CHAPTER 3

MAIN RESULTS

In this chapter we studied some property of $e^{\alpha t}\delta_T^{(k)}$ and used its to investigate the behavior of charge and current in the electrical circuit.

3.1 Some Properties of $e^{\alpha t}\delta_T^{(k)}$

Property 3.1.1 *$e^{\alpha t}\delta_T^{(k)} = (D-\alpha)^k\delta_T$ where $D \equiv \frac{d}{dt}$ and $e^{\alpha t}\delta_T^{(k)}$ is a periodic distribution of order k with period T.*

Proof. By the definition of periodic distribution and δ_T,

$$\begin{aligned} e^{\alpha t}\delta_T^{(k)} \odot \theta(t) &= \delta_T^{(k)} \odot e^{\alpha t}\theta(t) \\ &= \delta_T \odot (-1)^k \sum_{\nu=0}^{k} \binom{k}{\nu} (e^{\alpha t})^{(\nu)}(\theta(t))^{(k-\nu)} \end{aligned}$$

for every $\theta \in \mathcal{P}_T$ and also $e^{\alpha t}\theta(t) \in \mathcal{P}_T$ where $\mathcal{P}_T$ is the space of periodic function of infinitely differentiable with period T. Hence

$$\begin{aligned} \delta_T^{(k)} \odot e^{\alpha t}\theta(t) &= (-1)^k \sum_{\nu=0}^{k} \binom{k}{\nu} \alpha^\nu \theta^{(k-\nu)}(0) \\ &= \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \alpha^\nu \delta_T^{(k-\nu)} \odot \theta \\ &= (D-\alpha)^k \delta_T \odot \theta, \end{aligned}$$

where $D \equiv \frac{d}{dt}$. It is follow that

$$e^{\alpha t}\delta_T^{(k)} = (D-\alpha)^k\delta_T.$$

Since δ_T is a periodic distribution, hence so is $(D-\alpha)^k\delta_T$ and it follows that $e^{\alpha t}\delta_T^{(k)}$ is a periodic as required. Now

$$\begin{aligned} e^{\alpha t}\delta_T^{(k)} &= (D-\alpha)^k\delta_T \\ &= \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \alpha^\nu \delta_T^{(k-\nu)}, \end{aligned}$$

this means that $e^{\alpha t}\delta_T^{(k)}$ is a finite linear combination of Dirac-delta periodic distribution and its derivative up to order k. Hence, by [6] we obtain that $e^{\alpha t}\delta_T^{(k)}$ is of order k with a point support $\{nT\}_{n=-\infty}^{\infty}$. □

Property 3.1.2 *(The T-convolution of $e^{\alpha t}\delta_T^{(k)}$ with some periodic distributions)*

(1) $\left(e^{\alpha t}\delta_T^{(k)}\right)\Delta f = (D-\alpha)^k f$ *where* $D \equiv \frac{d}{dt}$ *and* f *is some periodic distributions in the space* $\mathcal{P}'_T$ *of periodic distributions.*

(2) $\left[\left(e^{\alpha_1 t}\delta_T^{(k_1)}\right)\Delta\left(e^{\alpha_2 t}\delta_T^{(k_2)}\right)\Delta\ldots\Delta\left(e^{\alpha_n t}\delta_T^{(k_n)}\right)\right]\Delta f = (D-\alpha_1)^{k_1}(D-\alpha_2)^{k_2}\ldots(D-\alpha_n)^{k_n} f$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are complex constants,* $D \equiv \frac{d}{dt}$ *and* $k_1, k_2, \ldots, k_n$ *are positive integers and* $f \in \mathcal{P}'_T$.

Proof. (1) By Property 3.1.1, we have $\left(e^{\alpha t}\delta_T^{(k)}\right)\Delta f = (D-\alpha)^k \delta_T \Delta f$, thus we obtain

$$
\begin{aligned}
\left(e^{\alpha t}\delta_T^{(k)}\right)\Delta f &= \left[\sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu}\alpha^\nu \delta_T^{(k-\nu)}\right]\Delta f \\
&= \sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu}\alpha^\nu \left(\delta_T^{(k-\nu)}\Delta f\right) \\
&= \sum_{\nu=0}^{k}(-1)^\nu \binom{k}{\nu}\alpha^\nu f^{(k-\nu)} \\
&= (D-\alpha)^k f, \qquad \text{since } \delta_T^{(k)}\Delta f = f^{(k)}.
\end{aligned}
$$

