CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some notations and definitions and theorems will be used in our research.

2.1 Distributions

2.1.1 The Space $\mathcal{D}$ of Testing Functions

Before we can describe distributions, we must define the *testing functions*, on which distributions operate.Throughout this and the next section, the independent real variable t will be assumed to be one-dimensional. When a function has continuous derivatives of all orders on some set of points, we shall say that the function is *infinitely smooth on that set.* If this is true for all points, we shall merely say that the function is *infinitely smooth.* Moreover, whenever we refer to a *complex number* or a *complex-valued function*, it is understood that the number may be real or the function may be real-valued.

The space of testing functions, which is denoted by $\mathcal{D}$, consists of all complex-valued function $\phi(t)$ that are infinitely smooth with compact support, where the support of continuous function $\phi(t)$ is now defined as $U = \{t \in \mathbb{R} : \phi(t) \neq 0\}$. Then U is an open set. The support of ϕ denote by *supp* $\phi(t)$ and define *supp* $\phi = \overline{U}$ (the closure of U).

An example of a testing function in $\mathcal{D}$ is

$$\zeta(t) = \begin{cases} 0 & |\,t| \geq 1 \\ exp\,\frac{1}{t^2-1} & |\,t| < 1\ . \end{cases}$$

It can be shown that every derivative of this function exists and is zero at $t = \pm 1$. More generally, then, this function has continuous derivatives of all orders for every t, and they are all equal to zero for $|\,t| \geq 1$ and *supp* $\zeta(t) = [-1, 1]$.

2.1.2 Distributions

A *functional* is a rule that assigns a number to every member of a certain set of functions. For our purposes, the set of functions will be taken to be the space $\mathcal{D}$ and we shall consider functionals that assign a complex number to every member of $\mathcal{D}$. Denoting a functional by the symbol f, we designate the number that f assigns to a particular testing function ϕ by $\langle f, \phi \rangle$.

Distributions, which we shall describe in this section, are functionals on the space $\mathcal{D}$ that possess, in addition, two essential properties. The first of these is *linearity*. A functional f on $\mathcal{D}$ is said to be linear if, for any two testing functions ϕ_1 and ϕ_2 in $\mathcal{D}$ and any complex number α, the following conditions are satisfied:

$$\begin{aligned} \langle f, \phi_1 + \phi_2 \rangle &= \langle f, \phi_1 \rangle + \langle f, \phi_2 \rangle \\ \langle f, \alpha\phi_1 \rangle &= \alpha \langle f, \phi_1 \rangle \, . \end{aligned} \tag{2.1}$$

The second property is *continuity*. A functional f on $\mathcal{D}$ is said to be continuous if, for any sequence of testing functions $\{\phi_\nu(t)\}_{\nu=1}^{\infty}$ that converges in $\mathcal{D}$ to $\phi(t)$, the sequence of numbers $\{\langle f, \phi_\nu \rangle\}_{\nu=1}^{\infty}$ converges to the number $\langle f, \phi \rangle$ in the ordinary sense. If f is known to be linear, the definition of continuity may be somewhat simplified. In this case, f will be continuous if the numerical sequence $\{\langle f, \phi_\nu \rangle\}_{\nu=1}^{\infty}$ converges to zero whenever the sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{D}$ to zero.

Thus, we may state the following definition of a distribution defined over the one-dimensional real euclidean space $\mathbb{R}^1$:

A continuous linear functional on the space $\mathcal{D}$ is a distribution.

The space of all such distributions is denoted by $\mathcal{D}'$ and $\mathcal{D}'$ is called the dual space of $\mathcal{D}$.

We can generate distributions by the regular function as follows. Let $f(t)$ be a locally integrable function (i.e., a function that is integrable in the Lebesgue sense over every finite interval). Corresponding to such $f(t)$, we can define a distribution f through the convergent integral

$$\langle f, \phi \rangle \;=\; \langle f(t), \phi(t) \rangle \;\triangleq\; \int_{-\infty}^{\infty} f(t)\phi(t)dt \tag{2.2}$$

where ϕ is any testing function with compact support.

2.1.3 Example of Distribution

An example of a distribution that is not a regular distribution is the so-called *Dirac delta function* δ, which is defined by the equation

$$\langle \delta, \phi \rangle \triangleq \phi(0)\ . \qquad (2.3)$$

Clearly, (2.3) is a continuous linear functional on $\mathcal{D}$. However, this distribution cannot be obtained from a locally integrable function through the use of (2.2). Indeed, if there were such a function $\delta(t)$, then we would have

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)dt \ = \ \phi(0) \qquad (2.4)$$

for all $\phi(t)$ in $\mathcal{D}$. Moreover, we can conjecture a new singular distribution, the first derivative $\delta^{(1)}(t)$ of the delta functional, the following definition suggests itself:

$$\langle \delta^{(1)}(t), \phi(t) \rangle \ \triangleq \ -\phi^{(1)}(0).$$

Next, an example of a distribution is so-called *Heaviside unit step function* $H(t)$:

$$H(t) = \begin{cases} 0 & \text{for } \ t < 0 \\ \frac{1}{2} & \text{for } \ t \ = \ 0 \\ 1 & \text{for } \ t > 0. \end{cases}$$

Then for any continuous function ϕ with compact support we have the result

$$\int_{-\infty}^{\infty} H(t)\phi(t)dt \ = \ \int_{0}^{\infty} \phi(t)dt. \qquad (2.5)$$

We use the symbol H to re present the mapping

$$H : \phi \to \int_{0}^{\infty} \phi(t)dt$$

which is well defined by (2.5) for all continuous testing functions of compact support. This means that H represents something other than an ordinary function.

