CHAPTER 3

MAIN RESULTS

In this chapter we consider controlling chaos and synchronization of perturbed Chen chaotic dynamical system.

3.1 Chen Chaotic Dynamical System

In this section, we give some sufficient conditions of parameters which ensure that the equilibrium points of Chen chaotic dynamical system are asymptotically stable.

Consider,

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - xz + cy \\ \dot{z} &= xy - bz \end{aligned} \tag{3.1}$$

where

x, y and z are the state variables.

a, b and C are positive real constants.

The equilibrium points of the system (3.1) are

$$E_1 = (0,0,0),\ E_2 = (\alpha,\alpha,\gamma),\ E_3 = (-\alpha,-\alpha,\gamma)$$

where $\alpha = \sqrt{b\gamma}$ and $\gamma = 2c - a$.

Theorem 3.1.1 *The equilibrium point $E_1 = (0,0,0)$ is asymptotically stable if $a > 2c$ and $ac < b^2 < 2ac$.*

Proof The Jacobian matrix of the system (3.1) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c \\ a_2 &= b(a-c) + a(a-2c) \\ a_3 &= ab(a-2c) \\ a_1a_2 - a_3 &= (ab+a^2)(a-2c) + a(b^2-ac) + c(2ac-b^2) + bc^2 \end{aligned}$$

We see that when $a > 2c > c$ and $ac < b^2 < 2ac$ then a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria. Therefore the equilibrium point $E_1 = (0,0,0)$ is asymptotically stable.

Theorem 3.1.2 *The equilibrium point $E_2 = (\sqrt{b\gamma}, \sqrt{b\gamma}, \gamma)$ is asymptotically stable if $2a > 3c$ and $b > c$.*

Proof The Jacobian matrix of the system (3.1) at the equilibrium point $E_2 = (\sqrt{b\gamma}, \sqrt{b\gamma}, \gamma)$ is given by

$$J_2 = \begin{bmatrix} -a & a & 0 \\ -c & c & -\sqrt{b\gamma} \\ \sqrt{b\gamma} & \sqrt{b\gamma} & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_2 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c \\ a_2 &= bc \\ a_3 &= 2ab(2c-a) \\ a_1a_2 - a_3 &= bc(b-c) + ab(2a-3c) \end{aligned}$$

We see that when $2a > 3c$ and $b > c$ then $a > \frac{3}{2}c > c$, a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria. Therefore the equilibrium point $E_2 = (\sqrt{b\gamma}, \sqrt{b\gamma}, \gamma)$ is asymptotically stable.

Theorem 3.1.3 *The equilibrium point $E_3 = (-\sqrt{b\gamma}, -\sqrt{b\gamma}, \gamma)$ is asymptotically stable if $2a > 3c$ and $b > c$.*

Proof The Jacobian matrix of the system (3.1) at the equilibrium point $E_3 = (-\sqrt{b\gamma}, -\sqrt{b\gamma}, \gamma)$ is given by

$$J_3 = \begin{bmatrix} -a & a & 0 \\ -c & c & \sqrt{b\gamma} \\ -\sqrt{b\gamma} & -\sqrt{b\gamma} & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c \\ a_2 &= bc \\ a_3 &= 2ab(2c-a) \\ a_1a_2 - a_3 &= bc(b-c) + ab(2a-3c) \end{aligned}$$

We see that when $2a > 3c$ and $b > c$ then $a > \frac{3}{2}c > c$, a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria. Therefore, the equilibrium point $E_3 = (-\sqrt{b\gamma}, -\sqrt{b\gamma}, \gamma)$ is asymptotically stable.

3.1.1 Numerical Simulations

Numerical experiments are carried out to investigate Chen chaotic dynamical system by using fourth-order Runge-Kutta method with time step 0.001. The parameters a, b and c are chosen as $a = 35$, $b = 3$ and $c = 28$. The initial states are taken as $x = 0.5$, $y = 1$ and $z = 5$. Fig. 3.1 shows the behavior of the states x, y and z of the system (3.1) with time in xy-plane. Fig. 3.2 shows the behavior of the states x, y and z of the system (3.1) with time in xz-plane. Fig. 3.3 shows the behavior of the states x, y and z of the system (3.1) with time in yz-plane.

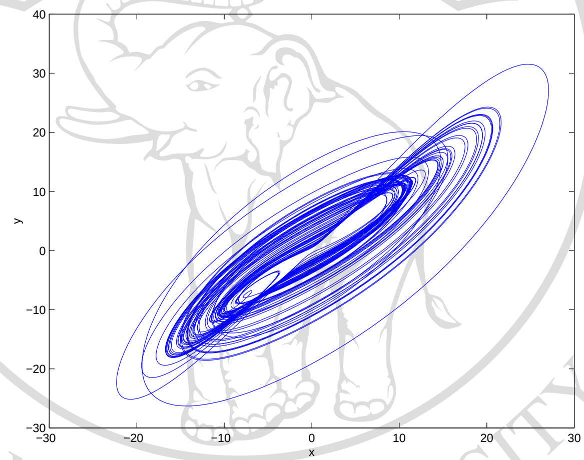

Figure 3.1: The chaotic attractor of Chen chaotic dynamical system (3.1) in the xy-plane.

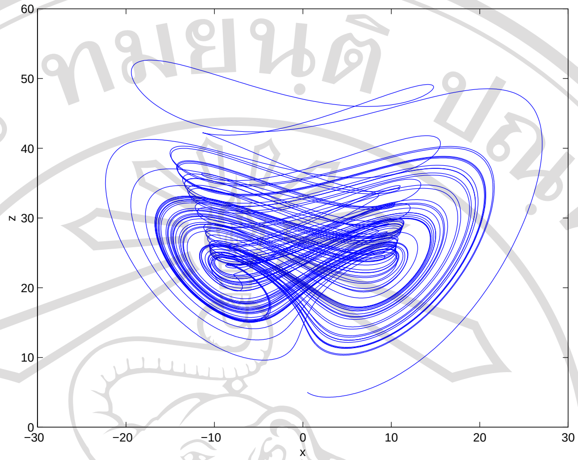

Figure 3.2: The chaotic attractor of Chen chaotic dynamical system (3.1) in the xz-plane.

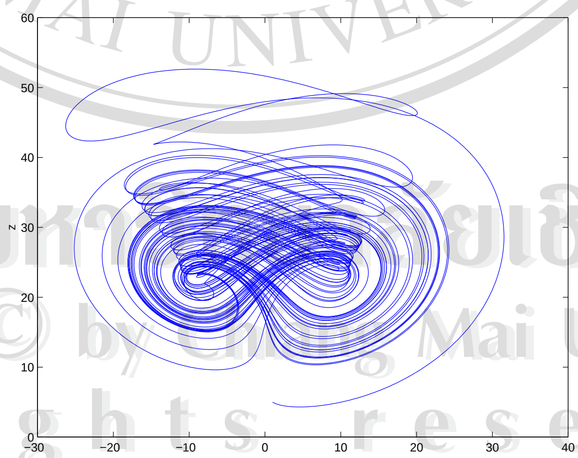

Figure 3.3: The chaotic attractor of Chen chaotic dynamical system (3.1) in the yz-plane.

3.2 The Perturbed Chen Chaotic Dynamical System

We will study the perturbed Chen chaotic dynamical system that is described by system of ordinary differential equations.

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - xz + cy \\ \dot{z} &= xy - bz + dx^2 \end{aligned} \tag{3.2}$$

where

x, y and z are the state variables.

a, b, c and d are positive real constants.

The equilibrium points of the system (3.2) are

$$E_1 = (0,0,0),\ E_2 = (\beta,\beta,\gamma),\ E_3 = (-\beta,-\beta,\gamma)$$

where $\beta = \sqrt{\dfrac{b\gamma}{1+d}}$ and $\gamma = 2c - a$.

Theorem 3.2.1 *The equilibrium point $E_1 = (0,0,0)$ is*

(i) asymptotically stable if $a > 2c$ and $ac < b^2 < 2ac$.

(ii) unstable if $2c > a$.