(2) Since $e^{\alpha_i t}\delta_T^{(k_i)}$ is a periodic, then we can take the finite Fourier transform to the convolution $\left(e^{\alpha_1 t}\delta_T^{(k_1)}\right)\Delta\left(e^{\alpha_2 t}\delta_T^{(k_2)}\right)\Delta\ldots\Delta\left(e^{\alpha_n t}\delta_T^{(k_n)}\right)$, that is

$$
\begin{aligned}
&\mathcal{F}_T\Big\{\left(e^{\alpha_1 t}\delta_T^{(k_1)}\right)\Delta\left(e^{\alpha_2 t}\delta_T^{(k_2)}\right)\Delta\ldots\Delta\left(e^{\alpha_n t}\delta_T^{(k_n)}\right)\Big\} \\
&= \frac{1}{T}\left(e^{\alpha_1 t}\delta_T^{(k_1)} \odot e^{-i\nu\omega t}\right)\left(e^{\alpha_2 t}\delta_T^{(k_2)} \odot e^{-i\nu\omega t}\right)\cdots\left(e^{\alpha_n t}\delta_T^{(k_n)} \odot e^{-i\nu\omega t}\right) \\
&= \frac{1}{T}(i\nu\omega-\alpha_1)^{k_1}(i\nu\omega-\alpha_2)^{k_2}\ldots(i\nu\omega-\alpha_n)^{k_n}.
\end{aligned}
$$

Take the inverse finite Fourier transform, we obtain

$$\left(e^{\alpha_1 t}\delta_T^{(k_1)}\right)\Delta\left(e^{\alpha_2 t}\delta_T^{(k_2)}\right)\Delta\ldots\Delta\left(e^{\alpha_n t}\delta_T^{(k_n)}\right)$$

$$\begin{aligned}
&= \mathcal{F}_T^{-1}\Big\{\frac{1}{T}(i\nu\omega-\alpha_1)^{k_1}(i\nu\omega-\alpha_2)^{k_2}\ldots(i\nu\omega-\alpha_n)^{k_n}\Big\}\\
&= \sum_{\nu=-\infty}^{\infty}\frac{1}{T}(i\nu\omega-\alpha_1)^{k_1}(i\nu\omega-\alpha_2)^{k_2}\ldots(i\nu\omega-\alpha_n)^{k_n}e^{i\nu\omega t}\\
&= (D-\alpha_n)\sum_{\nu=-\infty}^{\infty}\frac{1}{T}(i\nu\omega-\alpha_1)^{k_1}(i\nu\omega-\alpha_2)^{k_2}\ldots(i\nu\omega-\alpha_n)^{k_n-1}e^{i\nu\omega t}\\
&= (D-\alpha_n)^2\sum_{\nu=-\infty}^{\infty}\frac{1}{T}(i\nu\omega-\alpha_1)^{k_1}(i\nu\omega-\alpha_2)^{k_2}\ldots(i\nu\omega-\alpha_n)^{k_n-2}e^{i\nu\omega t}\\
&\ \ \vdots\\
&= (D-\alpha_n)^{k_n}\sum_{\nu=-\infty}^{\infty}\frac{1}{T}(i\nu\omega-\alpha_1)^{k_1}(i\nu\omega-\alpha_2)^{k_2}\ldots(i\nu\omega-\alpha_{n-1})^{k_{n-1}}e^{i\nu\omega t}\\
&\ \ \vdots\\
&= (D-\alpha_1)^{k_1}(D-\alpha_2)^{k_2}\ldots(D-\alpha_n)^{k_n}\sum_{\nu=-\infty}^{\infty}\frac{1}{T}e^{i\nu\omega t}\\
&= (D-\alpha_1)^{k_1}(D-\alpha_2)^{k_2}\ldots(D-\alpha_n)^{k_n}\delta_T
\end{aligned}$$

where $\frac{1}{T}\sum_{\nu=-\infty}^{\infty}e^{i\nu\omega t}=\delta_T$ [6] and similarly, as Property 3.1.1,

$$\left[\left(e^{\alpha_1 t}\delta_T^{(k_1)}\right)\Delta\left(e^{\alpha_2 t}\delta_T^{(k_2)}\right)\Delta\ldots\Delta\left(e^{\alpha_n t}\delta_T^{(k_n)}\right)\right]\Delta f$$
$$=(D-\alpha_1)^{k_1}(D-\alpha_2)^{k_2}\ldots(D-\alpha_n)^{k_n}f.$$