2.1.4 Multiplication of Distribution by Infinitely Smooth Function

An operation that would be useful in analyses involving distributions would be the multiplication of two arbitrary distributions. Unfortunately, it is not possible to define such an operation in general. It turns out that the product does not always exist within the system of distributions. As an example, for the one-dimensional variable t, let $f(t) = 1/\sqrt{|t|}$. Then, $f(t)$ represents a regular distribution as well as a locally integrable function. Now, $[f(t)]^2$ is a function of t defined for all nonzero t. But it is not integrable over any interval that includes the origin. This means that it cannot define a distribution through the expression

$$\left\langle \frac{1}{|t|}, \phi \right\rangle = \int_{-\infty}^{\infty} \frac{\phi(t)}{|t|}\, dt$$

since the integral does not converge for every ϕ in $\mathcal{D}$. In short, the product of $1/\sqrt{|t|}$ with itself does not exist as a distribution.

It is, however, possible to define the product of distributions in special cases. For instance, if f and g are locally integrable functions over $\mathbb{R}^n$ and if their product fg is also locally integrable, then the product of the corresponding regular distributions exists as a regular distribution defined by

$$\langle fg, \phi \rangle = \int_{\mathbb{R}^n} f(t)g(t)\phi(t)\, dt \qquad \phi \in \mathcal{D}.$$

A more important case arises when one of the distributions ψ is a regular distribution corresponding to an infinitely smooth function. The product of ψ with any distribution f in $\mathcal{D}'$ exists and is defined by

$$\langle \psi f, \phi \rangle \triangleq \langle f, \psi\phi \rangle \qquad \phi \in \mathcal{D}. \tag{2.6}$$

For every ϕ in $\mathcal{D}$ the function $\psi\phi$ is infinitely smooth everywhere and zero whenever ϕ is zero. Hence, $\psi\phi$ is also in $\mathcal{D}$. Thus (2.6) defines that functional on $\mathcal{D}$ which assigns to each ϕ in $\mathcal{D}$ the number $\langle f, \psi\phi \rangle$.

2.1.5 The Derivative of Distribution

Distributions, on the other hand, always possess derivatives, and these derivatives are again distributions. In order to explain this statement, we must, of course, define what we mean by the *derivative of a distribution.* Let us restrict ourselves for the moment to the case when the independent variable t has only one dimension. An appropriate definition can be constructed by considering a regular distribution $f(t)$ generated by a function which is differentiable everywhere and whose derivative is continuous. The derivative again generates a regular distribution $f^{(1)}(t)$ and, for each ϕ in $\mathcal{D}$, an integration by part yields

$$\begin{aligned} \langle f^{(1)}, \phi \rangle &= \int_{-\infty}^{\infty} f^{(1)}(t)\phi(t)\,dt \\ &= -\int_{-\infty}^{\infty} f(t)\phi^{(1)}(t)\,dt \;=\; \langle f, -\phi^{(1)} \rangle. \end{aligned} \tag{2.7}$$

Note that $\phi^{(1)}$ is in $\mathcal{D}$ whenever ϕ is in $\mathcal{D}$. Thus, a knowledge of f (and, therefore, of $\langle f, -\phi^{(1)} \rangle$) determines $\langle f^{(1)}, \phi \rangle$. In other words, (2.7) defines $f^{(1)}$ as a functional on $\mathcal{D}$. This result is generalized in the following definition.

The first derivative $f^{(1)}(t)$ of any distribution $f(t)$, where t is one-dimensional, is the functional on $\mathcal{D}$ given by

$$\langle f^{(1)}(t), \phi(t) \rangle \;=\; \langle f(t), -\phi^{(1)}(t) \rangle \qquad \phi \in \mathcal{D}.$$

At times, the conventional notation df/dt will also be used for the derivative of a distribution defined over $\mathbb{R}^1$.

A simple illustration is provided by the first derivative of the delta functional $\delta^{(1)}$, which is defined by the equation

$$\langle \delta^{(1)}, \phi \rangle \;=\; \langle \delta, -\phi^{(1)} \rangle \;=\; -\phi^{(1)}(0)$$

and in general the pth derivative, $\delta^{(p)}$, of the delta distribution is given by the mapping

$$\phi \to \langle \delta^{(p)}, \phi \rangle \;=\; (-1)^p \phi^{(p)}(0).$$

Example 2.1.1 The unit step function $H(t)$ is the function that equals zero for $t < 0$, $1/2$ for $t = 0$, and 1 for $t > 0$. Its first distributional derivative is $\delta(t)$. For, with ϕ in $\mathcal{D}$,

$$\begin{aligned}\langle H^{(1)}(t), \phi(t)\rangle &= \langle H(t), -\phi^{(1)}(t)\rangle \\ &= -\int_0^\infty \phi^{(1)}(t)\,dt \\ &= \phi(0) = \langle \delta(t), \phi(t)\rangle.\end{aligned}$$

On the other hand, the ordinary derivative of $H(t)$ is the function that is zero everywhere except at the origin, where it dose not exist.