Proof The Jacobian matrix of the system (3.2) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c \\ a_2 &= b(a-c)+a(a-2c) \\ a_3 &= ab(a-2c) \\ a_1a_2-a_3 &= (ab+a^2)(a-2c)+a(b^2-ac)+c(2ac-b^2)+bc^2 \end{aligned}$$

We see that a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria when $a>2c$ and $ac<b^2<2ac$, thus the equilibrium point $E_1=(0,0,0)$ is asymptotically stable.

The solution of characteristic equation are

$$\begin{aligned} \lambda_1 &= -b \\ \lambda_2 &= \frac{-(a-c)+\sqrt{(a-c)^2-4a(a-2c)}}{2} \\ \lambda_2 &= \frac{-(a-c)-\sqrt{(a-c)^2-4a(a-2c)}}{2} \end{aligned}$$

We see that $\lambda_1<0$ and when $2c>a$ we have either λ_2 or λ_3 has negative real part. Therefore the equilibrium point $E_1=(0,0,0)$ is unstable.

Theorem 3.2.2 *The equilibrium point $E_2=(\beta,\beta,\gamma)$ is*

(i) asymptotically stable if $\frac{3}{2}c<a<2c$, $b>6c$ and $\frac{1}{3}<d<1$.

(ii) unstable if $b<c<a$ and $a<\frac{4}{3}c$.

Proof The Jacobian matrix of the system (3.2) at the equilibrium point $E_2=(\beta,\beta,\gamma)$ is given by

$$J_2=\begin{bmatrix} -a & a & 0 \\ -c & c & -\beta \\ \beta+2d\beta & \beta & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_2 has the form

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$$

where

$$\begin{aligned}
a_1 &= a+b-c \\
a_2 &= b(a-c)+\beta^2 \\
a_3 &= 2ab(2c-a) \\
a_1a_2-a_3 &= \frac{abd(b-6c)+ab(2a-3c)+b(3a^2d-c^2)+b^2c(1-d)+bc^2d}{(1+d)} \\
&= \frac{abd(3a-4c)+abd(b-c)+bcd(c-a)+bc(b-c)+ab(2a-3c)-b^2cd}{(1+d)}
\end{aligned}$$

We see that a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria when $\frac{3}{2}c < a < 2c$, $b > 6c$ and $\frac{1}{3} < d < 1$, thus the equilibrium point $E_2 = (\beta, \beta, \gamma)$ is asymptotically stable.

But when $b < c < a$ and $a < \frac{4}{3}c$, we have $a_1a_2 - a_3 < 0$ which does not satisfy the Routh-Hurwitz criteria and so the equilibrium point $E_2 = (\beta, \beta, \gamma)$ is unstable.

Theorem 3.2.3 *The equilibrium point $E_3 = (-\beta, -\beta, \gamma)$ is*

(i) asymptotically stable if $\frac{3}{2}c < a < 2c$, $b > 6c$ and $\frac{1}{3} < d < 1$.

(ii) unstable if $b < c < a$ and $a < \frac{4}{3}c$.

Proof The Jacobian matrix of the system (3.2) at the equilibrium point $E_3 = (-\beta, -\beta, \gamma)$ is given by

$$J_3 = \begin{bmatrix} -a & a & 0 \\ -c & c & \beta \\ -\beta - 2d\beta & -\beta & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$
\begin{aligned}
a_1 &= a+b-c \\
a_2 &= b(a-c)+\beta^2 \\
a_3 &= 2ab(2c-a) \\
a_1a_2-a_3 &= \frac{abd(b-6c)+ab(2a-3c)+b(3a^2d-c^2)+b^2c(1-d)+bc^2d}{(1+d)} \\
&= \frac{abd(3a-4c)+abd(b-c)+bcd(c-a)+bc(b-c)+ab(2a-3c)-b^2cd}{(1+d)}
\end{aligned}
$$

We see that a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria when $\frac{3}{2}c<a<2c$, $b>6c$ and $\frac{1}{3}<d<1$, thus the equilibrium point $E_3=(-\beta,-\beta,\gamma)$ is asymptotically stable.

But when $b<c<a$ and $a<\frac{4}{3}c$, $a_1a_2-a_3<0$ does not satisfy the Routh-Hurwitz criteria, and so the equilibrium point $E_3=(-\beta,-\beta,\gamma)$ is unstable.

3.2.1 Numerical Simulations

Numerical experiments are carried out to investigate perturbed Chen chaotic dynamical system by using fourth-order Runge-Kutta method with time step 0.001. The parameters a, b, c and d are chosen as $a=35$, $b=3$, $c=28$ and $d=2$. The initial states are taken as $x=0.5$, $y=1$ and $z=5$. Fig. 3.4 shows the behavior of the states x, y and z of the system (3.2) with time in xy-plane. Fig. 3.5 shows the behavior of the states x, y and z of the system (3.2) with time in xz-plane. Fig. 3.6 shows the behavior of the states x, y and z of the system (3.2) with time in yz-plane.

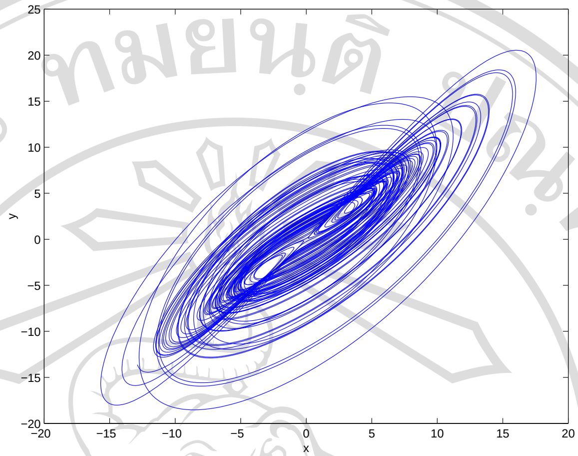

Figure 3.4: The chaotic attractor of perturbed Chen chaotic dynamical system (3.2) in the xy-plane.

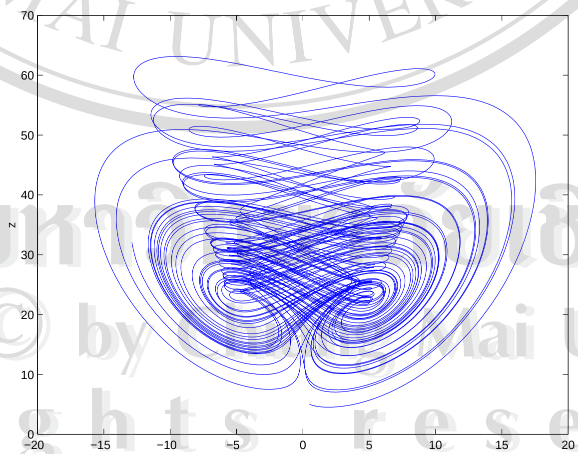

Figure 3.5: The chaotic attractor of perturbed Chen chaotic dynamical system (3.2) in the xz-plane.

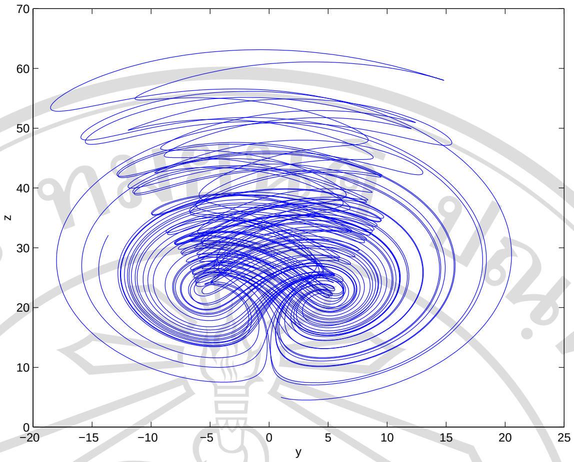

Figure 3.6: The chaotic attractor of perturbed Chen chaotic dynamical system (3.2) in the yz-plane.

3.3 Controlling Chaos of Perturbed Chen System to Equilibrium Point

In this section, the chaos of system (3.2) is controlled trajectory to one of three equilibrium points of the system. Feedback and bounded feedback control are applied to achieve this goal. We shall study in the case that equilibrium points of (3.2) are unstable, therefore $b < c < a$ and $a < \frac{4}{3}c$.