That completes the proof. □

Corollary 3.1.3 *The inverse of* $(e^{\omega_1 t}\delta_T^{(1)})\Delta(e^{\omega_2 t}\delta_T^{(1)})$ *is*

$$\frac{1}{\omega_1-\omega_2}\left[\frac{e^{\omega_1 t}}{1-e^{\omega_1 T}}-\frac{e^{\omega_2 t}}{1-e^{\omega_2 T}}\right] \tag{3.1}$$

Proof. Let $f(t)=(e^{\omega_1 t}\delta_T^{(1)})\Delta(e^{\omega_2 t}\delta_T^{(1)})$ for $0\le t<T$, we have finite Fourier transform F_ν of $f(t)$ is given by

$$F_\nu = \frac{1}{T}(i\nu\omega-\omega_1)(i\nu\omega-\omega_2).$$

By (2.34) we can write the inverse $f^{\Delta-1}(t)$ of $f(t)$ for $0\le t<T$, is of the form

Fourier series

$$f^{\Delta-1}(t) = \frac{1}{T^2}\sum_{\nu=-\infty}^{\infty}\frac{1}{F_\nu}\,e^{i\nu\omega t}$$
$$= \sum_{\nu=-\infty}^{\infty}\frac{1}{T(\omega_1 - i\nu\omega)(\omega_2 - i\nu\omega)}\,e^{i\nu\omega t}. \tag{3.2}$$

We will show that by taking the finite Fourier transform to (3.1), that is

$$\mathcal{F}_T\Big\{\frac{1}{\omega_1-\omega_2}\Big[\frac{e^{\omega_1 t}}{1-e^{\omega_1 T}} - \frac{e^{\omega_2 t}}{1-e^{\omega_2 T}}\Big]\Big\}$$
$$= \frac{1}{T(\omega_1-\omega_2)}\Big[\frac{1}{1-e^{\omega_1 T}}\,e^{\omega_1 t}\odot e^{-i\nu\omega t} - \frac{1}{1-e^{\omega_2 T}}\,e^{\omega_2 t}\odot e^{-i\nu\omega t}\Big]$$
$$= \frac{1}{T(\omega_1-\omega_2)}\Big[\frac{e^{\omega_1 T - i\nu\omega T}-1}{(1-e^{\omega_1 T})(\omega_1 - i\nu\omega)} - \frac{e^{\omega_2 T - i\nu\omega T}-1}{(1-e^{\omega_2 T})(\omega_2 - i\nu\omega)}\Big]$$
$$= \frac{1}{T(\omega_1-\omega_2)}\Big[\frac{1}{\omega_2 - i\nu\omega} - \frac{1}{\omega_1 - i\nu\omega}\Big]$$
$$= \frac{1}{T(\omega_1 - i\nu\omega)(\omega_2 - i\nu\omega)},$$

since the finite Fourier transform of (3.1) is Fourier coefficient of (3.2).This completes the proof. □

3.2 The Application of $e^{\alpha t}\delta_T^{(k)}$

Recall that the equation

$$L\frac{d^2}{dt^2}Q(t) + R\frac{d}{dt}Q(t) + \frac{1}{C}\,Q(t) = \sum_{k=0}^{m} c_k\delta_T^{(k)}(t). \tag{3.3}$$

Now (3.3) can be written as the form

$$(D^2 + \frac{R}{L}D + \frac{1}{LC})Q(t) = \frac{1}{L}\sum_{k=0}^{m} c_k\delta_T^{(k)}(t),$$

where $D \equiv \frac{d}{dt}$, or

$$\left(D - (-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}})\right)\left(D - (-\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}})\right)Q(t) = \frac{1}{L}\sum_{k=0}^{m} c_k\delta_T^{(k)}(t).$$

For simplicity, let

$$\omega_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad\text{and}\quad \omega_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}.$$

By applying the Property 3.1.2(2) to (3.3) with $k_1 = k_2 = 1$ and $\alpha_1 = \omega_1$, $\alpha_2 = \omega_2$ then (3.3) can be written as the form

$$\left[\left(e^{\omega_1 t}\delta_T^{(1)}\right) \Delta \left(e^{\omega_2 t}\delta_T^{(1)}\right)\right] \Delta Q(t) = \frac{1}{L}\sum_{k=0}^{m} c_k \delta_T^{(k)}(t). \tag{3.4}$$

Actually, $e^{\omega_1 t}$ and $e^{\omega_2 t}$ are the solutions of the homogenous equation of (3.3) with the right-hand side vanishes.

