# CHAPTER 2

# PRELIMINARIES

In this chapter, we give some notations and definitions that will be used in the later chapters.

## 2.1 Stability

### 2.1.1 Definition

Consider the system described by

$$\dot{x} = f(x,t) \tag{2.1}$$

where $x \in \mathbb{R}^n$, $\dot{x} = \left[\dfrac{dx_1}{dt}, \dfrac{dx_2}{dt}, \ldots, \dfrac{dx_n}{dt}\right]$ and $f$ is a vector having components $f_i(x_1, ..., x_n, t)$, $i = 1, 2, ..., n$. We shall assume that the $f_i$ are continuous and satisfy standard conditions, such as having continuous first partial derivatives so that the solution of (2.1) exists and is unique for given initial conditions. If $f_i$ do not depend explicitly on $t$, (2.1) is called autonomous. (otherwise, nonautonomous). If $f(c,t) = 0$ for all $t$,where $c$ is some constant vector,then it follow at once from (2.1) that if $x(t_0) = c$ then $x(t) = c$, all $t \geq t_0$. Thus solutions starting at $c$ remain there, and $c$ is said to be an *equilibrium* or *critical point*. Clearly, by introducing new variables $\acute{x_i} = x_i - c_i$ we can arrange for the equilibrium point to be transferred to the origin; we shall assume that this has been done for any equilibrium point under consideration (there may well be several for a given system (2.1) )so that we then have $f(0,t) = 0$, $t \geq t_0$.

An equilibrium state $x = 0$ is said to be

1. **Stable** if for any positive scalar $\varepsilon$ there exists a positive scalar $\delta$ such that $\|x(t_0)\|_e < \delta$ implies $\|x(t)\|_e < \varepsilon$, $t \geq t_0$, where $\|.\|_e$ is a standard Eucledian norm

2. **Asymptotically stable** if it is stable and if in addition $x(t) \to 0$ as $t \to \infty$.

3. **Unstable** if it is not stable; that is, there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exist an $x(t_0)$ with $\|x(t_0)\|_e < \delta$ so that $\|x(t_1)\|_e \geq \varepsilon$ for some $t_1 > t_0$. If this holds for every $x(t_0)$ in $\|x(t_0)\|_e < \delta$ the equilibrium is completely unstable.

### 2.1.2 Algebraic Criteria for Linear Systems

Before studying nonlinear systems we return to the general continuous time linear system.

$$\dot{x} = Ax, \tag{2.2}$$

where $A$ is a constant $n \times n$ matrix.

**Proposition 2.1.1** *The system (2.2) is asymptotically stable if and only if $A$ is a stability matrix, i.e. all the characteristic roots $\lambda_k$ of $A$ have negative real parts; (2.2) is unstable if for some characteristic roots $\lambda_k$, $\Re e(\lambda_k) > 0$; and completely unstable if for all characteristic roots $\lambda_k$, $\Re e(\lambda_k) > 0$.*

### 2.1.3 Lyapunov Theory

Consider autonomous system of nonlinear equations,

$$\dot{x} = f(x), \quad f(0) = 0. \tag{2.3}$$

We define a Lyapunov function $V(x)$ as follows:

1. $V(x)$ and all its partial derivatives $\dfrac{\partial V}{\partial x_i}$ are continuous.

2. $V(x)$ is positive definite, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighbourhood $\| x \| \leq k$ of the origin.

3. The derivative of $V$ with respect to (2.3), namely

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + ... + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + ... + \frac{\partial V}{\partial x_n}f_n \end{aligned} \tag{2.4}$$

is negative semidefinite i.e. $\dot{V}(0) = 0$, and for all $x$ in $\|x\| \leq k$, $V\dot{(x)} \leq 0$.

Notice that in (2.4) the $f_i$ are the components of $f$ in (2.3), so $\dot{V}$ can be determined directly from the system equations.

**Proposition 2.1.2** *The origin of (2.3) is stable if there exists a Lyapunov function defined as above.*

**Proposition 2.1.3** *The origin of (2.3) is asymptotically stable if there exists a Lyapunov function whose derivative (2.4) is negative definite.*

### 2.1.4 Application of Lyapunov Theory to Linear Systems

The usefulness of linear theory can be extended by using the idea of linearization. Suppose the components of $f$ in (2.1) are such that we can apply Taylor's theorem to obtain

$$f(x) = \acute{A}x + g(x), \tag{2.5}$$

using $f(0) = 0$. In (2.5) $\acute{A}$ denotes the $n \times n$ constant matrix having elements $(\partial f_i/\partial x_j)_{x=0}$, $g(0) = 0$ and the components of $g$ have power series expansions in $x_1, x_2, ..., x_n$ beginning with terms of at least second degree. The system

$$\dot{x} = \acute{A}x \tag{2.6}$$

is called the *first approximation* to (2.1). We then have:



**Proposition 2.1.4** *(Lyapunov's linearization theorem) If (2.6) is asymptotically stable, or unstable, then the origin for $\dot{x} = f(x)$, where $f(x)$ is given by (2.5), has the same stability property.*