3.3.1 Feedback Control Method

The goal of linear feedback control is to control the chaotic behavior of the system (3.2) to one of three unstable equilibrium points (E_1, E_2 or E_3). We assume that the controlled system is given by

$$
\begin{aligned}
\dot{x} &= a(y-x) + u_1 \\
\dot{y} &= (c-a)x - xz + cy + u_2 \\
\dot{z} &= xy - bz + dx^2 + u_3.
\end{aligned}
$$

where u_1, u_2 and u_3 are controllers that satisfy the following control law i.e.

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}(x-\bar{x}) \\ \dot{y} &= (c-a)x - xz + cy - k_{22}(y-\bar{y}) \\ \dot{z} &= xy - bz + dx^2 - k_{33}(z-\bar{z}). \end{aligned} \tag{3.3}$$

where $E = (\bar{x}, \bar{y}, \bar{z})$ is an equilibrium point of (3.2).

Stability of the Equilibrium Point $E_1 = (0, 0, 0)$

In this case $E = E_1$ and the controlled system (3.3) is in the form of

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}x \\ \dot{y} &= (c-a)x - xz + cy - k_{22}y \\ \dot{z} &= xy - bz + dx^2 - k_{33}z. \end{aligned} \tag{3.4}$$

Theorem 3.3.1 *The equilibrium point* $E_1 = (0, 0, 0)$ *is asymptotically stable if* $k_{11} = 0$, $k_{33} > 0$ *and* $k_{22} > 3c$.

Proof The Jacobian matrix of the system (3.4) at the equilibrium point $E_1 = (0, 0, 0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ c-a & c-k_{22} & 0 \\ 0 & 0 & -b-k_{33} \end{bmatrix}$$

The characteristic equation of the Jacobian J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$
\begin{aligned}
a_1 &= a+b-c+k_{22}+k_{33} \\
a_2 &= a(a-c)+a(k_{22}-c)+b(a-c)+bk_{22}+(a-c)k_{33}+k_{22}k_{33} \\
a_3 &= ab(k_{22}-2c)+ak_{33}(k_{22}-2c)+a^2b+a^2k_{33} \\
a_1a_2-a_3 &= (a-c)(b^2+2bk_{22}+k_{33}^2+2bk_{33})+(k_{22}-3c)(a^2+ak_{22})+2ac(c-b) \\
&\quad +k_{33}(k_{22}^2-2ac)+a^3+bc^2+a^2b+b^2k_{22}+c^2k_{33}+2(a+b-c)k_{22}k_{33} \\
&\quad +bk_{22}^2+k_{22}k_{33}^2.
\end{aligned}
$$

We see that when $k_{11}=0$, $k_{33}>0$ and $k_{22}>3c$, a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria. Therefore, the equilibrium point $E_1=(0,0,0)$ is asymptotically stable.

Stability of the Equilibrium Point $E_2=(\beta,\beta,\gamma)$

In this case $E=E_2$ and the controlled system (3.3) is in the form of

$$
\begin{aligned}
\dot{x} &= a(y-x)-k_{11}(x-\beta) \\
\dot{y} &= (c-a)x-xz+cy-k_{22}(y-\beta) \\
\dot{z} &= xy-bz+dx^2-k_{33}(z-\gamma).
\end{aligned} \tag{3.5}
$$

Theorem 3.3.2 *The equilibrium point* $E_2=(\beta,\beta,\gamma)$ *is asymptotically stable if* $k_{11},k_{33}>0$ *and* $k_{22}>2c$.

Proof The Jacobian matrix of the system (3.5) at the equilibrium point $E_2=(\beta,\beta,\gamma)$ is given by

$$
J_2=\begin{bmatrix} -a-k_{11} & a & 0 \\ -c & c-k_{22} & -\beta \\ \beta+2d\beta & \beta & -b-k_{33} \end{bmatrix}
$$

The characteristic equation of the Jacobian J_2 has the form

$$\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$$

where

$$
\begin{aligned}
a_1 &= a + b - c + k_{11} + k_{22} + k_{33} \\
a_2 &= \beta^2 + b(a - c) + k_{11}(k_{22} - c) + k_{33}(k_{22} - c) + bk_{11} + (a + b)k_{22} + ak_{33} \\
&\quad + k_{11}k_{33} \\
a_3 &= (k_{22} - c)(bk_{11} + k_{11}k_{33}) + abk_{22} + ak_{22}k_{33} + 2ab(2c - a) + \beta^2 k_{11} \\
a_1a_2 - a_3 &= (a - c)(3ab + b^2 + 2bk_{11} + 2bk_{22} + ak_{22} + 2bk_{33} + k_{33}^2) + a(k_{22}^2 - 2bc) \\
&\quad + (k_{22}^2 - ac)(b + k_{11}) + k_{33}(k_{22}^2 - 2ac) + k_{11}^2(k_{22} - c) + bc^2 + (b^2 + c^2)k_{11} \\
&\quad + (a + b - c)(\beta^2 + 2k_{11}k_{22} + 2k_{11}k_{33} + 2k_{22}k_{33}) + (\beta^2 + b^2)k_{22} + bk_{11}^2 \\
&\quad + (a^2 + c^2 + \beta^2)k_{33} + 2k_{11}k_{22}k_{33} + k_{11}k_{33}^2 + k_{11}^2k_{33} + k_{22}k_{33}^2.
\end{aligned}
$$

We see that when $k_{11}, k_{33} > 0$ and $k_{22} > 2c$, a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria. Therefore, the equilibrium point $E_2 = (\beta, \beta, \gamma)$ is asymptotically stable.

Stability of the Equilibrium Point $E_3 = (-\beta, -\beta, \gamma)$

In this case $E = E_3$ and the controlled system (3.3) is in the form of

$$
\begin{aligned}
\dot{x} &= a(y - x) - k_{11}(x + \beta) \\
\dot{y} &= (c - a)x - xz + cy - k_{22}(y + \beta) \qquad (3.6) \\
\dot{z} &= xy - bz + dx^2 - k_{33}(z - \gamma).
\end{aligned}
$$

Theorem 3.3.3 *The equilibrium point* $E_3 = (-\beta, -\beta, \gamma)$ *is asymptotically stable if* $k_{11}, k_{33} > 0$ *and* $k_{22} > 2c$.

Proof The Jacobian matrix of the system (3.6) at the equilibrium point $E_3 = (-\beta, -\beta, \gamma)$ is given by

$$
J_3 = \begin{bmatrix} -a - k_{11} & a & 0 \\ -c & c - k_{22} & \beta \\ -\beta - 2d\beta & -\beta & -b - k_{33} \end{bmatrix}
$$

The characteristic equation of the Jacobian J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned}
a_1 &= a + b - c + k_{11} + k_{22} + k_{33} \\
a_2 &= \beta^2 + b(a-c) + k_{11}(k_{22} - c) + k_{33}(k_{22} - c) + bk_{11} + (a+b)k_{22} + ak_{33} \\
&\quad + k_{11}k_{33} \\
a_3 &= (k_{22} - c)(bk_{11} + k_{11}k_{33}) + abk_{22} + ak_{22}k_{33} + 2ab(2c - a) + \beta^2 k_{11} \\
a_1a_2 - a_3 &= (a-c)(3ab + b^2 + 2bk_{11} + 2bk_{22} + ak_{22} + 2bk_{33} + k_{33}^2) + a(k_{22}^2 - 2bc) \\
&\quad + (k_{22}^2 - ac)(b + k_{11}) + k_{33}(k_{22}^2 - 2ac) + k_{11}^2(k_{22} - c) + bc^2 + (b^2 + c^2)k_{11} \\
&\quad + (a + b - c)(\beta^2 + 2k_{11}k_{22} + 2k_{11}k_{33} + 2k_{22}k_{33}) + (\beta^2 + b^2)k_{22} + bk_{11}^2 \\
&\quad + (a^2 + c^2 + \beta^2)k_{33} + 2k_{11}k_{22}k_{33} + k_{11}k_{33}^2 + k_{11}^2k_{33} + k_{22}k_{33}^2.
\end{aligned}$$

We see that when $k_{11}, k_{33} > 0$ and $k_{22} > 2c$, a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria. Therefore, the equilibrium point $E_3 = (\beta, \beta, \gamma)$ is asymptotically stable.

