Chapter 3

The Diamond Kernel of Marcel Riesz

In this chapter, we consider some properties of the Diamond kernel of Marcel Riesz $K_{2k,2k}$. At first we consider the value of $K_{2k,2k}$ acting on φ, that is $< K_{2k,2k}, \varphi >$ and second $K_{2k,2k} * \varphi$ where φ is a testing function of the space $\mathcal{D}$.

3.1 The Value of the Diamond Kernel $K_{2k,2k}$ of Marcel Riesz

In this section, we shall evaluate the Diamond kernel $< K_{\alpha,\beta}, \varphi >$ of Marcel Riesz, where $\varphi \in \mathcal{D}$.

Proposition 3.1.1 *Let φ be continuous and infinitely differentiable with compact support. If n is even, $n \geq 2(k+1)$ and p is odd, then*

$$< K_{2k,2k}(x), \varphi(x) >= C < \delta_1^{(\frac{n}{2}-k-1)}(U), \phi >,$$

where k is a positive integer, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, n is a dimension of the Euclidean space $\mathbb{R}^n$, $K_{2k,2k}(x)$ is given by (2.27) with $\alpha = \beta = 2k$, ϕ is given by (3.6),

$$U = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2, \; p + q = n,$$

$< \delta_1^{(\frac{n}{2}-k-1)}(U), \phi >$ is given by (3.24) with $m = \frac{n}{2} - k$ and

$$C = \frac{4^{1-2k} \pi^{\frac{1-n}{2}} (\frac{n-2k-2}{2})! \sin(\frac{p}{2}\pi)}{(\frac{n-2}{2})! [(k-1)!]^2}.$$

Proof. Let us begin by considering

$$
\begin{aligned}
< K_{\alpha,\beta}(x), \varphi(x) > &=< R^H_\alpha(x) * R^\ell_\beta(x), \varphi(x) > \\
&= \int_{\mathbb{R}^n} R^H_\alpha(x) * R^\ell_\beta(x)\varphi(x)\, dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} R^H_\alpha(y) R^\ell_\beta(x-y)\varphi(x)\, dy\, dx \\
&= \int_{\mathbb{R}^n} R^H_\alpha(y)\{\int_{\mathbb{R}^n} R^\ell_\beta(x-y)\varphi(x)\, dx\}\, dy. \qquad (3.1)
\end{aligned}
$$

Now we consider $\int_{\mathbb{R}^n} R^\ell_\beta(x-y)\varphi(x)\,dx$ and transform it into bipolar coordinates defined by

$$x_1 - y_1 = r\omega_1, x_2 - y_2 = r\omega_2, \ldots, x_n - y_n = r\omega_n, \sum_{i=1}^{n} \omega_i^2 = 1.$$

Thus $r = ((x_1-y_1)^2 + (x_2-y_2)^2 + \ldots + (x_n-y_n)^2)^{1/2}$ and the element of volume is given by

$$dx = r^{n-1}\, dr\, d\omega, \qquad (3.2)$$

where $d\omega$ is the element of surface area on the unit sphere Ω in $\mathbb{R}^n$. Therefore by (2.25), we obtain

$$\int_{\mathbb{R}^n} R^\ell_\beta(x-y)\varphi(x)\, dx = \frac{1}{W_n(\beta)}\int_{r=0}^{\infty} r^{\beta-n}\{\int_\Omega \varphi(y+r\omega)\, d\omega\} r^{n-1}\, dr, \qquad (3.3)$$

where $W_n(\beta)$ is given by (2.26). By the mean value for integral, we have

$$\int_\Omega \varphi(y+r\omega)\, d\omega = \Omega_n S_\varphi(y,r), \qquad (3.4)$$

where

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

is the hypersurface area of the unit sphere imbedded in the Euclidean space of n dimension and $S_\varphi(y,r)$ is the mean value of $\varphi(y+r\omega)$ on the sphere of radius r. Thus, from (3.3) and (3.4), we obtain

$$\int_{\mathbb{R}^n} R^\ell_\beta(x-y)\varphi(x)\, dx = \frac{\Omega_n}{W_n(\beta)}\int_{r=0}^{\infty} r^{\beta-1} S_\varphi(y,r)\, dr. \qquad (3.5)$$