The rule for the differentiation of the product of a distribution f and a function ψ, which is infinitely smooth, is the same as that for the product of two differentiable functions:

$$\frac{\partial}{\partial t_i}(\psi f) = \psi\frac{\partial f}{\partial t_i} + f\frac{\partial \psi}{\partial t_i}. \tag{2.8}$$

This is established as follows. For any ϕ in $\mathcal{D}$,

$$\begin{aligned}\left\langle \frac{\partial}{\partial t_i}(\psi f), \phi\right\rangle &= \left\langle \psi f, -\frac{\partial \phi}{\partial t_i}\right\rangle = \left\langle f, -\psi\frac{\partial \phi}{\partial t_i}\right\rangle \\ &= \left\langle f, -\frac{\partial(\psi\phi)}{\partial t_i}\right\rangle + \left\langle f, \phi\frac{\partial(\psi)}{\partial t_i}\right\rangle \\ &= \left\langle \frac{\partial f}{\partial t_i}, \psi\phi\right\rangle + \left\langle f, \phi\frac{\partial \psi}{\partial t_i}\right\rangle \\ &= \left\langle \psi\frac{\partial f}{\partial t_i}, \phi\right\rangle + \left\langle f\frac{\partial \psi}{\partial t_i}, \phi\right\rangle.\end{aligned}$$

Let the partial differential operator D^k, when acting on a distribution, is sufficiently specified by writing

$$D^k = \prod_{i=1}^{n}\left(\frac{\partial}{\partial t_i}\right)^{k_i}.$$

Two important properties of the differentiation of distributions are given by

Theorem 2.1.2 *Differentiation is a continuous linear operation in the space $\mathcal{D}'$ in the following sense:*

Linearity. For any two distributions f and g and for any number α,

$$D^k(f+g) = D^k f + D^k g$$

and

$$D^k(\alpha f) = \alpha D^k f.$$

Continuity. For any sequence of distributions $\{f_\nu\}_{\nu=1}^\infty$ *that converges in* $\mathcal{D}'$ *to a distribution* f*, the corresponding sequence of partial derivatives* $\{D^k f_\nu\}_{\nu=1}^\infty$ *also converges in* $\mathcal{D}'$ *to* $D^k f$

See [6] for more details.

Example 2.1.3 Let $f(t)$ be a bounded function on $\mathbb{R}^1$ that is piecewise-continuous and has a piecewise-continuous first derivative in the following way. The points t_ν ($\nu = \ldots, -2, -1, 0, 1, 2, \ldots; t_\nu < t_{\nu+1}$), at which $f(t)$ or $f^{(1)}(t)$ is discontinuous or $f^{(1)}(t)$ fails to exist, are finite in number in every finite interval. At each such point $f(t)$ has at most a finite jump

$$\triangle f \triangleq f(t_\nu+) - f(t_\nu-) \tag{2.9}$$

and its right-hand and left-hand derivatives both exist. Then we may define the continuous function $f_c(t)$ through

$$f(t) \triangleq f_c(t) - \sum_{\nu=-1}^{-\infty} \triangle f_\nu H(t_\nu - t) + \sum_{\nu=0}^{\infty} \triangle f_\nu H(t - t_\nu). \tag{2.10}$$

The infinite series certainly converges in $\mathcal{D}'$, since in every finite interval it possesses only a finite number of nonzero terms. By differentiating term by term and invoking the result developed in Example (2.1.1), we obtain

$$f^{(1)}(t) = f_c^{(1)}(t) + \sum_{\nu=-\infty}^{\infty} \triangle f_\nu \delta(t - t_\nu) \tag{2.11}$$

where $f_c^{(1)}(t)$ is a locally integrable function. Here again, distributional differentiation generates a delta functional at each ordinary discontinuity.

2.1.6 The Convolution Applied to Ordinary Linear Differential Equations with Constant Coefficients

Let $f(t)$ and $g(t)$ be two continuous functions with bounded support. Their convolution produces a third function $h(t)$, which is denoted by $f * g$ and defined by

$$h(t) \triangleq f(t) * g(t) \triangleq \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau.$$

Thus the rule that defines the convolution $f * g$ of two distributions $f(t)$ and $g(t)$ is suggested by this expression to be

$$\langle f * g, \phi \rangle \triangleq \langle f(t), \langle g(\tau), \phi(t+\tau) \rangle \rangle.$$

Example 2.1.4 The convolution of the delta functional with any distribution yields that distribution again; the convolution of the mth derivative of the delta functional with any distribution yields the mth derivative of that distribution. In symbols,

$$\begin{aligned} \delta * f &= f \\ \delta^{(m)} * f &= f^{(m)}. \end{aligned} \tag{2.12}$$

Note that these convolutions are valid for every distribution f in $\mathcal{D}'$ because $\delta^{(m)}$ has a bounded support. The more general expression (2.12) may be justified as follows. For every ϕ in $\mathcal{D}$,

$$\begin{aligned} \langle \delta^{(m)} * f, \phi \rangle &= \langle f * \delta^{(m)}, \phi \rangle = \langle f(t), \langle \delta^{(m)}(\tau), \phi(t+\tau) \rangle \rangle \\ &= \langle f(t), (-1)^m \phi^{(m)}(t) \rangle = \langle f^{(m)}(t), \phi(t) \rangle. \end{aligned}$$

An important consequence of (2.12) is that every linear differential operator with constant coefficients can be represented as a convolution. That is, with the a_ν being constants, we have

$$\begin{aligned} & a_n f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f \\ & \qquad = (a_n \delta^{(n)} + a_{n-1}\delta^{(n-1)} + \ldots + a_0 \delta) * f. \end{aligned}$$

Note that this statement could not be made if we restricted ourselves to the ordinary convolution of functions.