Numerical Simulations

Numerical experiments are carried out to investigate controlled systems by using fourth-order Runge-Kutta method with time step 0.001. The parameters a, b, c and d are chosen as $a = 35$, $b = 3$, $c = 28$ and $d = 2$ to ensure the existence of chaos in the absence of control. The initial states are taken as $x = 0.1$, $y = 0.2$ and $z = 0.3$. The equilibrium point $E_1 = (0,0,0)$ of the system (3.2) is stabilized for $k_{11} = 0$, $k_{22} = 85$ and $k_{33} = 5$. Fig. 3.7 shows the behavior of the states x, y and z of the controlled system (3.4) with time. The control is active at $t = 10$. The equilibrium point $E_2 = (\sqrt{21}, \sqrt{21}, 21)$ of the system (3.2) is stabilized for $k_{11} = 1$, $k_{22} = 60$ and $k_{33} = 5$. Fig. 3.8 shows the behavior of the states x,y and z of the controlled system (3.5) with time. The control is active at $t = 10$. The equilibrium point $E_3 = (-\sqrt{21}, -\sqrt{21}, 21)$ of the system (3.2) is stabilized for

$k_{11} = 1$, $k_{22} = 60$ and $k_{33} = 5$. Fig. 3.9 shows the behavior of the states x, y and z of the controlled system (3.6) with time. The control is active at $t = 10$.

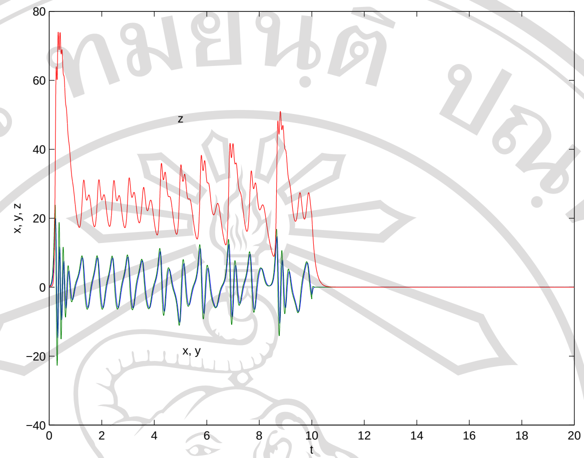

Figure 3.7: The time responses for the states x, y and z of the controlled system (3.4) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 0$, $k_{22} = 85$ and $k_{33} = 5$.

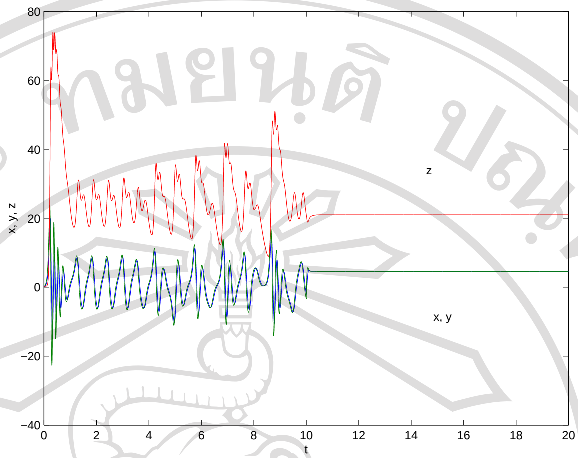

Figure 3.8: The time responses for the states x, y and z of the controlled system (3.5) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 1$, $k_{22} = 60$ and $k_{33} = 5$.

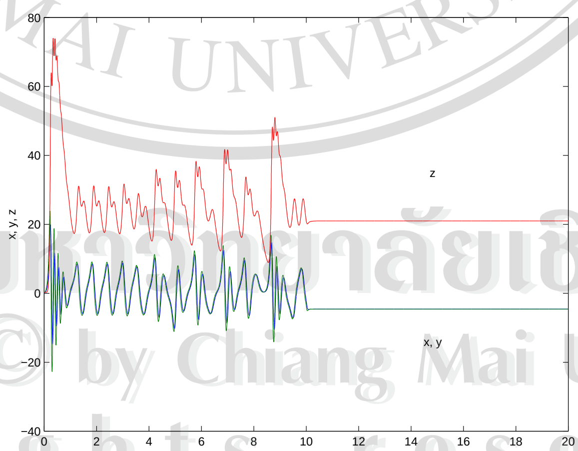

Figure 3.9: The time responses for the states x, y and z of the controlled system (3.6) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 1$, $k_{22} = 60$ and $k_{33} = 5$.

3.3.2 Bounded Feedback Control Method

In this case, we control chaos with bounded controller that vanishes after the stabilization is achieved.

Stability of the Equilibrium Point $E_1 = (0, 0, 0)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.2) as follows:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - xz + cy + u(t) \\ \dot{z} &= xy - bz + dx^2 \end{aligned} \tag{3.7}$$

where $u(t) = -k(a(x+y)), k > 0$.

Theorem 3.3.4 *The equilibrium point* $E_1 = (0, 0, 0)$ *is asymptotically stable if* $k > \dfrac{2c}{a}$.

Proof The Jacobian matrix of the system (3.7) at the equilibrium point $E_1 = (0, 0, 0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ c-a-ka & c-ka & 0 \\ 0 & 0 & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c + ka \\ a_2 &= (a-c)(a+b) + a(2ka - c) + kab \\ a_3 &= ab(a-c) + ab(2ka - c) \\ a_1a_2 - a_3 &= (a-c)(2kab + 2ka^2 + a^2 + b^2) + 2ka^2(ka - c) + a^2(ka - 2c) \\ &\quad +2ac(c-b) + a^2b(k^2+1) + kab^2 + bc^2. \end{aligned}$$

We see that under the assumption that $k > \frac{2c}{a}$, a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria, thus the equilibrium point $E_1 = (0,0,0)$ is asymptotically stable.

Stability of the Equilibrium Point $E_2 = (\beta, \beta, \gamma)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.2) as follows:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (c-a)x - xz + cy + u(t) \\ \dot{z} &= xy - bz + dx^2 \end{aligned} \tag{3.8}$$

where $u(t) = -k(a(y-x)), k > 0$.

Theorem 3.3.5 *The equilibrium point* $E_2 = (\beta, \beta, \gamma)$ *is asymptotically stable if* $k > \sqrt{2}$.

Proof The Jacobian matrix of the system (3.8) at the equilibrium point $E_2 = (\beta, \beta, \gamma)$ is given by

$$J_2 = \begin{bmatrix} -a & a & 0 \\ -c+ka & c-ka & -\beta \\ \beta + 2d\beta & \beta & -b \end{bmatrix}$$

The characteristic equation of the Jacobian J_2 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b + ka - c \\ a_2 &= \beta^2 + kab + b(a-c) \\ a_3 &= 2ab(2c-a) \\ a_1a_2 - a_3 &= (a-c)(3ab - bc + 2kab + \beta^2 + b^2) + ab(k^2a - 2c) + kab^2 + (b+ka)\beta^2 \end{aligned}$$

We see that under the assumption that $k > \sqrt{2}$, a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria, thus the equilibrium point $E_2 = (\beta, \beta, \gamma)$ is asymptotically stable.

Stability of the Equilibrium Point $E_3 = (-\beta, -\beta, \gamma)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.2) as follows:

$$\begin{aligned}\dot{x} &= a(y-x)\\ \dot{y} &= (c-a)x - xz + cy + u(t)\\ \dot{z} &= xy - bz + dx^2\end{aligned} \tag{3.9}$$

where $u(t) = -k(a(y-x)), k > 0$.