Now we can find the charge $Q(t)$ in (3.4) by convolving both sides of (3.4) with the inverse of $\left(e^{\omega_1 t}\delta_T^{(1)}\right) \Delta \left(e^{\omega_2 t}\delta_T^{(1)}\right)$. By the Corollary 3.1.3 the inverse of $\left(e^{\omega_1 t}\delta_T^{(1)}\right) \Delta \left(e^{\omega_2 t}\delta_T^{(1)}\right)$ is

$$\frac{1}{\omega_1 - \omega_2}\left[\frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}}\right].$$

Now convolve both sides of (3.4) by $\frac{1}{\omega_1 - \omega_2}\left[\frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}}\right]$, we then obtain the charge

$$Q(t) = \frac{1}{\omega_1 - \omega_2}\left[\frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}}\right] \Delta \left(\frac{1}{L}\sum_{k=0}^{m} c_k \delta_T^{(k)}(t)\right) \tag{3.5}$$

which is the solution of (3.3).

Before convolving by $\frac{1}{L}\sum_{k=0}^{m} c_k \delta_T^{(k)}(t)$ on equation (3.5). We will consider some technique of $\delta_T^{(k)}(t) \Delta e^{\omega t}$ for $0 \le t < T$. Let $f(t) = e^{\omega t}$ where ω is a complex constant and $0 \le t < T$, by (2.9) we obtain

$$\begin{aligned} \triangle f &= f(0) - f(T) \\ &= 1 - e^{\omega T}. \end{aligned}$$

Then, by (2.11) and $\nu = n$, $t_\nu = nT$, we have

$$\begin{aligned} f^{(1)}(t) &= \omega e^{\omega t} + (1 - e^{\omega T}) \sum_{\nu=-\infty}^{\infty} \delta(t - t_\nu) \\ &= \omega e^{\omega t} + (1 - e^{\omega T}) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \omega e^{\omega t} + (1 - e^{\omega T})\, \delta_T(t) \\ &= \delta_T^{(1)} \Delta\, e^{\omega t} \end{aligned}$$

for $0 \leq t < T$. Similarly for second- derivative

$$f^{(2)}(t) = \delta_T^{(2)} \Delta\, e^{\omega t} = \omega^2 e^{\omega t} + (1 - e^{\omega T})(\delta_T^{(1)} + \omega \delta_T)$$

and go on to order k-derivatives. Then we obtain the formula of convolving $e^{\omega t}$ by $\delta_T^{(k)}$ for $0 \leq t < T$, that is

$$\delta_T^{(k)} \Delta\, e^{\omega t} = \omega^k e^{\omega t} + (1 - e^{\omega T}) \sum_{r=0}^{k-1} \omega^r \delta_T^{(k-1-r)}.$$

By computing directly

$$\begin{aligned}
Q(t) &= \frac{1}{L(\omega_1 - \omega_2)(1 - e^{\omega_1 T})} \sum_{k=0}^{m} c_k \omega_1^k e^{\omega_1 t} + \frac{1}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k \omega_1^r \delta_T^{(k-1-r)}(t) \\
&\quad + \frac{-1}{L(\omega_1 - \omega_2)(1 - e^{\omega_2 T})} \sum_{k=0}^{m} c_k \omega_2^k e^{\omega_2 t} + \frac{-1}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k \omega_2^r \delta_T^{(k-1-r)}(t) \\
&= \frac{1}{L(\omega_1 - \omega_2)} \sum_{k=0}^{m} c_k \left[\omega_1^k \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \omega_2^k \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right] \\
&\quad + \frac{1}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k \left[\omega_1^r - \omega_2^r \right] \delta_T^{(k-1-r)}(t). \qquad (3.6)
\end{aligned}$$

Now consider the following cases.

(1) If $m \geq 2$ then the right-hand side of (3.6) contains the Dirac-delta periodic distribution and its derivatives. That means that the charge $Q(t)$ is not an ordinary periodic function but it is the periodic distribution in the space $\mathcal{P}'_T$.