## 2.2 Routh-Hurwitz Theorem

Consider the characteristic equation of matrix $A$

$$det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n = 0 \tag{2.7}$$

determining the $n$ eigenvalues $\lambda$ of a real $n \times n$ square matrix $A$,

where $I$ is the identity matrix. Then the eigenvalues $\lambda$ all have negative real parts if

$$\Delta_i > 0, \quad i = 1, 2, \ldots, n-1 \tag{2.8}$$

where

$$\Delta_k = \begin{vmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2k-1} & a_{2k-2} & a_{2k-3} & a_{2k-4} & a_{2k-5} & a_{2k-6} & \ldots & a_k \end{vmatrix} \tag{2.9}$$

If $n = 3$ then

$$|\lambda I - A| = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{2.10}$$

In this case all of the eigenvalues $\lambda$ have negative real parts if

$$\Delta_1 > 0, \Delta_2 > 0, \tag{2.11}$$

or

$$\begin{array}{ll} (1) & a_1 > 0, \\ \text{and} \quad (2) & \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0 \text{ or } a_1a_2 - a_3 > 0. \end{array}$$



## 2.3 Fourth-Order Runge-Kutta Method



In order to solve an initial-value problem



$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0 \tag{2.12}$$

where $x = [x_1, x_2, \ldots, x_n]^T$ and $f = [f_1, f_2, \ldots, f_n]^T$

The best known Runge-Kutta method of the first stage and fourth order is given by

$$X_{i+1} = X_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.13}$$

where

$$\begin{aligned} k_1 &= hf(t_i, X_i) \\ k_2 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_1}{2}) \\ k_3 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_2}{2}) \\ k_4 &= hf(t_i + h, X_i + k_3) \end{aligned}$$

where $X_i$ is an approximation of $x(t_i)$ when $X_i = [X_{i1}, X_{i2}, \ldots, X_{in}]^T$, $t_i = t_0 + ih$, $h$ is step size and $k_i = [k_{i1}, k_{i2}, \ldots, k_{in}]^T \forall i = 1, \ldots 4$.

## 2.4 Matrix Types

### 2.4.1 Hermitian Matrix

A real $n \times n$ matrix $A$ is called *Hermitian* if

$$A^T = A.$$

### 2.4.2 Positive Definite Matrix

Consider a real $n \times n$ matrix $A$ , $A$ is called *positive definite* if

$$x^T Ax > 0$$

for all nonzero vectors $x \in \mathbb{R}^n$, where $x^T$ denotes the transpose.

Or Hermitian matrix $A$ is called *positive definite* if and only if $D_i > 0$, $i = 1, 2, ..., n$, where $D_i$ denotes leading principal minors.



### 2.4.3 Positive Semidefinite Matrix

A *Positive semidefinite* matrix is a Hermitian matrix in which all of whose eigenvalues are nonnegative.

Or Hermitian matrix $A$ is called *positive semidefinite* if and only if $det(A) = 0$ and $P_i \geq 0$, $i = 1, 2, ..., n$, where $P_i$ denotes principal minors.

### 2.4.4 Negative Definite Matrix

A *Negative definite* matrix is a Hermitian matrix in which all of whose eigenvalues are negative.

Or Hermitian matrix $A$ is called *negative definite* if and only if $(-1)^i D_i > 0$, $i = 1, 2, ..., n$, where $D_i$ denotes leading principal minors.

### 2.4.5 Negative Semidefinite Matrix

A *Positive semidefinite* matrix is a Hermitian matrix in which all of whose eigenvalues are nonpositive.

Or Hermitian matrix $A$ is called *negative semidefinite* if and only if $det(A) = 0$ and $(-1)^i P_i \geq 0$, $i = 1, 2, ..., n$, where $P_i$ denotes principal minors.

If $A$ satisfies none of the above then it is indefinite.

## 2.5 Synchronization

Consider the system of differential equations

$$\dot{x} = f(x) \tag{2.14}$$

$$\dot{y} = g(y, x) \tag{2.15}$$

where $x, y \in \mathbb{R}^n$, $f$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be analytic functions.

Let $x(t, x_0)$ and $y(t, y_0)$ be solutions to (2.14) and (2.15) respectively. The solutions $x(t, x_0)$ and $y(t, y_0)$ are said to be *synchronized* if

$$\lim_{t\to\infty} \| x(t, x_0) - y(t, y_0) \| = 0$$

## 2.6 Terminology

- *Chaos* is characterized by three simple ideas. First, chaotic systems are deterministic, meaning they obey some simple rules. In general this means that we can predict their behavior of short times. Second, chaotic systems have sensitively dependence on initial conditions, which means we can't predict their behavior for long times. Third, chaotic systems generally have underlying patterns, sometimes called attractors.

- *Chaotic behavior* is the behavior of a system whose final state depends so sensitively on the system's precise initial state the behavior is in effect unpredictable and can not be distinguished from a random process, even though it is strictly determinate in a mathematical sense. Also known as chaos.

- *Dynamical system* is means of describing how one state develops into another state over the course of time.

- The sequence of solution value of differential equation or difference equation generated by this iteration procedure will be called the *trajectory*.

- *Attractor* is the set of points to which trajectories approach as the number of iterations goes to infinity.

- The notation of *equilibrium points* (also called *fixed points* or *singular points* or *critical points*).

- $u(t) = f[x(t)]$, this equation is called the *control rule* or *control law*.