Theorem 3.3.6 *The equilibrium point $E_3 = (-\beta, -\beta, \gamma)$ is asymptotically stable if $k > \sqrt{2}$.*

Proof The Jacobian matrix of the system (3.9) at the equilibrium point $E_3 = (\beta, \beta, \gamma)$ is given by

$$J_3 = \begin{bmatrix} -a & a & 0\\ -c & c & \beta\\ -\beta - 2d\beta + ka & -\beta - ka & -b\end{bmatrix}$$

The characteristic equation of the Jacobian J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b + ka - c\\ a_2 &= \beta^2 + kab + b(a-c)\\ a_3 &= 2ab(2c-a)\\ a_1a_2 - a_3 &= (a-c)(3ab - bc + 2kab + \beta^2 + b^2) + ab(k^2a - 2c) + kab^2 + (b+ka)\beta^2\end{aligned}$$

We see that under the assumption that $k > \sqrt{2}$, a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria, thus the equilibrium point $E_3 = (-\beta, -\beta, \gamma)$ is asymptotically stable.

Numerical Simulations

We will show a series of numerical experiments by using the fourth-order Runge-Kutta method with step size 0.001. The parameters a, b, c and d are chosen as $a = 35$, $b = 3$, $c = 28$ and $d = 2$ to ensure the existence of chaos in the absence of control. The control is active at $t = 10$ for all simulations. In the first numerical experiment, we intend to control the chaos to equilibrium point $E_1 = (0, 0, 0)$ of system (3.2). Fig. 3.10-3.12 shows the time response of the states x, y and z of system (3.7) and the controller $u(t)$ with time for $k = 1.6$. The initial condition are $x = 0.1$, $y = 0.2$ and $z = 0.3$. In the second numerical experiment, we intend to control the chaos to equilibrium point $E_2 = (\sqrt{21}, \sqrt{21}, 21)$ of system (3.2). Fig. 3.13-3.15 shows the time response of the states x, y and z of system (3.8) and the controller $u(t)$ with time for $k = 2$. The initial condition are $x = -2.5$, $y = -2.5$ and $z = 3$. In the third numerical experiment, we intend to control the chaos to equilibrium point $E_3 = (-\sqrt{21}, -\sqrt{21}, 21)$ of system (3.2). Fig. 3.16-3.18 shows the time response of the states x, y and z of system (3.9) and the controller $u(t)$ with time for $k = 2$. The initial condition are $x = 2.5$, $y = 2.5$ and $z = 3$.

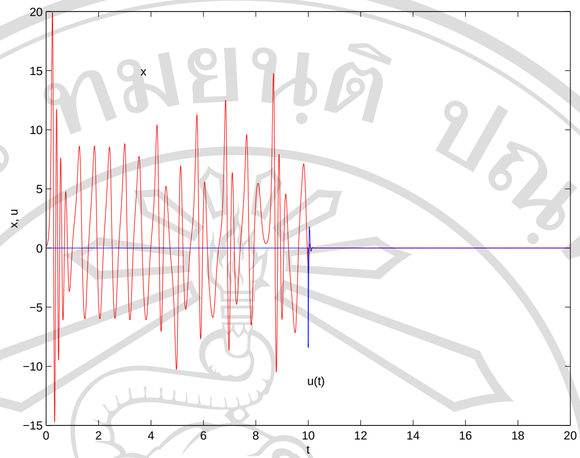

Figure 3.10: The states x of the controlled system (3.7) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10$, $k = 1.6$.

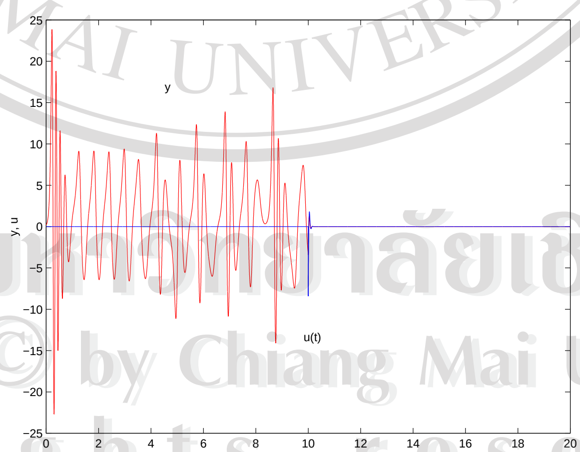

Figure 3.11: The states y of the controlled system (3.7) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10$, $k = 1.6$.

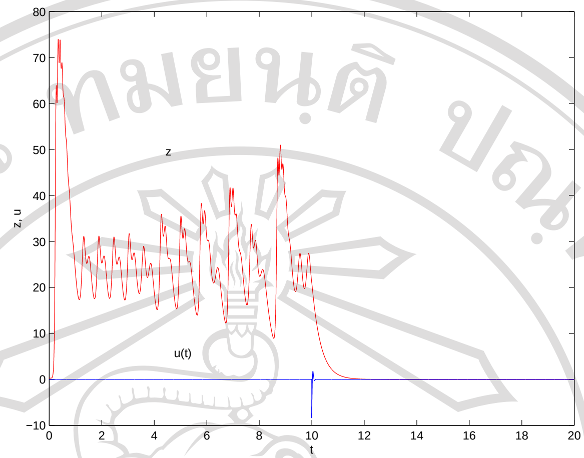

Figure 3.12: The states z of the controlled system (3.7) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10,\ k = 1.6$.

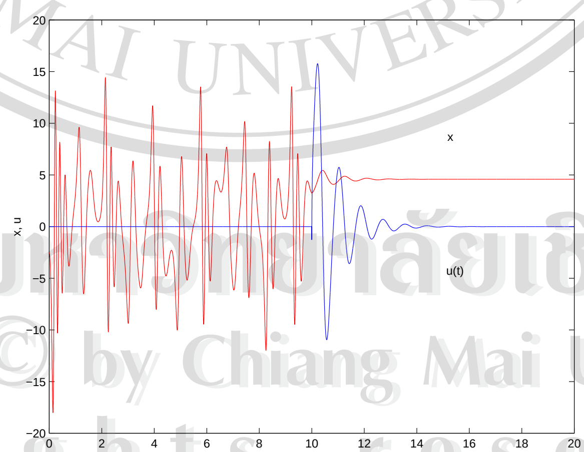

Figure 3.13: The states x of the controlled system (3.8) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10,\ k = 2$.

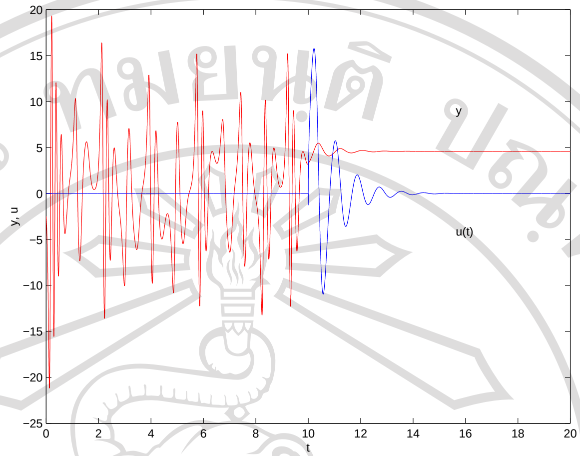

Figure 3.14: The states y of the controlled system (3.8) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10,\ k = 2$.

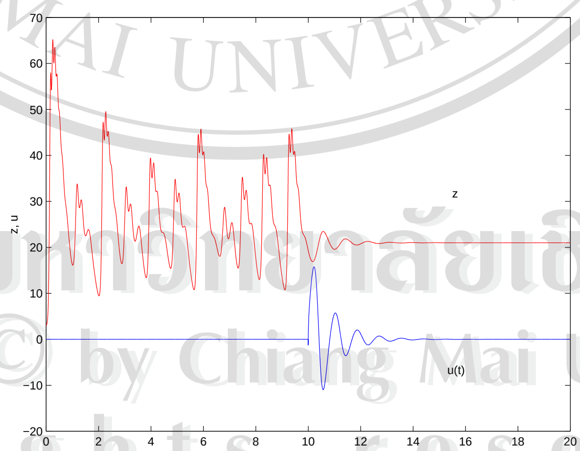

Figure 3.15: The states z of the controlled system (3.8) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10,\ k = 2$.