Let L denote the general differential operator of the form

$$L \triangleq a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1 \frac{d}{dt} + a_0$$

where the a_ν ($\nu = 1, 2, \ldots, n$) are constants, $a_n \neq 0$, and $n \geq 1$. We wish to resolve the equation

$$Lu = g \tag{2.13}$$

where g is a known distribution in $\mathcal{D}'_R$ and u is unknown but also required to be in $\mathcal{D}'_R$ and (2.13) may be written as a convolution equation:

$$(L\delta) * u = g$$
$$L\delta = a_n\delta^{(n)} + a_{n-1}\delta^{(n-1)} + \ldots + a_1\delta^{(1)} + a_0\delta. \tag{2.14}$$

Thus, the technique developed for the convolution algebra $\mathcal{D}'_R$ may be applied here and, as we have shown, the problem becomes simply that of finding in $\mathcal{D}'_R$ an inverse for $L\delta$.

Theorem 2.1.5 *The distribution $L\delta$, given by (2.14) with the a_ν ($\nu = 0, 1, \ldots, n; n \geq 1$) being constants and $a_n \neq 0$, has an inverse in $\mathcal{D}'_R$. This inverse is $H(t)y(t)$, where $y(t)$ is that classical solution of the homogeneous equation $Lu = 0$ which satisfies the initial conditions*

$$y(0) = y^{(1)}(0) = \ldots = y^{(n-2)} = 0 \qquad \text{and} \quad y^{(n-1)}(0) = \frac{1}{a_n}.$$

See [6] for more details.

2.2 Periodic Distribution

2.2.1 The Space $\mathcal{P}_T$ of Periodic Testing Functions

An ordinary function $f(t)$ is said to be *periodic* if there exists some real positive number T such that $f(t) = f(t - T)$ for all values of t and T is called a *period* of $f(t)$. It follows that, if a function is periodic, it will possess an infinity of periods, since nT ($n = 1, 2, 3, \ldots$) will be a period whenever T is a period. Note that by our definition all periods are positive numbers.

A function $\theta(t)$ will be called a *periodic testing function* if it is periodic and infinitely smooth. The space of all such periodic testing functions having the *common* period T (T being a *fixed* positive number) will be denoted by $\mathcal{P}_T$ and $\mathcal{P}_T$ is a linear space.

Any testing function ϕ in $\mathcal{D}$ generates a unique testing function θ in $\mathcal{P}_T$ through the expression

$$\theta(t) = \sum_{n=-\infty}^{\infty} \phi(t - nT). \tag{2.15}$$

Actually, over any bounded t interval there are only a finite number of nonzero terms in this summation because ϕ has a bounded support. Thus, we may differentiate term by term to get

$$\theta^{(k)}(t) = \sum_{n=-\infty}^{\infty} \phi^{(k)}(t - nT) \qquad k = 1, 2, 3, \ldots. \tag{2.16}$$

Another type of function that we shall make use of is the so-called *unitary function*. A function $\xi(t)$ is said to be unitary if it is an element of $\mathcal{D}$ and if there exists a real number T for which

$$\sum_{n=-\infty}^{\infty} \xi(t - nT) \;=\; 1 \tag{2.17}$$

for all t. The space of all functions that are unitary with respect to some fixed real number T will be denoted by u_T.

Clearly, if θ is in $\mathcal{P}_T$ and ξ is in u_T, then $\xi\theta$ is in $\mathcal{D}$. Each θ can be related to $\xi\theta$ according to (2.15) because the periodicity of θ may be employed to write

$$\sum_{n=-\infty}^{\infty} \xi(t - nT)\theta(t - nT) = \theta(t) \sum_{n=-\infty}^{\infty} \xi(t - nT) = \theta(t). \tag{2.18}$$

This shows that *every* θ in $\mathcal{P}_T$ can be generated through (2.15) from some ϕ in $\mathcal{D}$.

2.2.2 The Space $\mathcal{P}'_T$ of Periodic Distribution

A periodic distribution is defined in the same way as is a periodic function. In particular, the distribution f is said to be *periodic* if there exists a real positive number T such that

$$f(t) = f(t - T)$$

for all t. This means, of course, that for every ϕ in $\mathcal{D}$

$$\langle f(t), \phi(t)\rangle = \langle f(t-T), \phi(t)\rangle \tag{2.19}$$

T is called a *period* of $f(t)$. As with ordinary functions, a distribution will have an infinity of periodic nT ($n = 1, 2, 3, \dots$) so long as it has at least one period T. Obviously, every constant distribution is a periodic distribution and each positive number is one of its periods.

Now, let T be a given (fixed) positive number. The space of all periodic distributions possessing T as one of its periods will be denoted by $\mathcal{P}'_T$.

