Chapter 4

On the Nonlinear Diamond Operator Related to the Wave Equation

In this chapter, we shall give a solution of the equation (4.1) with boundary condition (4.3) under some assumptions. This solution relates to the wave equation.

Theorem 4.1 *Consider nonlinear equation*

$$\Diamond^k u(x) = f(x, \triangle^{k-1}\Box^k u(x)), \tag{4.1}$$

with the boundary condition

$$\triangle^{k-1}\Box^k u(x) = 0 \quad \text{for all} \;\; x \in \partial\Omega, \tag{4.2}$$

where $\Diamond^k$ *is the Diamond operator iterated k-times defined by (1.1),* $\triangle^{k-1}$ *is the Laplacian operator iterated* $k-1$*-times defined by(1.3) and* $\Box^k$ *is the ultra-hyperbolic operator iterated k-times defined by (1.4). Let f be defined and have continuous first derivatives for all* $x \in \Omega \cup \partial\Omega$*,* Ω *is an open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *denotes the boundary of* Ω *and n is even with* $n \geq 4$*. If f be a bounded function, that is, there exists an* $N > 0$ *such that*

$$|f(x, \triangle^{k-1}\Box^k u(x))| \leq N \quad \text{for all} \;\; x \in \Omega, \tag{4.3}$$

then there exists unique continuous function $W(x)$ *such that*

$$u(x) = (-1)^{k-1} R^{\ell}_{2(k-1)}(x) * R^{H}_{2k}(x) * W(x) \tag{4.4}$$

as a solution of (4.1) with the boundary condition (4.2) or

$$u(x) = R^{H}_{2k}(x) * (-1)^{k-2}\left(R^{\ell}_{2(k-2)}(x)\right)^{(m)}$$

for $x \in \partial\Omega$, $m = (n-4)/2$, $k = 2, 3, 4, \ldots$, $R^{\ell}_{2(k-2)}(x)$ and $R^{H}_{2k}(x)$ are given by (2.25) and (2.21) respectively, with $\beta = 2(k-2)$ and $\alpha = 2k$. Moreover, for $k = 1$ we obtain

$$u(x) = R^{H}_{2}(x) * W(x)$$

as a solution of (4.1) with the boundary condition

$$u(x) = \delta^{(m)}(U) \quad \text{for all} \quad x \in \partial\Omega$$

where $U = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$, $\delta^{(m)}(U)$ is the Dirac-delta function with m derivative and $m = (n-4)/2$. Also, if $k = 1$, $p = 1$ and $q = n-1$, we obtain $u(x) = M_2(s) * W(x)$ as a solution of the nonhomogeneous wave equation

$$\Box^* u(x) = W(x) \tag{4.5}$$

with the boundary condition $u(x) = \delta^{(m)}(s)$ for $x \in \partial\Omega$, where

$$\Box^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

and $M_2(s)$ is defined by (2.24) with $\alpha = 2$ and $s = x_1^2 - x_2^2 - \ldots - x_n^2$.

Proof. We have

$$\Diamond^k u(x) = \triangle(\triangle^{k-1}\Box^k u(x)) = f(x, \triangle^{k-1}\Box^k u(x)). \tag{4.6}$$

Since $u(x)$ has continuous derivatives up to order $4k$ for $k = 1, 2, 3, \ldots$, we can assume

$$\triangle^{k-1}\Box^k u(x) = W(x) \quad \text{for all } x \in \Omega. \tag{4.7}$$

Thus, (4.6) can be written in the form

$$\Diamond^k u(x) = \triangle W(x) = f(x, W(x)). \tag{4.8}$$

Since by (4.3),

$$|f(x, W(x))| \leq N \quad \text{for all } x \in \Omega \tag{4.9}$$

and by (4.2), $W(x) = 0$ or

$$\triangle^{k-1}\Box^k u(x) = 0, \tag{4.10}$$

for $x \in \partial\Omega$, by Proposition 2.3.3 with $c = 1$ there exists unique solution $W(x)$ of (4.8) which is continuous and satisfies (4.10). Now consider the equation (4.7), we have $\triangle^{k-1}(-1)^{k-1}R^{\ell}_{2(k-1)}(x) = \delta$ and $\Box^k R^{H}_{2k}(x) = \delta$ where δ is the Dirac-delta function, that is, $R^{H}_{2k}(x)$ and $(-1)^{k-1}R^{\ell}_{2(k-1)}(x)$ are the elementary solutions of the operators $\Box^k$ and $\triangle^{k-1}$ respectively, see Proposition 2.3.14 and 2.3.15 with $c_1 = 1$. The functions $R^{H}_{2k}(x)$ and $R^{\ell}_{2(k-1)}(x)$ are defined by (2.21) and (2.25) respectively with $\alpha = 2k$ and $\beta = 2(k-1)$. Thus, convolving both sides of (4.7) by $(-1)^{k-1}R^{\ell}_{2(k-1)}(x) * R^{H}_{2k}(x)$, we obtain

