Chapter 5

On the Product of the Nonlinear Diamond Operators Related to the Elastic Wave

In this chapter, we develop the operator defined by (1.1) to be

$$\diamond_{c_1}^k = \left[\frac{1}{c_1^4}(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2})^2 - (\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2})^2\right]^k \tag{5.1}$$

and

$$\diamond_{c_2}^k = \left[\frac{1}{c_2^4}(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2})^2 - (\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2})^2\right]^k, \tag{5.2}$$

where $p+q=n$, $c_1$and c_2 are positive constants and k is a nonnegative integer. Then we shall give a solution of the equation (5.3) with the boundary condition (5.5) under some assumptions. This solution relates to the elastic wave.

Theorem 5.1 *Let c_1 and c_2 be positive numbers. Consider the nonlinear equation*

$$\diamond_{c_1}^k \diamond_{c_2}^k u(x) = f\left(x, \Delta_{c_1}^{k-1}\Box_{c_1}^k \diamond_{c_2}^k u(x)\right) \tag{5.3}$$

with the boundary condition

$$\Delta_{c_1}^{k-1}\Box_{c_1}^k \diamond_{c_2}^k u(x) = 0, \quad \text{for all } x \in \partial\Omega, \tag{5.4}$$

where $\diamond_{c_1}^k$, $\diamond_{c_2}^k$, $\Delta_{c_1}^{k-1}$, and $\Box_{c_1}^k$ are defined by (5.1), (5.2), (2.42), and (2.40) respectively. Let f be defined and having continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of $\mathbb{R}^n$, $\partial\Omega$ denotes the boundary of Ω and n is even with $n \geq 4$. If f is bounded function, that is, there exists an $N > 0$ such that

$$\left|f\left(x, \Delta_{c_1}^{k-1}\Box_{c_1}^k \diamond_{c_2}^k u(x)\right)\right| \leq N, \quad \text{for all } x \in \Omega, \tag{5.5}$$

then there exists unique continuous function $U(x)$ *such that*

$$u(x) = (-1)^{k-1} A^{\ell}_{2(k-1)}(x) * S^H_{2k}(x) * (-1)^k L^{\ell}_{2k}(x) * T^H_{2k}(x) * U(x) \tag{5.6}$$

as a solution of (5.4) with the boundary condition

$$u(x) = S^H_{2k}(x) * (-1)^k L^{\ell}_{2k}(x) * T^H_{2k}(x) * (-1)^{k-2} \left(A^{\ell}_{2(k-2)}(x)\right)^{(m)} \tag{5.7}$$

for all $x \in \partial\Omega$, $m = \frac{n-4}{2}$, $S^H_{2k}(x)$, $T^H_{2k}(x)$, $A^{\ell}_{2(k-2)}(x)$ *and* $L^{\ell}_{2k}(x)$ *are given by (2.33), (2.34), (2.38) and (2.39) respectively with* $\alpha = \beta = \nu = 2k$ *and* $\gamma = 2(k-2)$. *Moreover,*

$$W(x) = (-1)^{k-1} A^{\ell}_{2(1-k)} * (-1)^k L^{\ell}_{-2k}(x) * u(x) \tag{5.8}$$

is a solution of the equation

$$\square^k_{c_1} \square^k_{c_2} W(x) = U(x), \tag{5.9}$$

where $\square^k_{c_1}$ *and* $\square^k_{c_2}$ *are defined by (2.40) and (2.41) respectively and* $u(x)$ *is obtained from (5.6). Furthermore, if we put* $p = k = 1$, *then* $W(x)$ *reduces to* $W(x) = M^H_2(x) * N^H_2(x) * U(x)$ *which is a solution of the inhomogeneous elastic wave equation* $(\frac{1}{c_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2})(\frac{1}{c_2^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2})W(x) = U(x)$ *where* $M^H_2(x)$ *and* $N^H_2(x)$ *are defined by (2.36) and (2.37) respectively with* $\alpha = \beta = 2$ *and* $c_1 \neq c_2$.

Proof We have

$$\diamond^k_{c_1} \diamond^k_{c_2} u(x) = \triangle_{c_1} \triangle^{k-1}_{c_1} \square^k_{c_1} \diamond^k_{c_2} u(x) = f\left(x, \triangle^{k-1}_{c_1} \square^k_{c_1} \diamond^k_{c_2} u(x)\right). \tag{5.10}$$

Since $u(x)$ has continuous derivative up to order $8k$ for $k = 1, 2, 3, \dots$, thus we can assume

$$\triangle^{k-1}_{c_1} \square^k_{c_1} \diamond^k_{c_2} u(x) = U(x) \quad \text{for all } x \in \Omega. \tag{5.11}$$