Every element f of $\mathcal{P}'_T$ can also be identified as a continuous linear functional on the space $\mathcal{P}_T$. The (complex) number that f assigns to any θ in $\mathcal{P}_T$ will be denoted by the *dot product* $f \odot \theta$ in order to avoid confusion with the number $\langle f, \phi\rangle$ that f assigns to any ϕ in $\mathcal{D}$. This number $f \odot \theta$ is defined by

$$f \odot \theta \;\triangleq\; \langle f, \xi\theta\rangle \tag{2.20}$$

where ξ is any unitary function in u_T.

As usual, f is said to be a *linear* functional on the space $\mathcal{P}_T$ if, for any θ_1 and θ_2 in $\mathcal{P}_T$ and for any two complex numbers α and β, we have

$$f \odot (\alpha\theta_1 + \beta\theta_2) \;=\; \alpha(f \odot \theta_1) + \beta(f \odot \theta_2)$$

Similarly, f is said to be a *continuous* functional on $\mathcal{P}_T$ if, for any sequence $\{\theta_\nu\}_{\nu=1}^{\infty}$ that converges in $\mathcal{P}_T$ to θ, the sequence of number $\{f \odot \theta_\nu\}_{\nu=1}^{\infty}$ converges to the number $f \odot \theta$.

Theorem 2.2.1 *If f is a periodic distribution with period T, then (2.20) defines it as a continuous linear functional on $\mathcal{P}_T$.*

See [6] for more details.

Any ϕ in $\mathcal{D}$ will generate a θ in $\mathcal{P}_T$ through the expression (2.15).Then, by knowing f as a functional on $\mathcal{P}_T$, we define the number $\langle f, \phi\rangle$ by

$$\langle f, \phi\rangle \;\triangleq\; f \odot \theta. \tag{2.21}$$

Theorem 2.2.2 *If f is a continuous linear functional on $\mathcal{P}_T$ and if θ and ϕ are related by (2.15), then (2.21) defines f as a periodic distribution with period T .*

See [6] for more details.

Example 2.2.3 Let

$$\delta_T(t) \triangleq \sum_{n=-\infty}^{\infty} \delta(t-nT) \qquad T>0.$$

Clearly, δ_T is in $\mathcal{P}'_T$. For θ in $\mathcal{P}_T$ and ξ in u_T, we have

$$\delta_T \odot \theta = \langle \delta_T, \xi\theta\rangle = \langle \sum_n \delta(t-nT), \xi(t)\theta(t)\rangle$$

$$= \sum_n \xi(nT)\theta(nT) = \theta(0)\sum_n \xi(nT).$$

Since $\sum_n \xi(nT) = 1$, we have established that

$$\delta_T(t) \odot \theta(t) = \theta(0).$$

A similar analysis shows that

$$\delta_T(t-\tau) \odot \theta(t) = \theta(\tau).$$

Example 2.2.4 More generally, consider

$$\delta_T^{(k)}(t) = \sum_{n=-\infty}^{\infty} \delta^{(k)}(t-nT) \qquad k = 0,1,2,3\,\ldots.$$

This is also in $\mathcal{P}'_T$ and, as before, we have

$$\delta_T^{(k)} \odot \theta = \sum_{n=-\infty}^{\infty} (-1)^k \frac{d^k}{dt^k}[\xi(t)\theta(t)]\Big|_{t\,=\,nT}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^k \sum_{p=0}^{k} \binom{k}{p} \xi^{(p)}(nT)\,\theta^{(k-p)}(nT)$$

$$= (-1)^k \sum_{p=0}^{k} \binom{k}{p} \theta^{(k-p)}(0) \sum_{n=-\infty}^{\infty} \xi^{(p)}(nT).$$

In view of (2.20) and (2.15) of the preceding section, this equation yields

$$\delta_T^{(k)} \odot \theta(t) = (-1)^k \theta^{(k)}(0) \qquad k = 0,1,2,3\,\ldots$$

2.2.3 T-Convolution

Let f and g be arbitrary distribution in $\mathcal{P}'_T$, ξ and κ arbitrary unitary functions in u_T, and θ any testing function in $\mathcal{P}_T$. Then, the T-convolution of f with g, which we shall denote by $f\Delta\, g$, is the functional on $\mathcal{P}_T$ defined by

$$\begin{aligned}(f\Delta\, g)\odot\theta &\triangleq f(t)\odot[g(\tau)\odot\theta(t+\tau)] \\ &= f(t)\odot\langle g(\tau),\kappa(\tau)\theta(t+\tau)\rangle. \end{aligned} \tag{2.22}$$

As a consequence of Theorem 2.7 − 2[6],

$$\langle g(\tau),\kappa(\tau)\theta(t+\tau)\rangle$$

is a function of t that is infinitely smooth. Also, it is clearly periodic with period T. Hence, it is in $\mathcal{P}_T$. Thus, definition (2.22) may be replaced by the equivalent definition

$$\begin{aligned}(f\Delta\, g)\odot\theta &\triangleq \langle f(t),\langle g(\tau),\xi(t)\kappa(\tau)\theta(t+\tau)\rangle\rangle \\ &= \langle f(t)\otimes g(\tau),\xi(t)\kappa(\tau)\theta(t+\tau)\rangle. \end{aligned} \tag{2.23}$$

Theorem 2.2.5 *The T-convolution of two distributions in $\mathcal{P}'_T$ yields a distribution in $\mathcal{P}'_T$. In other words, the space $\mathcal{P}'_T$ is closed under T-convolution.*

See [6] for more details.