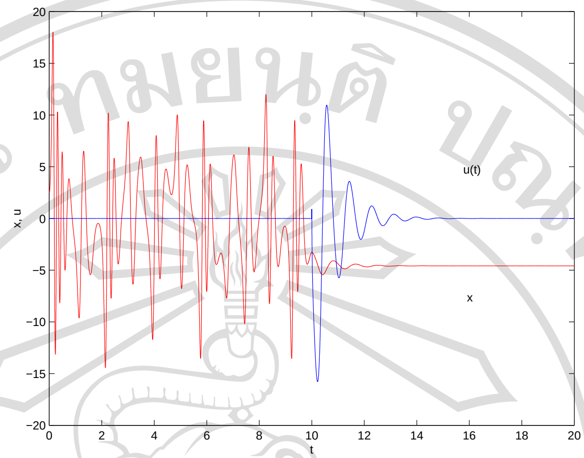

Figure 3.16: The states x of the controlled system (3.9) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10,\ k = 2$.

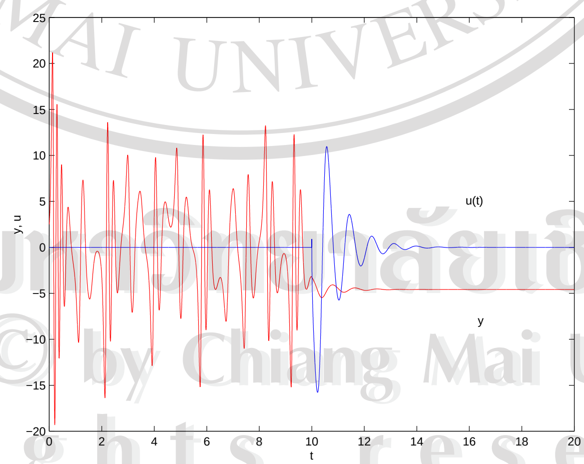

Figure 3.17: The states y of the controlled system (3.9) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10,\ k = 2$.

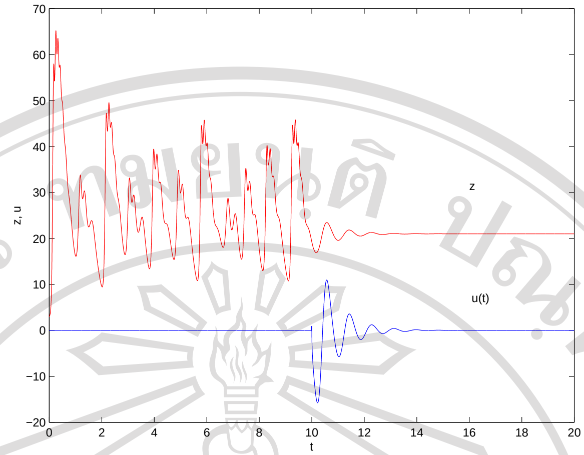

Figure 3.18: The states z of the controlled system (3.9) and the control $u(t)$ respond with time before and after control activation. The control is activated at $t = 10$, $k = 2$.

3.4 Synchronization of Perturbed Chen Chaotic Dynamical System

To begin with, the definition of chaos synchronization used in this thesis is given below.

For two nonlinear chaotic system:

$$\dot{x} = f(t, x) \tag{3.10}$$

$$\dot{y} = g(t, y) + u(t, x, y) \tag{3.11}$$

where $x, y \in \mathbb{R}^n$, $f, g \in C^r[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $u \in C^r[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $r \geq 1$, $\mathbb{R}^+$ is the set of non-negative real numbers. Assume that (3.10) is the drive system, and (3.11) is the response system, $u(t, x, y)$ is the control vector. Response system and drive system are said to be *synchronic* if for $\forall x(t_0), y(t_0) \in \mathbb{R}^n$,

$$\lim_{t\to\infty} \| x(t) - y(t) \| = 0$$

3.4.1 Synchronization of Perturbed Chen Chaotic Dynamical System Using Active Control

In this section, we will give some particular active control which ensures synchronization of drive system and response system of perturbed Chen chaotic dynamical system. System (3.2) has chaotic behavior at the parameters values $a = 35$, $b = 3$, $c = 28$ and $d = 2$. Our aim is to make synchronization of system (3.2) by using active control. The drive system is defined as follows,

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1) \\ \dot{y}_1 &= (c-a)x_1 - x_1 z_1 + c y_1 \\ \dot{z}_1 &= x_1 y_1 - b z_1 + d x_1^2 \end{aligned} \tag{3.12}$$

and the response system is given by

$$\begin{aligned} \dot{x}_2 &= a(y_2 - x_2) + \mu_1(t) \\ \dot{y}_2 &= (c-a)x_2 - x_2 z_2 + c y_2 + \mu_2(t) \\ \dot{z}_2 &= x_2 y_2 - b z_2 + d x_2^2 + \mu_3(t). \end{aligned} \tag{3.13}$$

We have introduced three control functions $\mu_1(t), \mu_2(t)$ and $\mu_3(t)$ in (3.13). These functions are to be determined. Let the error states be

$$\begin{aligned} x_3 &= x_2 - x_1 \\ y_3 &= y_2 - y_1 \\ z_3 &= z_2 - z_1. \end{aligned}$$

Using this notation, we obtain the error system.

$$\begin{aligned} \dot{x}_3 &= a(y_3 - x_3) + \mu_1(t) \\ \dot{y}_3 &= (c-a)x_3 + c y_3 - x_2 z_2 + x_1 z_1 + \mu_2(t) \\ \dot{z}_3 &= -b z_3 - x_1 y_1 + x_2 y_2 + d x_2^2 - d x_1^2 + \mu_3(t). \end{aligned} \tag{3.14}$$

We define the active control functions $\mu_1(t), \mu_2(t)$ and $\mu_3(t)$ as

$$\begin{aligned}
\mu_1(t) &= V_1(t) \\
\mu_2(t) &= x_2z_2 - x_1z_1 + V_2(t) \\
\mu_3(t) &= x_1y_1 - x_2y_2 - dx_2^2 + dx_1^2 + V_3(t).
\end{aligned} \tag{3.15}$$

Hence,

$$\begin{aligned}
\dot{x}_3 &= a(y_3 - x_3) + V_1(t) \\
\dot{y}_3 &= (c-a)x_3 + cy_3 + V_2(t) \\
\dot{z}_3 &= -bz_3 + V_3(t).
\end{aligned} \tag{3.16}$$

The control inputs $V_1(t), V_2(t)$ and $V_3(t)$ are functions of x_3, y_3 and z_3 and are chosen as

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = A \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \tag{3.17}$$

where the matrix A is given by

$$A = \begin{bmatrix} a-1 & -a & 0 \\ a-c & -(1+c) & 0 \\ 0 & 0 & b-1 \end{bmatrix},$$

With this particular choice of A, (3.16) has eigenvalues which are found to be -1, -1 and -1. The choice will lead to the error states x_3, y_3 and z_3 converge to zero as time t tends to infinity and this implies that the synchronization of perturbed Chen system is achieved.

Numerical Simulations

Fourth-order Runge-Kutta method of differential equations (3.12) and (3.13) with time step size 0.001 are used in all numerical simulations.

The parameters are selected in (3.12) as follow: $a = 35$, $b = 3$, $c = 28$ and $d = 2$ to ensure the chaotic behavior of perturbed Chen system. The initial value of the drive system are $x_1(0) = 0.5$, $y_1(0) = 1$ and $z_1(0) = 1$ and the initial value of the response system are $x_2(0) = 10.5$, $y_2(0) = 1$ and $z_2(0) = 38$. Then the initial value of the error system are $x_3(0) = 10$, $y_3(0) = 0$ and $z_3(0) = 37$.