Now by I.M. Gelfand and G.E. Shilov [5, p71], we can assert that $S_\varphi(y,r)$ (defined for $r \geq 0$) has bounded support and infinitely differentiable . Thus if we put

$$\phi(y) =< r^{\beta-1}, S_\varphi(y,r) >= \int_{r=0}^{\infty} r^{\beta-1} S_\varphi(y,r)\, dr, \qquad (3.6)$$

then by Proposition 2.2.6, $\phi(y)$ has bounded support and infinitely differentiable and therefore $\int_{\mathbb{R}^n} R^{\ell}_{\beta}(x-y)\varphi(x)\,dx$ has bounded support and infinitely differentiable. Now we substitute (3.6) into (3.5) and by (3.1) and (2.21), we obtain

$$< K_{\alpha,\beta}(x), \varphi(x) >= C_n(\alpha,\beta) \int_{U>0} [U(y)]^{\frac{\alpha-n}{2}} \phi(y)\, dy, \tag{3.7}$$

where

$$C_n(\alpha,\beta) = \frac{\Omega_n}{W_n(\beta)K_n(\alpha)}.$$

Let us transform into bipolar coordinates defined by

$$y_1 = s\omega_1, y_2 = s\omega_2, \ldots, y_p = s\omega_p, \sum_{i=1}^{p} \omega_i^2 = 1,$$

$$y_{p+1} = t\omega_{p+1}, y_{p+2} = t\omega_{p+2}, \ldots, y_{p+q} = t\omega_{p+q}, \sum_{i=p+1}^{p+q} \omega_i^2 = 1,$$

where $p+q=n$.

In these coordinates the element of volume is given by

$$dy = s^{p-1}t^{q-1}\, ds\, dt\, d\Omega^{(p)}\, d\Omega^{(q)},$$

where $d\Omega^{(p)}$ and $d\Omega^{(q)}$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.

We put

$$\begin{aligned} U(y) &= y_1^2 + y_2^2 + \ldots + y_p^2 \\ &\quad - y_{p+1}^2 - \ldots - y_{p+q}^2 \\ &= s^2 - t^2. \end{aligned}$$

Then the equation (3.7) becomes

$$< K_{\alpha,\beta}(x), \varphi(x) >= C_n(\alpha,\beta) \int_{U>0} (s^2-t^2)^{\frac{\alpha-n}{2}} \phi(y)\, s^{p-1}t^{q-1}\, dt\, ds\, d\Omega^{(p)}\, d\Omega^{(q)}. \tag{3.8}$$

Let us write

$$\psi(s,t) = \int \phi(y)\, d\Omega^{(p)}\, d\Omega^{(q)}. \tag{3.9}$$

Thus we have

$$< K_{\alpha,\beta}(x), \varphi(x) >= C_n(\alpha,\beta) \int_{s=0}^{\infty} \int_{t=0}^{s} (s^2-t^2)^{\frac{\alpha-n}{2}} \psi(s,t)\, s^{p-1}t^{q-1}\, dt\, ds. \tag{3.10}$$

Now, by I.M. Gelfand and G.E. Shilov [5, p249-252] since $\phi(y) \in \mathcal{D}$, we can affirm that $\psi(s,t)$ is an infinitely differentiable function of s^2 and t^2 with bounded support.