Then (5.10) can be written in the form

$$\diamond^k_{c_1} \diamond^k_{c_2} u(x) = \triangle_{c_1} U(x) = f(x, U(x)). \tag{5.12}$$

Thus by (5.3),

$$|f(x, U(x))| \leq N \quad \text{for all } x \in \Omega \tag{5.13}$$

and by (5.5), $U(x) = 0$ or

$$\triangle^{k-1}_{c_1} \square^k_{c_1} \diamond^k_{c_2} u(x) = 0 \quad \text{for all } x \in \partial\Omega, \tag{5.14}$$

we obtain a unique solution $U(x)$ of (5.12) which is continuous and satisfies (5.14) by Proposition 2.3.3 with $c = c_1$. Note that the functions $A^{\ell}_{2(k-1)}(x)$,

$S^H_{2k}(x)$, $L^\ell_{2k}(x)$ and $T^H_{2k}(x)$ are the elementary solutions of the operators $\Delta^{k-1}_{c_1}, \Box^k_{c_1}, \Delta^k_{c_2}$ and $\Box^k_{c_2}$ respectively,
i.e., $\Delta^{k-1}_{c_1}(-1)^{k-1}A^\ell_{2(k-1)}(x) = \delta$, $\Box^k_{c_1}S^H_{2k}(x) = \delta$, $\Delta^k_{c_2}(-1)^k L^\ell_{2k}(x) = \delta$ and $\Box^k_{c_2}T^H_{2k}(x) = \delta$, by Proposition 2.3.14 and 2.3.15. The functions $A^\ell_{2(k-1)}(x)$, $S^H_{2k}(x)$, $L^\ell_{2k}(x)$ and $T^H_{2k}(x)$ are defined by (2.38), (2.33), (2.39) and (2.34) respectively with $\gamma = 2(k-1)$ and $\alpha = \beta = \nu = 2k$. Thus convolving both sides of (5.11) by $(-1)^{k-1}A^\ell_{2(k-1)}(x)*S^H_{2k}(x)*(-1)^k L^\ell_{2k}(x)*T^H_{2k}(x)$ and by Proposition 2.2.15, we obtain

$$u(x) = (-1)^{k-1}A^\ell_{2(k-1)}(x) * S^H_{2k}(x) * (-1)^k L^\ell_{2k}(x) * T^H_{2k}(x) * U(x) \qquad (5.15)$$

as required. Now consider the boundary condition (5.14). By Proposition 2.3.16, we have

$$\Box^k_{c_1}\Diamond^k_{c_2}u(x) = (-1)^{k-2}\left(A^\ell_{2(k-2)}(x)\right)^{(m)},$$

where $m = \frac{n-4}{2}, n \geq 4$ and n is even. Convolving both sides of above equation by $S^H_{2k}(x) * (-1)^k L^\ell_{2k}(x) * T^H_{2k}(x)$, we obtain

$$u(x) = S^H_{2k}(x) * (-1)^k L^\ell_{2k}(x) * T^H_{2k}(x) * (-1)^{k-2}\left(A^\ell_{2(k-2)}(x)\right)^{(m)},$$

for $x \in \partial\Omega$ by proposition 2.3.14 and 2.3.15.

Lastly, convolving both sides of (5.15) by $(-1)^{k-1}A^\ell_{2(1-k)}(x)*(-1)^k L^\ell_{-2k}(x)$,we obtain

$$(-1)^{k-1}A^\ell_{2(1-k)}(x) * (-1)^k L^\ell_{-2k}(x) * u(x) = S^H_{2k}(x) * T^H_{2k}(x) * U(x),$$

by Proposition 2.3.17. Thus by Proposition 2.3.14, 2.3.15, and 2.3.17, we obtain

$$W(x) = (-1)^{k-1}A^\ell_{2(1-k)}(x) * (-1)^k L^\ell_{-2k}(x) * u(x)$$

as a solution of the equation $\Box^k_{c_1}\Box^k_{c_2}W(x) = U(x)$. If we put $p = 1$, then $S^H_{2k}(x)$ and $T^H_{2k}(x)$ reduce to $M^H_{2k}(x)$ and $N^H_{2k}(x)$ defined by (2.36) and (2.37), respectively. Moreover, if we put $p = k = 1$ then the operators $\Box^k_{c_1}$ and $\Box^k_{c_2}$ reduces to $\frac{1}{c_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$ and $\frac{1}{c_2^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$ respectively and then the solution $W(x)$ reduces to $W(x) = M^H_2(x) * N^H_2(x) * U(x)$ which is the solution of the inhomogeneous elastic wave equation $(\frac{1}{c_1^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2})(\frac{1}{c_2^2}\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2})W(x) = U(x)$ where c_1 and c_2 are positive constants and $c_1 \neq c_2$. This completes the proof. □