text{for } x \notin \Gamma_+(V), \end{cases} \tag{2.36}$$

and

$$N_\beta^H(x) = \begin{cases} \frac{W^{\frac{\beta-n}{2}}}{H_n(\beta)} & \text{for } x \in \Gamma_+(W), \\ 0 & \text{for } x \notin \Gamma_+(W), \end{cases} \tag{2.37}$$

respectively, where $H_n(a) = \pi^{\frac{n-2}{2}} 2^{a-1} \Gamma(\frac{a-n+2}{2}) \Gamma(\frac{a}{2})$ for $a = \alpha, \beta$, $V = c_1^2 x_1^2 - x_2^2 - \ldots - x_n^2$ and $W = c_2^2 x_1^2 - x_2^2 - \ldots - x_n^2$. The functions $M_\alpha^H(x)$ and $N_\beta^H(x)$ are precisely called *the Hyperbolic kernel of Marcel Riesz.*

Definition 2.3.12 *Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and the functions $A_\gamma^\ell(x)$ and $L_\nu^\ell(x)$ be defined by*

$$A_\gamma^\ell(x) = \frac{X^{\frac{\gamma-n}{2}}}{P_n(\gamma)}, \tag{2.38}$$

and

$$L_\nu^\ell(x) = \frac{Y^{\frac{\nu-n}{2}}}{P_n(\nu)}, \tag{2.39}$$

where $X = c_1^2(x_1^2 + x_2^2 + \ldots + x_p^2) + x_{p+1}^2 + \ldots + x_{p+q}^2$, $Y = c_2^2(x_1^2 + x_2^2 + \ldots + x_p^2) + x_{p+1}^2 + \ldots + x_{p+q}^2$, $p + q = n$, $P_n(\gamma) = \frac{\pi^{\frac{n}{2}} 2^\gamma \Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}$, $P_n(\nu) = \frac{\pi^{\frac{n}{2}} 2^\nu \Gamma(\frac{\nu}{2})}{\Gamma(\frac{n-\nu}{2})}$, γ and ν are complex numbers.

Definition 2.3.13 *Let c_1 and c_2 be positive numbers, $p + q = n$ and k is a nonnegative integer. The ultra-hyperbolic operators iterated k times $\Box_{c_1}^k$ and $\Box_{c_2}^k$ are defined by*

$$\Box_{c_1}^k = \left(\frac{1}{c_1^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \tag{2.40}$$

and

$$\Box_{c_2}^k = \left(\frac{1}{c_2^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k. \tag{2.41}$$

The Laplacian operators iterated k times $\triangle_{c_1}^k$ and $\triangle_{c_2}^k$ are defined by

$$\triangle_{c_1}^k = \left(\frac{1}{c_1^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k, \tag{2.42}$$

and

$$\triangle_{c_2}^k = \left(\frac{1}{c_2^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k. \tag{2.43}$$

Proposition 2.3.14 *Let c_1 and c_2 be positive numbers. Given the equations*

$$\Box_{c_1}^k u(x) = \delta, \tag{2.44}$$

and

$$\Box_{c_2}^k v(x) = \delta, \tag{2.45}$$

where $\Box_{c_1}^k$ *and* $\Box_{c_2}^k$ *are defined by (2.40) and (2.41) respectively,* $x \in \mathbb{R}^n$ *and* δ *is the Dirac-delta function. Then we obtain* $u(x) = S_{2k}^H(x)$ *and* $v(x) = T_{2k}^H(x)$ *as an elementary solution of (2.44) and (2.45) respectively. Here* $S_{2k}^H(x)$ *and* $T_{2k}^H(x)$ *are defined by (2.33) and (2.34) respectively with* $\alpha = \beta = 2k$.

Proof. See[12, p11]. □

Proposition 2.3.15 *Let* c_1 *and* c_2 *be positive numbers. Given the equations*

$$\triangle_{c_1}^k u(x) = \delta, \tag{2.46}$$

and

$$\triangle_{c_2}^k v(x) = \delta, \tag{2.47}$$

where $\triangle_{c_1}^k$ *and* $\triangle_{c_2}^k$ *are defined by (2.42) and (2.43) respectively. Then we obtain* $u(x) = (-1)^k A_{2k}^\ell(x)$ *and* $v(x) = (-1)^k L_{2k}^\ell(x)$ *as an elementary solution of (2.46) and (2.47) respectively. Here* $A_{2k}^\ell(x)$ *and* $L_{2k}^\ell(x)$ *are defined by (2.38) and (2.39) respectively with* $\gamma = \nu = 2k$.

Proof. See[3, p118]. □

Proposition 2.3.16 *Let* c_1 *is a positive number. Given the equation*

$$\triangle_{c_1}^k u(x) = 0 \tag{2.48}$$

where $\triangle_{c_1}^k$ *is defined by (2.42). Then we obtain* $u(x) = (-1)^{k-1}\left(A_{2(k-1)}^\ell(x)\right)^{(m)}$ *as a solution of (2.48) where* m *is a nonnegative integer with* $m = \frac{n-4}{2}$, $n \geq 4$, n *is even and* $\left(A_{2(k-1)}^\ell(x)\right)^{(m)}$ *is a function defined by (2.38) with* m *derivative with* $\gamma = 2(k-1)$.