We make the change of variable $u = s^2$ and $v = t^2$ in (3.10), writing

$$\psi(s,t) = \psi_1(u,v)$$

to obtain

$$< K_{\alpha,\beta}(x), \varphi(x) >= \frac{1}{4} C_n(\alpha,\beta) \int_{u=0}^{\infty} \int_{v=0}^{u} (u-v)^{\frac{\alpha-n}{2}} \psi_1(u,v)\, u^{\frac{p-2}{2}} v^{\frac{q-2}{2}}\, dv\, du. \tag{3.11}$$

By putting $v = uw$, we have

$$< K_{\alpha,\beta}(x), \varphi(x) >= C_n(\alpha,\beta) \{ \frac{1}{4} \int_{u=0}^{\infty} u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1} \left[\int_0^1 (1-w)^{\frac{\alpha-n}{2}} w^{\frac{q-2}{2}} \psi_1(u,wu)\, dw \right] du \}. \tag{3.12}$$

Now we shall consider the expression

$$\frac{1}{4} \int_{u=0}^{\infty} u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1} \left[\int_0^1 (1-w)^{\frac{\alpha-n}{2}} w^{\frac{q-2}{2}} \psi_1(u,wu)\, dw \right] du. \tag{3.13}$$

This expression has two sets of poles. The first of those consists of the poles of

$$\phi(\frac{\alpha-n}{2}, u) = \frac{1}{4} \int_0^1 (1-w)^{\frac{\alpha-n}{2}} w^{\frac{q-2}{2}} \psi_1(u,wu)\, dw. \tag{3.14}$$

This function, by [5, p254], is regular for all $\frac{\alpha-n}{2}$ except

$$\frac{\alpha-n}{2} = -1, -2, \ldots, -m, \ldots, \tag{3.15}$$

where it has simple poles.

At these poles, by [5, p254, formula (12)], we have

$$\underset{\frac{\alpha-n}{2}=-m}{\text{res}} \phi(\frac{\alpha-n}{2}, u) = \frac{(-1)^{m-1}}{4(m-1)!} \left[\frac{\partial^{m-1}}{\partial w^{m-1}} \{ w^{\frac{q-2}{2}} \psi_1(u,wu) \} \right]_{w=1}. \tag{3.16}$$

Thus $\underset{\frac{\alpha-n}{2}=-m}{\text{res}} \phi(\frac{\alpha-n}{2}, u)$ is a functional concentrated on the surface of the $U=0$ cone.

On the other hand, even at regular points of $\phi(\frac{\alpha-n}{2}, u)$ the integral

$$\int_{u=0}^{\infty} u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1} \phi(\frac{\alpha-n}{2}, u)\, du$$

may also have poles. This occurs at

$$\frac{\alpha-n}{2} = -\frac{n}{2}, -\frac{n}{2}-1, \ldots, -\frac{n}{2}-m, \ldots, \tag{3.17}$$

where $p+q=n$.

At these points

$$\operatorname*{res}_{\frac{\alpha-n}{2}=-\frac{n}{2}-m}\int_{u=0}^{\infty}u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi(\frac{\alpha-n}{2},u)\,du=\frac{1}{m!}\left[\frac{\partial^m}{\partial u^m}\phi(-\frac{n}{2}-m,u)\right]_{u=0}. \tag{3.18}$$

Thus the residue of $\int_{u=0}^{\infty}u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi(\frac{\alpha-n}{2},u)\,du$ at $\frac{\alpha-n}{2}=-\frac{n}{2}-m$, is a functional concentrated on the vertex of the $U=0$ cone. Our purpose is to obtain the explicit value of $< K_{\alpha,\beta}(x),\varphi(x) >$ where $\alpha=\beta=2k$.

We can put

$$< K_{2k,2k}(x),\varphi(x) >=\lim_{\substack{\alpha\to 2k\\ \beta\to 2k}}< K_{\alpha,\beta}(x),\varphi(x) > \tag{3.19}$$

From (3.15) and (3.17), we know that if

$$\frac{2k-n}{2}=-m\quad ;\quad m=1,2,3,\ldots,$$

and

$$\frac{2k-n}{2}=\frac{-n}{2}-m\quad ;\quad m=1,2,3,\ldots,$$

the expression (3.19) has two cases of singularities.