Moreover, we have shown that in this special case

$$f\Delta\, g=\int_a^{a+T} f(\tau)(t-\tau)d\tau$$

the right-hand side again being a locally integrable periodic function of period T. This is just like an ordinary convolution except that now the integration is over a *finite* interval of length T. This is the reason why T-convolution is also called *finite convolution.*

Example 2.2.6 Consider

$$\delta_T(t) \;=\; \sum_{n=-\infty}^{\infty}\delta(t-nT)$$

the distribution in $\mathcal{P}'_T$ that was discussed in Example (2.2.3). If f is also in $\mathcal{P}'_T$ and θ is any element in $\mathcal{P}_T$, then

$$(f\Delta\,\delta_T)\odot\theta = f(t)\odot[\delta_T(\tau)\odot\theta(t+\tau)] = f(t)\odot\theta(t)$$

and therefore

$$f\Delta\,\delta_T = f.$$

Thus, δ_T is the unit element in the T-convolution algebra, which we shall describe later on. In accordance with the Theorem $11.4-3$[6] , we may also write

$$f^{(1)} \;=\; \frac{d}{dt}(f\Delta\,\delta_T) \;=\; f\Delta\,\delta_T^{(1)}$$

and, more generally,

$$f^{(k)} \;=\; f\Delta\,\delta_T^{(k)} \qquad k \;=\; 0,\,1,\,2,\,\ldots. \tag{2.24}$$

T-convolution is clearly a linear operation in the sense that, if f, g, and h are in $\mathcal{P}'_T$ and if α and β are real numbers, then

$$f\Delta\,(\alpha g+\beta h) \;=\; \alpha(f\Delta\,g)+\beta(f\Delta\,h)$$

By virtue of this and (2.24) any linear differential expression with constant coefficients can be represented as a T-convolution so long as we restrict ourselves to the distributions in $\mathcal{P}'_T$:

$$a_nf^{(n)} \;+\; a_{n-1}f^{(n-1)} \;+\; \ldots \;+\; a_0f$$
$$= (a_n\delta_T^{(n)} \;+\; a_{n-1}\delta_T^{(n-1)} \;+\; \ldots \;+\; a_0\delta_T)\Delta\,f, \qquad f\in\mathcal{P}'_T.$$

2.2.4 The T-Convolution Algebra

A *T-convolution equation* is an equation of the form

$$f\Delta\,u \;=\; g \tag{2.25}$$

where f and g are given elements of $\mathcal{P}'_T$ and u is an unknown element that is also required to be in $\mathcal{P}'_T$. The technique that was described in Sec.6.2[6] for solving a convolution equation in some convolution algebra can be applied here to solve (2.25). As before, the *inverse* of an element f is an element $f^{\Delta-1}$ of $\mathcal{P}'_T$ such that

$$f^{\Delta-1}\Delta f = \delta_T. \tag{2.26}$$

Theorem 2.2.7 *Let f be a given distribution in $\mathcal{P}'_T$. A necessary and sufficient condition for (2.25) to have at least one solution in $\mathcal{P}'_T$ for every g in $\mathcal{P}'_T$ is that f possess an inverse $f^{\Delta-1}$ in $\mathcal{P}'_T$. When f dose have an inverse in $\mathcal{P}'_T$, this inverse is unique and (2.25) possesses a unique solution in $\mathcal{P}'_T$ given by*

$$u = f^{\Delta-1}\Delta g \tag{2.27}$$

.

See [6] for more details.

2.2.5 The Finite Fourier Transformation

Our purpose in this section is to show that a distribution f is periodic and of period T if and only if it has the series expansion

$$f(t) = \sum_{\nu=-\infty}^{\infty} F_\nu e^{i\nu\omega t} \qquad \omega = \frac{2\pi}{T} \tag{2.28}$$

where the constant coefficients F_ν are given by

$$F_\nu = \frac{1}{T} f(t) \odot e^{-i\nu\omega t} \tag{2.29}$$

and have the property of being of *slow growth*. This latter property is defined as follows. A sequence of constant $\{F_\nu\}_{\nu=-\infty}^{\infty}$ is said to be of slow growth if there exist a constant M and an integer k such that $|F_\nu| \leq M|\nu|^k$ for all nonzero ν. The series (2.28) is called the *Fourier series* of $f(t)$, and the constants F_ν are called the *Fourier coefficients* of $f(t)$. On the other hand, one can apply the Fourier-coefficients formula (2.29) to any distribution in $\mathcal{P}'_T$ to convert it into a sequence of complex numbers. This defines the *direct finite Fourier transformation* $\mathcal{F}_T$, which is, therefore, a one-to-one mapping of the space $\mathcal{P}'_T$ onto the space of sequence of numbers of slow growth. The *inverse finite Fourier transformation* $\mathcal{F}_T^{-1}$

is defined in turn by the Fourier series representation (2.28). The symbolism for these transformations is

$$\mathcal{F}_T f \;=\; \{F_\nu\}_{\nu=-\infty}^{\infty}$$

and

$$\mathcal{F}_T^{-1}\{F_\nu\}_{\nu=-\infty}^{\infty} \;=\; f$$

where f is an element of $\mathcal{P}'_T$ and $\{F_\nu\}_{\nu=-\infty}^{\infty}$ is the corresponding sequence of Fourier coefficients.