Proof. We first show that the generalized function $u(x) = \delta^{(m)}(X)$ where $X = c_1^2(x_1^2 + x_2^2 + \ldots + x_p^2) + x_{p+1}^2 + \ldots + x_{p+q}^2$ is a solution of

$$\triangle_{c_1} u(x) = 0 \tag{2.49}$$

where $\triangle_{c_1} = \frac{1}{c_1^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}$, $\quad p + q = n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.
Now for $i = 1, 2, \ldots, p$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\delta^{(m)}(X) &= 2c_1^2 x_i^2 \delta^{(m+1)}(X), \\
\frac{\partial^2}{\partial x_i^2}\delta^{(m+1)}(X) &= 2c_1^2\delta^{(m+1)}(X) + 4c_1^4 x_i^2 \delta^{(m+2)}(X)
\end{aligned}$$

and for $j = p+1, p+2, \ldots, p+q$, we obtain

$$\begin{aligned}\frac{\partial}{\partial x_j}\delta^{(m)}(X) &= 2x_j^2\delta^{(m+1)}(X),\\ \frac{\partial^2}{\partial x_j^2}\delta^{(m)}(X) &= 2\delta^{(m+1)}(X) + 4x_j^2\delta^{(m+2)}(X).\end{aligned}$$

Therefore

$$\begin{aligned}\triangle_{c_1}\delta^{(m)}(X) &= \frac{1}{c_1^2}\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}(X) + \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}(X)\\ &= \left[2p\delta^{(m+1)}(X) + 4c_1^2\sum_{i=1}^{p}x_i^2\delta^{(m+2)}(X)\right]\\ &\quad + \left[2q\delta^{(m+1)}(X) + 4\sum_{j=p+1}^{p+q}x_j^2\delta^{(m+2)}(X)\right]\\ &= 2n\delta^{(m+1)}(X) + 4X\delta^{(m+2)}(X)\\ &= 2n\delta^{(m+1)}(X) - 4(m+2)\delta^{(m+1)}(X)\end{aligned}$$

by Proposition 2.2.9 with $P = X = c_1^2(x_1^2 + x_2^2 + \ldots + x_p^2) + x_{p+1}^2 + \ldots + x_{p+q}^2$. Thus $\triangle_{c_1}\delta^{(m)}(X) = [2n - 4(m+2)]\,\delta^{(m+1)}(X) = 0$ if $2n - 4(m+2) = 0$ or if $m = \frac{n-4}{2}, n \geq 4$ and n is even. Now $\triangle_{c_1}^k u(x) = \triangle_{c_1}\left(\triangle_{c_1}^{k-1}u(x)\right) = 0$ then from the above proof $\triangle_{c_1}^{k-1}u(x) = \delta^{(m)}(X)$ with $m = \frac{n-4}{2}, n \geq 4$ and n is even. Convolving both sides of this equation by the function $(-1)^{k-1}A_{2(k-1)}^{\ell}(x)$, we obtain $u(x) = (-1)^{k-1}A_{2(k-1)}^{\ell}(x) * \delta^{(m)}(X)$ by Proposition 2.3.15. Now from (2.38),

$$A_{2(k-1)}^{\ell}(x) = \frac{X^{\frac{2(k-1)-n}{2}}}{P_n(2(k-1))}.$$

Hence

$$\begin{aligned}A_{2(k-1)}^{\ell}(x) * \delta^{(m)}(X) &= \frac{X^{\frac{2(k-1)-n}{2}}}{P_n(2(k-1))} * \delta^{(m)}(X)\\ &= \left[\frac{X^{\frac{2(k-1)-n}{2}}}{P_n(2(k-1))}\right]^{(m)}\\ &= \left[A_{2(k-1)}^{\ell}(x)\right]^{(m)}.\end{aligned}$$

It follows that $u(x) = (-1)^{k-1}\left[A_{2(k-1)}^{\ell}(x)\right]^{(m)}$ is a solution of (2.49) with $m = \frac{n-4}{2}, n \geq 4$ and n is even. □

Proposition 2.3.17 *The functions $A^{\ell}_{-2k}(x)$ and $L^{\ell}_{-2k}(x)$ are the inverse of the convolutions algebra of $A^{\ell}_{2k}(x)$ and $L^{\ell}_{2k}(x)$ respectively, that is*

$$A^{\ell}_{-2k}(x) * A^{\ell}_{2k}(x) = A^{\ell}_{-2k+2k}(x) = A^{\ell}_{0}(x) = \delta,$$

and

$$L^{\ell}_{-2k}(x) * L^{\ell}_{2k}(x) = L^{\ell}_{-2k+2k}(x) = L^{\ell}_{0}(x) = \delta,$$

Proof. See[3, p158]. □