The first case occurs at

$$n=2(k+m)\quad ;\quad m=1,2,3,\ldots$$

but the another case does not occur since $2k\neq -m$ for each $m=1,2,3,\ldots$. Therefore the expression (3.19) has singularities for $n\geq 2(k+1)$ and n even. Thus, in the case n odd and these case n even with $n\leq 2k$, there is no problem and we obtain the explicit formula (3.19) by passing the limit to (3.10) as α and β tend to $2k$.

Now we shall evaluate its explicit expression in the case $n\geq 2(k+1)$ and n even. By (3.19) and (3.10), we obtain

$$\begin{aligned}< K_{2k,2k}(x),\varphi(x) > &=\left(\lim_{\substack{\alpha\to 2k\\ \beta\to 2k}}C_n(\alpha,\beta)\right)\\ &\quad\times\left(\lim_{\alpha\to 2k}\int_{s=0}^{\infty}\int_{t=0}^{s}(s^2-t^2)^{\frac{\alpha-n}{2}}\psi(s,t)s^{p-1}t^{q-1}\,dt\,ds\right).\end{aligned} \tag{3.20}$$

We note that it always true that $\frac{\alpha-n}{2}>-\frac{n}{2}$ for $\alpha=2k$. Since the singular point occurs at $\frac{\alpha-n}{2}=-m$ and n is even, we can write (3.14) in the neighborhood of $\frac{\alpha-n}{2}=-m$ in the form

$$\phi(\frac{\alpha-n}{2},u)=\frac{\phi_0(u)}{\alpha-n+2m}+\phi_1(\frac{\alpha-n}{2},u), \tag{3.21}$$

where $\phi_0(u) = \operatorname*{res}_{\frac{\alpha-n}{2}=-m} \phi(\frac{\alpha-n}{2}, u)$ and $\phi_1(\frac{\alpha-n}{2}, u)$ is regular at $\frac{\alpha-n}{2} = -m$. Thus

$$\int_0^\infty u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi(\frac{\alpha-n}{2}, u)\,du = \frac{1}{\alpha+n+2m}\int_0^\infty u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi_0(u)\,du + \int_0^\infty u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi_1(\frac{\alpha-n}{2}, u)\,du.$$

Therefore

$$\operatorname*{res}_{\alpha=n-2m}\int_0^\infty u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi(\frac{\alpha-n}{2}, u)\,du = \int_0^\infty u^{-m+\frac{p+q}{2}-1}\phi_0(u)\,du,$$

where for $m \geq \frac{1}{2}(p+q)$ the integral is understood in the sense of its regularization. Inserting (3.16) for $\phi_0(u)$, we arrive at

$$\begin{aligned}&\operatorname*{res}_{\alpha=n-2m}\int_0^\infty u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi(\frac{\alpha-n}{2}, u)\,du\\ &= \frac{(-1)^{m-1}}{4(m-1)!}\int_0^\infty \left[\frac{\partial^{m-1}}{\partial w^{m-1}}\{w^{\frac{q-2}{2}}\psi_1(u, wu)\}\right]_{w=1} u^{-m+\frac{p+q}{2}-1}\,du. \qquad (3.22)\end{aligned}$$

This residue is a generalized function concentrated on the $P=0$ cone. If we write $wu = v$, we obtain

$$\left[\frac{\partial^{m-1}}{\partial w^{m-1}}\{w^{\frac{q-2}{2}}\psi_1(u, wu)\}\right]_{w=1} = \left[\frac{\partial^{m-1}}{\partial v^{m-1}}\{v^{\frac{q-2}{2}}\psi_1(u, v)\}\right]_{v=u} u^{-\frac{1}{2}q+k}$$