Theorem 2.2.8 *If f and g are in $\mathcal{P}'_T$ and if F_ν and G_ν $(\nu = 0, \pm 1, \pm 2 \ldots)$ are their respective Fourier coefficients, then the Fourier coefficients of $f \Delta g$ are $TF_\nu G_\nu$.*

See [6] for more details.

An important application of the finite Fourier transformation is in the resolution of the T-convolution equation

$$f \Delta u \;=\; g$$

where f and g are given elements of $\mathcal{P}'_T$ and the solution u is also required to be in $\mathcal{P}'_T$. This problem can be solved quite simply if f has a T-convolution inverse $f^{\Delta-1}$, which is by definition an element of $\mathcal{P}'_T$ that satisfies the equation (2.26) ,

$$f \Delta f^{\Delta-1} \;=\; \delta_T.$$

By applying the direct finite Fourier transformation to (2.26), we obtain the following infinite set of equations, wherein the X_ν denote the Fourier cofficients of $f^{\Delta-1}$:

$$TF_\nu X_\nu = \frac{1}{T} \qquad \nu = 0, \pm 1, \pm 2, \ldots. \tag{2.30}$$

All these equations can be satisfied only if every F_ν is different from zero. In this case, we can divide both sides of (2.30) by TF_ν and then apply the inverse finite Fourier transformation to obtain

$$f^{\Delta-1} \;=\; \mathcal{F}_T^{-1}\{X_\nu\}_{\nu=-\infty}^{\infty} \;=\; \frac{1}{T^2}\, \mathcal{F}_T^{-1}\Big\{\frac{1}{F_\nu}\Big\}_{\nu=-\infty}^{\infty}$$

$$= \frac{1}{T^2} \sum_{\nu=-\infty}^{\infty} \frac{1}{F_\nu}\, e^{i\nu\omega t}. \tag{2.31}$$

However, the right-hand side will have a sense if $\{1/F_\nu\}_{\nu=-\infty}^{\infty}$ is a sequence of slow growth.

Under these conditions on the F_ν, we can employ Theorems 2.2.7 and 2.2.8 to obtain a solution to (2.25) for any g in $\mathcal{P}'_T$ by using the finite Fourier transformation. Indeed, with $\{G_\nu\}_{\nu=-\infty}^{\infty} = \mathcal{F}_T g$, the solution to (2.25) is

$$u(t) = \mathcal{F}_T^{-1}\left\{\frac{G_\nu}{TF_\nu}\right\}_{\nu=-\infty}^{\infty} = \frac{1}{T}\sum_{\nu=-\infty}^{\infty}\frac{G_\nu}{F_\nu}\,e^{i\nu\omega t} \tag{2.32}$$

Thus, we have arrived at

Theorem 2.2.9 *The T-convolution equation (2.25) has a unique solution in $\mathcal{P}'_T$ for every g in $\mathcal{P}'_T$ if none of the Fourier coefficients F_ν of f are zero and if $\{1/F_\nu\}_{\nu=-\infty}^{\infty}$ is a sequence of slow growth. In this case, the solution is given by (2.32).*

See [6] for more details.

On the other hand, if some of the F_ν are zero, then $f^{\Delta-1}$ dose not exist, since there is no set of X_ν for which all the equations (2.30) will be satisfied. This means that (2.25) will not have a solution for every g in $\mathcal{P}'_T$. However, for certain g, (2.25) will have a solution and, in fact, an infinite number of solutions.

2.2.6 The T-Convolution Applied to Ordinary Linear Differential Equations with Constant Coefficients

In this section, the theory of periodic distributions will be applied to solve in $\mathcal{P}'_T$ ordinary linear differential equations with constant coefficients. Consider the differential equation

$$a_n u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_0 u = g \qquad a_n \neq 0\,, n > 0 \tag{2.33}$$

or, equivalently,

$$(a_n\delta_T^{(n)} + a_{n-1}\delta_T^{(n-1)} + \ldots + a_0\delta_T)\,\Delta\, u = g$$

where the a_k ($k = 0, 1, \ldots, n$) are constants, g is a given element of $\mathcal{P}'_T$, and u is an unknown element in $\mathcal{P}'_T$ that we seek. The finite Fourier transformation converts (2.33) into

$$[a_n(i\nu\omega)^n + a_{n-1}(i\nu\omega)^{n-1} + \ldots + a_0]\, U_\nu = G_\nu$$
$$\omega = \frac{2\pi}{T}\,;\, \nu = 0, \pm 1, \pm 2, \ldots. \tag{2.34}$$

If it turns out that none of the roots of the polynomial

$$P(\zeta) = a_n\zeta^n + a_{n-1}\zeta^{n-1} + \ldots + a_0 \tag{2.35}$$

coincide with any of the values $i\nu\omega = i\nu 2\pi/T$ ($\nu = 0, \pm 1, \pm 2, \ldots$), then each U_ν is uniquely determined by (2.34). In this case $\{1/P(i\nu\omega)\}_{\nu=-\infty}^{\infty}$ is clearly a sequence of slow growth. Thus, according to Theorem 2.2.9 the unique solution to (2.33) is given by the following Fourier series :

$$u(t) = \sum_{\nu=-\infty}^{\infty} \frac{G_\nu}{P(i\nu\omega)}\, e^{i\nu\omega t}. \tag{2.36}$$