The results of the simulation of the two identical perturbed Chen systems without active control are shown in Fig. 3.19 displays x_1 and x_2, Fig. 3.20 displays y_1 and y_2, Fig. 3.21 displays z_1 and z_2. Fig. 3.22-3.24 show the synchronization is occurred after applying active control at $t = 5$

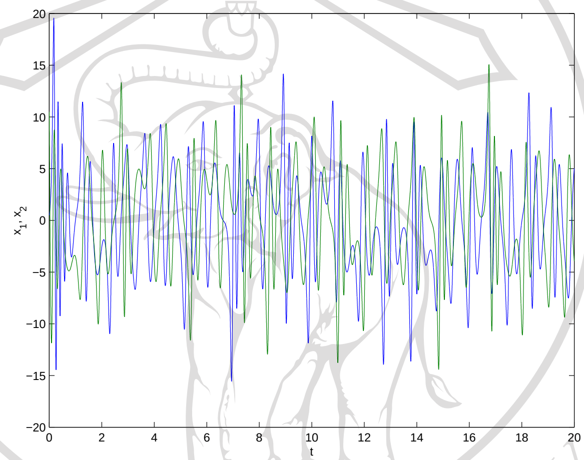

Figure 3.19: The states x_1, x_2 of the coupled perturbed Chen system of equations with the active control deactivated.

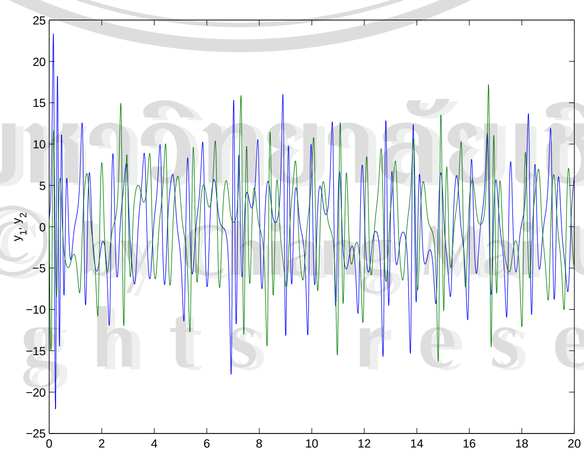

Figure 3.20: The states y_1, y_2 of the coupled perturbed Chen system of equations with the active control deactivated.

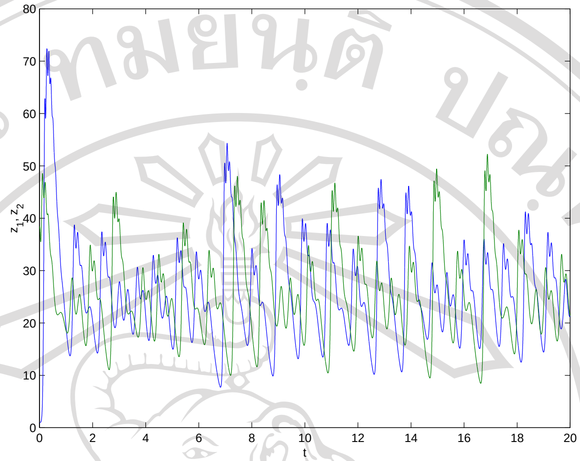

Figure 3.21: The states z_1, z_2 of the coupled perturbed Chen system of equations with the active control deactivated.

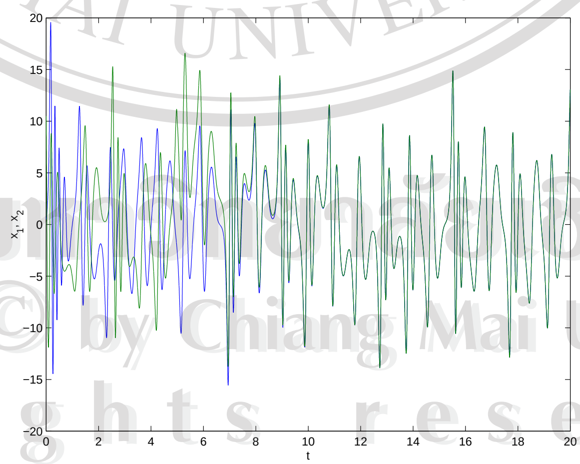

Figure 3.22: The states x_1, x_2 of the coupled perturbed Chen system of equations with the active control activated.

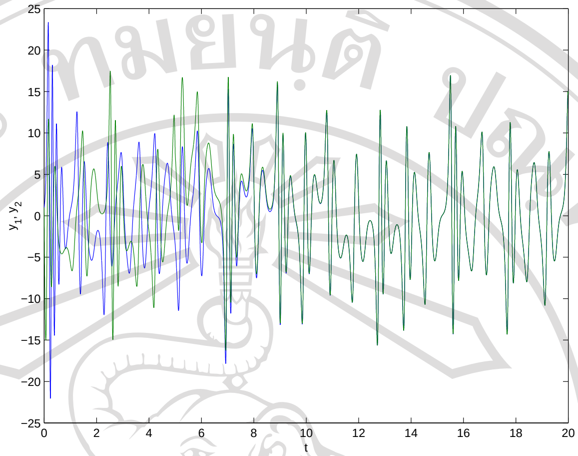

Figure 3.23: The states y_1, y_2 of the coupled perturbed Chen system of equations with the active control activated.

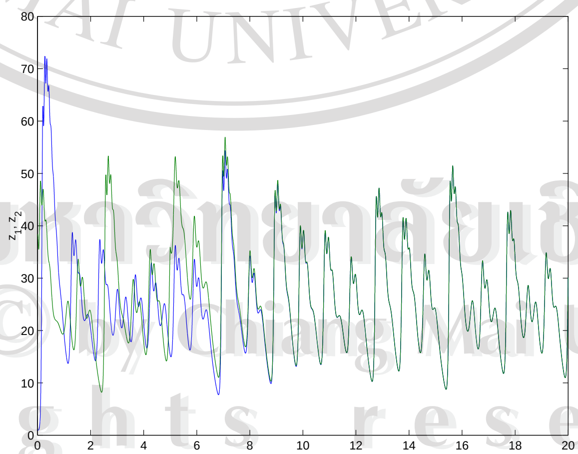

Figure 3.24: The states z_1, z_2 of the coupled perturbed Chen system of equations with the active control activated.

3.4.2 Adaptive Synchronization of Perturbed Chen Chaotic Dynamical System

This section considers adaptive synchronization of perturbed Chen system. This approach can synchronize the chaotic systems with fully unmatched parameters. The synchronization problem of perturbed Chen systems with fully unknown parameters will be studied in which the adaptive controller will be introduced.

Let system (3.2) be the drive system. Suppose that the parameters of the system (3.2) are unknown or uncertain, then the response system is given by

$$\begin{aligned} \dot{\tilde{x}} &= \hat{a}(\tilde{y}-\tilde{x}) - u_1 \\ \dot{\tilde{y}} &= (\hat{c}-\hat{a})\tilde{x} - \tilde{x}\tilde{z} + \hat{c}\tilde{y} - u_2 \\ \dot{\tilde{z}} &= \tilde{x}\tilde{y} - \hat{b}\tilde{z} + \hat{d}\tilde{x}^2 - u_3 \end{aligned} \tag{3.18}$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are parameters of the response system which need to be estimated. Suppose that

$$\begin{aligned} u_1 &= k_1 e_x \\ u_2 &= k_2 e_y \\ u_3 &= k_3 e_z + d\tilde{x}e_x \end{aligned} \tag{3.19}$$

where $e_x = \tilde{x} - x, e_y = \tilde{y} - y$ and $e_z = \tilde{z} - z$ and

$$\begin{aligned} \dot{\hat{a}} &= f_a = -\gamma(\tilde{y}-\tilde{x})\rho e_x + \gamma\tilde{x}e_y \\ \dot{\hat{b}} &= f_b = \theta\tilde{z}e_z \\ \dot{\hat{c}} &= f_c = -\beta(\tilde{x}+\tilde{y})e_y \\ \dot{\hat{d}} &= f_d = -\delta\tilde{x}^2 e_z \end{aligned} \tag{3.20}$$

where k_1, k_2, $k_3 \geq 0$ and ρ, γ, θ, β, $\delta > 0$ are constants.