so that we may rewrite (3.22) in the form

$$\begin{aligned}&\operatorname*{res}_{\alpha=n-2m}\int_0^\infty u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi(\frac{\alpha-n}{2}, u)\,du\\ &= \frac{(-1)^{m-1}}{4(m-1)!}\int_0^\infty \left[\frac{\partial^{m-1}}{\partial v^{m-1}}\{v^{\frac{q-2}{2}}\psi_1(u, v)\}\right]_{v=u} u^{\frac{p-2}{2}}\,du\\ &= \frac{(-1)^{m-1}}{(m-1)!} < \delta_1^{(m-1)}(U(y)), \phi(y) >, \qquad (3.23)\end{aligned}$$

where

$$< \delta_1^{(m-1)}(U(y)), \phi(y) >= \frac{1}{4}\int_{u=0}^\infty \left[\frac{\partial^{m-1}}{\partial v^{m-1}}\{v^{\frac{q-2}{2}}\psi_1(u, v)\}\right]_{v=u} u^{\frac{p-2}{2}}\,du. \qquad (3.24)$$

Therefore in case of $m = \frac{n}{2} - k$, the equation (3.23) becomes

$$\operatorname*{res}_{\alpha=2k}\int_0^\infty u^{\frac{\alpha-n}{2}+\frac{p+q}{2}-1}\phi(\frac{\alpha-n}{2}, u)\,du = \frac{(-1)^{\frac{n}{2}-k-1}}{(\frac{n-2k-2}{2})!} < \delta_1^{(\frac{n}{2}-k-1)}(U(y)), \phi(y) > . \qquad (3.25)$$

Thus by (3.10), (3.12), (3.13), and (3.25), we obtain

$$\operatorname*{res}_{\alpha=2k}\int_{s=0}^{\infty}\int_{t=0}^{s}(s^2-t^2)^{\frac{\alpha-n}{2}}\psi(s,t)s^{p-1}t^{q-1}\,dt\,ds=\frac{(-1)^{\frac{n}{2}-k-1}}{\left(\frac{n-2k-2}{2}\right)!}<\delta_1^{(\frac{n}{2}-k-1)}(U(y)),\phi(y)>. \tag{3.26}$$

Now, we consider

$$\begin{aligned}\lim_{\substack{\alpha\to 2k\\ \beta\to 2k}}C_n(\alpha,\beta)&=\lim_{\substack{\alpha\to 2k\\ \beta\to 2k}}\frac{\Omega_n}{W_n(\beta)K_n(\alpha)}\\ &=\frac{2^{1-2k}\Gamma(\frac{n-2k}{2})}{\Gamma(\frac{n}{2})\Gamma(k)}\lim_{\alpha\to 2k}\frac{1}{K_n(\alpha)}\\ &=\frac{2^{1-2k}\left(\frac{n-2k-2}{2}\right)!}{\left(\frac{n-2}{2}\right)!(k-1)!}\lim_{\alpha\to 2k}\frac{\pi^{\frac{1-n}{2}}\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}{\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}.\end{aligned} \tag{3.27}$$

Taking into account the Legendre's duplication formula

$$\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\frac{1}{2}),$$

we obtain

$$\lim_{\alpha\to 2k}\frac{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}{\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}=\lim_{\alpha\to 2k}\frac{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}{\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})2^{\alpha-1}\pi^{-1/2}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+1}{2})}. \tag{3.28}$$

We know from Proposition 2.1.4 that

$$\operatorname*{res}_{\lambda=-k}\Gamma(\lambda)=\frac{(-1)^k}{k!},\quad k=0,1,2,\ldots, \tag{3.29}$$

thus (n even and $n\geq 2(k+1)$)

$$\operatorname*{res}_{\alpha=2k}\Gamma\left(\frac{2+\alpha-n}{2}\right)=\frac{(-1)^{\frac{n-2k-2}{2}}}{(\frac{n-2k-2}{2})!}. \tag{3.30}$$

Now consider

$$\begin{aligned}\lim_{\alpha\to 2k}\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)&=\frac{(-1)^k\Gamma(\frac{1}{2})}{(k-\frac{1}{2})(k-\frac{3}{2})\ldots(\frac{1}{2})}\cdot(k-\frac{1}{2})(k-\frac{3}{2})\ldots(\frac{1}{2})\Gamma(\frac{1}{2})\\ &=(-1)^k\pi.\end{aligned} \tag{3.31}$$