Equation (2.33) can also be solved directly in the T-convolution algebra $\mathcal{P}'_T$ by using the T-convolution inverse of

$$f \triangleq a_n\delta_T^{(n)} + a_{n-1}\delta_T^{(n-1)} + \ldots + a_0\delta_T \tag{2.37}$$

instead of using the finite Fourier transformation. Let us first factor the polynomial (2.35) into the form

$$P(\zeta) = a_n(\zeta - \gamma_1)(\zeta - \gamma_2)\ldots(\zeta - \gamma_n)$$

where some or all of the roots γ_k may be equal to each other. Certainly, f can be written in the form

$$f = a_n(\delta_T^{(1)} - \gamma_1\delta_T)\ \Delta\ (\delta_T^{(1)} - \gamma_2\delta_T)\ \Delta\ \ldots\ \Delta\ (\delta_T^{(1)} - \gamma_n\delta_T).$$

Moreover, still assuming that none of the γ_k equal $i\nu\omega$ ($\nu = 0, \pm 1, \pm 2, \ldots$), one can easily verify that the T-convolution inverse of $\delta_T^{(1)} - \gamma_k\delta_T$ is the periodic function h_k defined by the equations

$$\begin{aligned} h_k(t) &\triangleq \frac{e^{\gamma_k t}}{1 - e^{\gamma_k T}} \qquad 0 \leq t < T \\ h_k(t) &\triangleq h_k(t - T) \qquad -\infty < t < \infty. \end{aligned} \tag{2.38}$$

The T-convolution inverse of (2.37) is

$$f^{\Delta -1} = \frac{1}{a_n} h_1 \Delta h_2 \Delta \ldots \Delta h_n$$

and, consequently, the solution to (2.33) is found to be

$$u = \frac{1}{a_n} h_1 \Delta h_2 \Delta \ldots \Delta h_n \Delta g. \tag{2.39}$$

Let us compare these two technique for solving (2.33). The first method is computationally easy to perform except that the evaluation of the Fourier coefficients G_ν is a possible stumbling block. Moreover, the solution is rendered as a Fourier series (2.36). The second method yields the solution in a closed form (2.39), but it requires the determination of the roots of the polynomial (2.35) and the evaluation of n T-convolutions.

2.3 Electric Circuit Problems

In this section we consider the application of differential equations to series circuits containing (1) an electromotive force, and (2) resistors, inductors, and capacitors. We assume that the reader is some what familiar with these items and so we shall avoid an extensive discussion. Let us simply recall that the electromotive force (for example, a battery or generator) produces a flow of current in a closed circuit and that this current produces a so-called voltage drop across each resistor, inductor, and capacitor. Further, the following three laws concerning the voltage drops across these various elements are known to hold:

1. The voltage drop across a resistor is given by

$$E_R = Ri, \tag{2.40}$$

where R is a constant of proportionality called the *resistance*, and i is the current.

2. The voltage drop across an inductor is given by

$$E_L = L\,\frac{di}{dt}, \tag{2.41}$$

where L is a constant of proportionality called the *inductance*, and i again denotes the current.

3. The voltage drop across a capacitor is given by

$$E_C = \frac{1}{C}\,q, \tag{2.42}$$

where C is a constant of proportionality called the *capacitance* and q is the instantaneous charge on the capacitor. Since $i = dq/dt$, this is often written as

$$E_C = \frac{1}{C}\int i\,dt.$$

The fundamental law in the study of electric circuits is the following:

Kirchhoff's Voltage Law (*Form 1*). The algebraic sum of the instantaneous voltage drops around a close circuit in a specific direction is zero.

Since voltage drops across resistors, inductors, and capacitors have the opposite sign from voltages arising from electromotive forces, we may state this law in the following alternative form:

Kirchhoff's Voltage Law (*Form 2*). The sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit.

Let us apply Kirchhoff's law to the series circuit. Letting E denote the electromotive force, and using the law $1, 2,$ and 3 for voltage drops that were given above, we are led at once to the equation

$$L\,\frac{di}{dt} + R\,i + \frac{1}{C}\,q = E. \tag{2.43}$$

This equation contains *two* dependent variables i and q. However, we recall that these two variables are related to each other by the equation

$$i = \frac{dq}{dt}\,. \tag{2.44}$$

Using this we may eliminate i from Equation (2.43) and write it in the form

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = E. \tag{2.45}$$

Equation (2.45) is a second-order linear differential equation in the single dependent variable q. On the other hand, if we differentiate Equation (2.43) with respect to t and make use of (2.44), we may eliminate q from Equation (2.43) and write

$$L\,\frac{d^2i}{dt^2} + R\,\frac{di}{dt} + \frac{1}{C}\,i = \frac{dE}{dt}\,. \tag{2.46}$$

This is a second-order linear differential equation in the single dependent variable i.

Thus we have the two second-order linear differential equation (2.45) and (2.46) for the charge q and current i, respectively. Further observe that in two very simple cases the problem reduces to a *first*-order linear differential equation. If the circuit contains no capacitor, Equation (2.43) itself reduces directly to

$$L\,\frac{di}{dt} + R\,i = E\,;$$

while if no inductor is present, Equation (2.45) reduces to

$$R\,\frac{dq}{dt} + \frac{1}{C}\,q = E\,.$$