(2) If $0 \leq m < 2$ $(m = 0, 1)$ then for $m = 1$, from (3.6) we obtain

$$\begin{aligned}
Q(t) = &\frac{c_0}{L(\omega_1 - \omega_2)} \left[\frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right] \\
&+ \frac{c_1}{L(\omega_1 - \omega_2)} \left[\omega_1 \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \omega_2 \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right]
\end{aligned}$$

That $Q(t)$ is the periodic function for $0 \leq t < T$ with period T. For $m = 0$, then

$$Q(t) = \frac{c_0}{L(\omega_1 - \omega_2)} \left[\frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right]$$

and also is the periodic function for $0 \leq t < T$ with period T. Now consider the current $I(t)$, we know that $I(t) = \frac{d}{dt} Q(t)$, hence by (3.5)

$$
\begin{aligned}
I(t) &= \frac{1}{\omega_1 - \omega_2} \frac{d}{dt} \left[\frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right] \Delta \left(\frac{1}{L} \sum_{k=0}^{m} c_k \delta_T^{(k)}(t) \right) \\
&= \frac{1}{\omega_1 - \omega_2} \left[\omega_1 \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} + \frac{1 - e^{\omega_1 T}}{1 - e^{\omega_1 T}} \, \delta_T - \omega_2 \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} - \frac{1 - e^{\omega_2 T}}{1 - e^{\omega_2 T}} \, \delta_T \right] \\
&\quad \Delta \left(\frac{1}{L} \sum_{k=0}^{m} c_k \delta_T^{(k)}(t) \right) \\
&= \frac{1}{\omega_1 - \omega_2} \left[\omega_1 \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \omega_2 \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right] \Delta \left(\frac{1}{L} \sum_{k=0}^{m} c_k \delta_T^{(k)}(t) \right).
\end{aligned}
$$

By computing directly,

$$
\begin{aligned}
I(t) &= \frac{\omega_1}{L(\omega_1 - \omega_2)(1 - e^{\omega_1 T})} \sum_{k=0}^{m} c_k \omega_1^k e^{\omega_1 t} + \frac{\omega_1}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k \omega_1^r \delta_T^{(k-1-r)}(t) \\
&\quad + \frac{-\omega_2}{L(\omega_1 - \omega_2)(1 - e^{\omega_2 T})} \sum_{k=0}^{m} c_k \omega_2^k e^{\omega_2 t} + \frac{-\omega_2}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k \omega_2^r \delta_T^{(k-1-r)}(t) \\
&= \frac{1}{L(\omega_1 - \omega_2)} \sum_{k=0}^{m} c_k \left[\omega_1^{k+1} \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \omega_2^{k+1} \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right] \\
&\quad + \frac{1}{L(\omega_1 - \omega_2)} \sum_{k=1}^{m} \sum_{r=0}^{k-1} c_k \left[\omega_1^{r+1} - \omega_2^{r+1} \right] \delta_T^{(k-1-r)}(t). \qquad (3.7)
\end{aligned}
$$

Now consider the following case.

(1) If $m \geq 2$ then we see that the current $I(t)$ contains the Dirac-delta periodic distribution and its derivatives, that means $I(t)$ is not an ordinary periodic function but it is the periodic distribution in the space $\mathcal{P}'_T$. It follows that the current $I(t)$ is not periodic continuous and it occurs impulse and its derivatives in every period T.

(2) If $0 \leq m < 2$ $(m = 0, 1)$ then for $m = 1$, (3.7) becomes

$$
\begin{aligned}
I(t) &= \frac{c_0}{L(\omega_1 - \omega_2)} \left[\omega_1 \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \omega_2 \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right] \\
&\quad + \frac{c_1}{L(\omega_1 - \omega_2)} \left[\omega_1^2 \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \omega_2^2 \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}} \right] + \frac{c_1}{L} \, \delta_T(t).
\end{aligned}
$$

It follows that the current $I(t)$ is the same the case (1). For $m = 0$, (3.7) becomes

$$I(t) = \frac{c_0}{L(\omega_1 - \omega_2)}\left[\omega_1 \frac{e^{\omega_1 t}}{1 - e^{\omega_1 T}} - \omega_2 \frac{e^{\omega_2 t}}{1 - e^{\omega_2 T}}\right]$$
$$= \frac{c_0 e^{-\frac{R}{2L}}}{\sqrt{R^2 - 4\frac{L}{C}}}\left[\left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\frac{e^{\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}}{1 - e^{\left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)T}}\right]$$
$$+ \frac{c_0 e^{-\frac{R}{2L}}}{\sqrt{R^2 - 4\frac{L}{C}}}\left[\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\frac{e^{-\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}}{1 - e^{\left(-\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)T}}\right]$$

by substitution for ω_1, ω_2.

That $I(t)$ is periodic function for $0 \leq t < T$ with period T. It follows that the current $I(t)$ flows periodic continuously for the period T.