Lastly, we suppose that p is odd. Then by Proposition 2.1.2(2),

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\,\pi z},$$

we have

$$\lim_{\alpha\to 2k}\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)=\frac{\pi}{\sin\,\pi\left(\frac{2+2k-p}{2}\right)}$$
$$=\frac{-\pi(-1)^{k+1}}{\sin(\frac{p\pi}{2})}$$
$$=(-1)^k\pi\sin(\frac{p\pi}{2}). \qquad (3.32)$$

Now, from (3.20),(3.26)-(3.28) and (3.30)-(3.32), we obtain

$$< K_{2k,2k}(x),\varphi(x) > = \frac{(-1)^{\frac{n}{2}-k-1}}{\left(\frac{n-2k-2}{2}\right)!}\cdot\frac{2^{1-2k}\left(\frac{n-2k-2}{2}\right)!}{\left(\frac{n-2}{2}\right)!(k-1)!}\pi^{\frac{1-n}{2}}$$
$$\times\frac{(-1)^k\pi\sin(\frac{p}{2}\pi)\left(\frac{n-2k-2}{2}\right)!}{(-1)^{\frac{n}{2}-k-1}2^{2k-1}(-1)^k\pi(k-1)!} < \delta_1^{(\frac{n}{2}-k-1)}(U(y)),\phi(y) >$$
$$=\frac{4^{(1-2k)}\pi^{\frac{1-n}{2}}\left(\frac{n-2k-2}{2}\right)!\sin(\frac{p}{2}\pi)}{\left(\frac{n-2}{2}\right)![(k-1)!]^2} < \delta_1^{(\frac{n}{2}-k-1)}(U(y)),\phi(y) > . \qquad (3.33)$$

This completes the proof. □

3.2 An Explicit Formula of the Solution of the Diamond Operator

The purpose of this section is to obtain an explicit form of the expression $K_{2k,2k}(x)*\varphi(x)$.

Proposition 3.2.1 *Let φ be continuous and infinitely differentiable with compact support. If n is even, $n\geq 2(k+1)$ and p is odd, then for any $y\in\mathbb{R}^n$,*

$$K_{2k,2k}(x)*\varphi(x)=C< \delta_1^{(\frac{n}{2}-k-1)}(U(x-y)),\phi(y) >, \qquad (3.34)$$

where k is a positive integer, $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, n is the dimension of the Euclidean space $\mathbb{R}^n$,$K_{2k,2k}(x)$ is given by (2.27), ϕ is given by (3.43),

$$U(x-y)=(x_1-y_1)^2+\ldots+(x_p-y_p)^2-(x_{p+1}-y_{p+1})^2-\ldots-(x_{p+q}-y_{p+q})^2,\;\; p+q=n,$$

$$C=\frac{4^{1-2k}\pi^{\frac{1-n}{2}}\left(\frac{n-2k-2}{2}\right)!\sin(\frac{p}{2}\pi)}{\left(\frac{n-2}{2}\right)![(k-1)!]^2}, \qquad (3.35)$$

and

$$< \delta_1^{(\frac{n}{2}-k-1)}(U(x-y)),\phi(y) >=\frac{1}{4}\int_{u=0}^{\infty}\left[\frac{\partial^{\frac{n}{2}-k-1}}{\partial v^{\frac{n}{2}-k-1}}\{v^{\frac{q-2}{2}}\psi_1(u,v,x)\}\right]_{v=u}u^{\frac{p-2}{2}}\,du, \qquad (3.36)$$

where $\psi_1(u,v,x)$ is given by (3.48).