$$[(-1)^{k-1}R^{\ell}_{2(k-1)}(x)*R^{H}_{2k}(x)]*\Delta^{k-1}\Box^{k}u(x) = [(-1)^{k-1}R^{\ell}_{2(k-1)}(x)*R^{H}_{2k}(x)]*W(x).$$

By Proposition 2.2.15, we obtain

$$(\Delta^{k-1}(-1)^{k-1}R^{\ell}_{2(k-1)}(x))*(\Box^{k}R^{H}_{2k}(x))*u(x) = (-1)^{k-1}R^{\ell}_{2(k-1)}(x)*R^{H}_{2k}(x)*W(x),$$

$$\delta * \delta * u(x) = (-1)^{k-1}R^{\ell}_{2(k-1)}(x) * R^{H}_{2k}(x) * W(x).$$

Thus

$$u(x) = (-1)^{k-1}R^{\ell}_{2(k-1)}(x) * R^{H}_{2k}(x) * W(x) \tag{4.11}$$

as required. Consider $\Delta^{k-1}\Box^{k}u(x) = 0$ for $x \in \partial\Omega$. By Proposition 2.3.16 with $c_1 = 1$, we have

$$\Box^{k}u(x) = (-1)^{k-2}(R^{\ell}_{2(k-2)}(x))^{(m)}.$$

Convoluting both sides of the above equation by $R^{H}_{2k}(x)$, we obtain

$$R^{H}_{2k}(x) * \Box^{k}u(x) = R^{H}_{2k}(x) * (-1)^{k-2}(R^{\ell}_{2(k-2)}(x))^{(m)}$$

or $\Box^{k}R^{H}_{2k}(x) * u(x) = R^{H}_{2k}(x) * (-1)^{k-2}(R^{\ell}_{2(k-2)}(x))^{(m)}$. It follows that $\delta * u(x) = u(x) = R^{H}_{2k}(x) * (-1)^{k-2}(R^{\ell}_{2(k-2)}(x))^{(m)}$ for $x \in \partial\Omega$ and $k = 2, 3, 4, \ldots$.

Now, for $k = 1$ by (4.11) and Proposition 2.3.17, we obtain

$$\begin{aligned} u(x) &= R^{\ell}_{0}(x) * R^{H}_{2}(x) * W(x) \\ &= \delta * R^{H}_{2}(x) * W(x) \\ &= R^{H}_{2}(x) * W(x). \end{aligned} \tag{4.12}$$

Now consider the boundary condition (4.10) for $k = 1$, we have $\Box u(x) = 0$ for $x \in \partial\Omega$. Thus by Proposition 2.3.8 with $k = 1$ and by Proposition 2.3.10, we obtain

$$u(x) = \delta^{(m)}(U) \quad \text{for } x \in \partial\Omega, \tag{4.13}$$

where $U = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$. Now consider the case $k = 1$, $p = 1$ and $q = n - 1$; thus from (4.12), $R^{H}_{2}(x)$ reduces to $M_2(s)$ where $M_2(s)$ is defined by (2.24) with $\alpha = 2$ and $s = x_1^2 - x_2^2 - \ldots - x_n^2$ and also the operator $\Box$ defined by (1.4) reduces to the wave operator

$$\Box^{*} = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \ldots - \frac{\partial^2}{\partial x_n^2}.$$

Thus, the solution $u(x)$ in (4.12) reduces to $u(x) = M_2(s) * W(x)$ which is the solution of the wave equation $\Box^{*}u(x) = W(x)$ with the boundary condition $\Box^{*}u(x) = 0$ for $x \in \partial\Omega$ or $u(x) = \delta^{(m)}(s)$ for $x \in \partial\Omega$ by (4.13). This completes the proof. $\Box$