Proof. Let us begin by considering the convolution

$$R^\ell_\beta(x) * \varphi(x), \tag{3.37}$$

where $R^\ell_\beta(x)$ given by (2.25) and β is a complex number. We suppose that $\mathrm{Re}(\beta)$ is large enough to be locally integrable. Thus

$$R^\ell_\beta(x) * \varphi(x) = \int_{\mathbb{R}^n} R^\ell_\beta(x-z)\varphi(z)\,dz. \tag{3.38}$$

Let us transform to bipolar coordinates defined by

$$x_1 - z_1 = r\omega_1, x_2 - z_2 = r\omega_2, \ldots, x_n - z_n = r\omega_n, \sum_{i=1}^{n} \omega_i^2 = 1.$$

Thus $r = ((x_1-z_1)^2 + (x_2-z_2)^2 + \ldots + (x_n-z_n)^2)^{1/2}$ and the element of volume is given by

$$dz = r^{n-1}\,dr\,d\omega, \tag{3.39}$$

where $d\omega$ is the element of surface area on the unit sphere Ω in $\mathbb{R}^n$. Therefore by (3.38) and (2.25), we obtain

$$R^e_\beta(x) * \varphi(x) = \frac{1}{W_n(\beta)} \int_{r=0}^{\infty} r^{\beta-n}\{\int_\Omega \varphi(x-r\omega)\,d\omega\} r^{n-1}\,dr, \tag{3.40}$$

where $W_n(\beta)$ is given by (2.26). By the mean value for integral, we have

$$\int_\Omega \varphi(x-r\omega)\,d\omega = \Omega_n S_\varphi(x,r), \tag{3.41}$$

where

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

is the hypersurface area of the unit sphere imbedded in the Euclidean space of n dimension and $S_\varphi(x,r)$ is the mean value of $\varphi(x-r\omega)$ on the sphere of radius r. Thus, from (3.40) and (3.41), we obtain

$$R^\ell_\beta(x) * \varphi(x) = \frac{\Omega_n}{W_n(\beta)} \int_{r=0}^{\infty} r^{\beta-1} S_\varphi(x,r)\,dr. \tag{3.42}$$

Now, by I.M. Gelfand and G.E. Shilov [5, p71], we can assert that $S_\varphi(x,r)$ (defined for $r \geq 0$) has bounded support and infinitely differentiable . Thus if we put

$$\phi(x) = < r^{\beta-1}, S_\varphi(x,r) > = \int_{r=0}^{\infty} r^{\beta-1} S_\varphi(x,r)\,dr, \tag{3.43}$$

then by Proposition 2.2.6, $\phi(x)$ has bounded support and infinitely differentiable and therefore $R^\ell_\beta(x) * \varphi(x)$ has bounded support and infinitely differentiable. Thus by (2.21),(3.42) and (3.43), we obtain

$$\begin{aligned}
K_{\alpha,\beta}(x) * \varphi(x) &= \left(R^H_\alpha(x) * R^\ell_\beta(x)\right) * \varphi(x) \\
&= R^H_\alpha(x) * \left(R^e_\beta(x) * \varphi(x)\right) \\
&= R^H_\alpha(x) * \frac{\Omega_n}{W_n(\beta)}\phi(x) \\
&= \frac{\Omega_n}{W_n(\beta)} \int_{\mathbb{R}^n} R^H_\alpha(x-y)\phi(y)\,dy \\
&= C_n(\alpha,\beta) \int_{U>0} [U(x-y)]^{\frac{\alpha-n}{2}}\,\phi(y)\,dy \qquad (3.44)
\end{aligned}$$

where

$$C_n(\alpha,\beta) = \frac{\Omega_n}{W_n(\beta)K_n(\alpha)}. \qquad (3.45)$$

Let us transform to bipolar coordinates defined by

$$x_1 - y_1 = s\omega_1, x_2 - y_2 = s\omega_2, \ldots, x_p - y_p = s\omega_p, \sum_{i=1}^{p} \omega_i^2 = 1,$$

$$x_{p+1} - y_{p+1} = t\omega_{p+1}, x_{p+2} - y_{p+2} = t\omega_{p+2}, \ldots, x_{p+q} - y_{p+q} = t\omega_{p+q}, \sum_{i=p+1}^{p+q} \omega_i^2 = 1,$$

where $p + q = n$.

In these coordinates the element of volume is given by

$$dy = s^{p-1}t^{q-1}\,ds\,dt\,d\Omega^{(p)}\,d\Omega^{(q)},$$

where $d\Omega^{(p)}$ and $d\Omega^{(q)}$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.

We put

$$\begin{aligned}
U(x-y) &= (x_1-y_1)^2 + (x_2-y_2)^2 + \ldots + (x_p-y_p)^2 \\
&\quad - (x_{p+1}-y_{p+1})^2 - \ldots - (x_{p+q}-y_{p+q})^2 \\
&= s^2 - t^2.
\end{aligned}$$

Then the equation (3.44) becomes

$$K_{\alpha,\beta}(x)*\varphi(x) = C_n(\alpha,\beta) \int_{U>0} (s^2-t^2)^{\frac{\alpha-n}{2}}\phi(y)\,s^{p-1}t^{q-1}\,dt\,ds\,d\Omega^{(p)}\,d\Omega^{(q)}. \quad (3.46)$$

Let us write

$$\psi(s,t,x) = \int \phi(y)\,d\Omega^{(p)}\,d\Omega^{(q)}.$$

Thus we have

$$K_{\alpha,\beta}(x) * \varphi(x) = C_n(\alpha,\beta) \int_{s=0}^{\infty} \int_{t=0}^{s} (s^2 - t^2)^{\frac{\alpha-n}{2}} \psi(s,t,x) s^{p-1} t^{q-1}\, dtds. \quad (3.47)$$

Now, $\psi(s,t,x)$ is an infinitely differentiable function of s^2 and t^2 with bounded support.

We make change of variable $u = s^2$ and $v = t^2$ in (3.47), writing

$$\psi(s,t,x) = \psi_1(u,v,x), \quad (3.48)$$

then we can prove by proceeding as the proof of Proposition 3.1.1 from the equation (3.11) to the end of the proof but we must substitute $\psi_1(u,v)$, $< K_{2k,2k}(x), \varphi(x) >$, $< K_{\alpha,\beta}(x), \varphi(x) >$, and $< \delta_1^{(m-1)}(U(y)), \phi(y) >$ into $\psi_1(u,v,x)$, $K_{2k,2k}(x) * \varphi(x)$,$K_{\alpha,\beta}(x) * \varphi(x)$, and $< \delta_1^{(m-1)}(U(x-y)), \phi(y) >$ respectively. This completes the proof. □

It is to note that if the hypothesis of Proposition 3.2.1 holds, then by Proposition 2.3.9, we have $u(x) = (-1)^k K_{2k,2k}(x) * \varphi(x)$ is a solution of the equation (2.30), where $K_{2k,2k}(x) * \varphi(x)$ is given by (3.34).

We remark that the formula we have obtained for the functions of the family $\mathcal{R}$ are also valid for functions of the class $\mathcal{A}$; except that p and q must be interchanged, and in all the formula $\delta_1^{(\frac{n}{2}-k-1)}(U(x-y))$ must be replaced by $\delta_1^{(\frac{n}{2}-k-1)}(-U(x-y)) = (-1)^{(\frac{n}{2}-k-1)} \delta_2^{(\frac{n}{2}-k-1)}(U(x-y))$, where

$$< \delta_2^{(\frac{n}{2}-k-1)}(U(x-y)), \phi(y) > = \frac{(-1)^{(\frac{n}{2}-k-1)}}{4} \times \int_{v=0}^{\infty} \left[\frac{\partial^{\frac{n}{2}-k-1}}{\partial u^{\frac{n}{2}-k-1}} \{ u^{\frac{p-2}{2}} \psi_1(u,v,x) \} \right]_{u=v} v^{\frac{q-2}{2}}\, dv.$$