Theorem 3.4.1 *Suppose that $M_{C_x} > |x|$, $M_{C_y} > |y|$, $M_{C_z} > |z|$, ρ, γ, θ, β, δ are positive constants. When k_1, k_2 and $k_3 \geq 0$ are properly chosen such that the following matrix inequality holds,*

$$P = \begin{bmatrix} \rho(k_1+a) & -\frac{1}{2}(\rho a - a + c + M_{C_z}) & -\frac{1}{2}(M_{C_y} + dM_{C_x}) \\ -\frac{1}{2}(\rho a - a + c + M_{C_z}) & k_2 - c & 0 \\ -\frac{1}{2}(M_{C_y} + dM_{C_x}) & 0 & k_3 + b \end{bmatrix} > 0 \quad (3.21)$$

or k_1, k_2 and k_3 can be chosen so that the following inequalities hold:

$$\begin{aligned} (i) \quad A &= \rho(k_1+a)(k_2-c) - \frac{1}{4}(\rho a - a + c + M_{C_z})^2 > 0 \\ (ii) \quad B &= A(k_3+b) - \frac{1}{4}(M_{C_y} + dM_{C_x})^2(k_2-c) > 0 \end{aligned} \quad (3.22)$$

then the two perturbed Chen systems (3.2) and (3.18) can be synchronized under the adaptive control of (3.19) and (3.20).

Proof It is easy to see from (3.2) and (3.18) that the error dynamics can be obtained as follow

$$\begin{aligned} \dot{e_x} &= \hat{a}(\tilde{y} - \tilde{x}) - a(y - x) - u_1 \\ \dot{e_y} &= -\hat{a}\tilde{x} + ax + \hat{c}\tilde{x} - cx + \hat{c}\tilde{y} - cy - \tilde{x}\tilde{z} + xz - u_2 \\ \dot{e_z} &= -\hat{b}\tilde{z} + bz + \tilde{x}\tilde{y} - xy - u_3 \end{aligned} \quad (3.23)$$

Let $e_a = \hat{a} - a, e_b = \hat{b} - b, e_c = \hat{c} - c, e_d = \hat{d} - d$. Choose the following Lyapunov function:

$$V(e_x, e_y, e_z) = \frac{1}{2}(\rho e_x^2 + e_y^2 + e_z^2 + \frac{1}{\gamma}e_a^2 + \frac{1}{\theta}e_b^2 + \frac{1}{\beta}e_c^2 + \frac{1}{\delta}e_d^2) \quad (3.24)$$

in which the differentiation of V along trajectories of (3.23) gives

$$
\begin{aligned}
\dot{V} &= \rho e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + \frac{1}{\gamma}e_a\dot{e}_a + \frac{1}{\theta}e_a\dot{e}_a + \frac{1}{\beta}e_b\dot{e}_b + \frac{1}{\delta}e_d\dot{e}_d \\
&= \rho e_x[\hat{a}(\tilde{y}-\tilde{x}) - a(y-x) - u_1] + e_y[-\hat{a}\tilde{x} + ax + \hat{c}\tilde{x} - cx + \hat{c}\tilde{y} - cy - \tilde{x}\tilde{z} + xz - u_2] \\
&\quad + e_z[-\hat{b}\tilde{z} + bz + \tilde{x}\tilde{y} - xy - u_3] + \frac{1}{\gamma}e_af_a + \frac{1}{\theta}e_bf_b + \frac{1}{\beta}e_cf_c + \frac{1}{\delta}e_df_d \\
&= [\rho\hat{a}(\tilde{y}-\tilde{x}) - \rho a(\tilde{y}-\tilde{x}) + \rho a(\tilde{y}-\tilde{x}) - \rho a(y-x)]e_x - \rho u_1e_x \\
&\quad + [-\hat{a}\tilde{x} + a\tilde{x} - a\tilde{x} + ax]e_y + [-\tilde{x}\tilde{z} + \tilde{x}z - \tilde{x}z + xz]e_y - u_2e_y \\
&\quad + [\hat{c}(\tilde{x}+\tilde{y}) - c(\tilde{x}+\tilde{y}) + c(\tilde{x}+\tilde{y}) - c(x+y)]e_y \\
&\quad + [-\hat{b}\tilde{z} + b\tilde{z} - b\tilde{z} + bz]e_z + [\tilde{x}\tilde{y} - \tilde{x}y + \tilde{x}y - xy]e_z \\
&\quad + [\hat{d}\tilde{x}^2 - d\tilde{x}^2 + d\tilde{x}^2 - dx^2]e_z - u_3e_z + \frac{1}{\gamma}e_af_a + \frac{1}{\theta}e_bf_b + \frac{1}{\beta}e_cf_c + \frac{1}{\delta}e_dfd \\
&= \rho(\tilde{y}-\tilde{x})e_ae_x + \rho a(e_y - e_x)e_x - \rho u_1e_x - \tilde{x}e_ae_y - ae_xe_y - \tilde{x}e_ye_z - ze_xe_y \\
&\quad - u_2e_y + e_ce_y(\tilde{x}+\tilde{y}) + c(e_x+e_y)e_y - \tilde{z}e_be_z - be_z^2 + \tilde{x}e_ye_z + ye_xe_z \\
&\quad + \tilde{x}^2e_de_z + de_xe_z(\tilde{x}+x) - u_3e_z + \frac{1}{\gamma}e_af_a + \frac{1}{\theta}e_bf_b + \frac{1}{\beta}e_cf_c + \frac{1}{\delta}e_df_d \\
&= \rho(\tilde{y}-\tilde{x})e_ae_x + \rho a(e_y - e_x)e_x - \rho k_1e_x^2 - \tilde{x}e_ae_y - ae_xe_y - \tilde{x}e_ye_z - ze_xe_y \\
&\quad - k_2e_y^2 + e_ce_y(\tilde{x}+\tilde{y}) + c(e_x+e_y)e_y - \tilde{z}e_be_z - be_z^2 + \tilde{x}e_ye_z + ye_xe_z + \tilde{x}^2e_de_z \\
&\quad + de_xe_z(\tilde{x}+x) - (k_3e_z + d\tilde{x}e_x)e_z + \frac{1}{\gamma}e_af_a + \frac{1}{\theta}e_bf_b + \frac{1}{\beta}e_cf_c + \frac{1}{\delta}e_df_d \\
&= -\rho(k_1+a)e_x^2 - (k_2-c)e_y^2 - (k_3+b)e_z^2 + (\rho a + c - a - z)e_xe_y + (y + xd)e_xe_z \\
&\quad + e_a[\frac{1}{\gamma}f_a + (\tilde{y}-\tilde{x})\rho e_x - \tilde{x}e_y] + e_b[\frac{1}{\theta}f_b - \tilde{z}e_z] + e_c[\frac{1}{\beta}f_c + (\tilde{x}+\tilde{y})e_y] \\
&\quad + e_d[\frac{1}{\delta}f_d + \tilde{x}^2e_z] \\
&< -\rho(k_1+a)e_x^2 - (k_2-c)e_y^2 - (k_3+b)e_z^2 + (\rho a + c - a - M_{C_z})|e_xe_y| \\
&\quad + (M_{C_y} + dM_{C_x})|e_xe_z| = -e^TPe
\end{aligned}
$$

where $e = \begin{bmatrix} |e_x| & |e_y| & |e_z| \end{bmatrix}^T$, P is as in (3.21). Thus the differentiation of $V(e_x, e_y, e_z)$ is negative definite, which implies that the origin of error system (3.23) is asymptotically stable. Therefore, the response system (3.18) is synchronizing with the drive system (3.2).

Numerical Simulations

The numerical simulations are carried out using the fourth-order Runge-Kutta method. The initial conditions of the drive and response systems are $(0.5, 1, 5)$ and $(10.5, 20, 38)$. The parameters of the drive system are $a = 35$, $b = 3$, $c = 28$ and $d = 2$.

In order to choose the control parameters, $M_{C_x} > |x|$, $M_{C_y} > |y|$ and $M_{C_z} > |z|$ must be estimated. Through simulations, we obtain $M_{C_x} \approx 20$, $M_{C_y} \approx 25$ and $M_{C_z} \approx 70$. Then we firstly choose $\rho = M_{C_y}^2/(ab)$. Then choose $\gamma = \theta = \beta = 1$ and then choose $k_1 = 25$, $k_2 = 88$, $k_3 = 50$ which satisfy (3.22) and the initial values of the parameters $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are all chosen to be 0, the response system synchronizes with the drive system as shown in Fig. 3.25-3.27 and the changing parameters of $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are shown in Fig. 3.28-3.31.

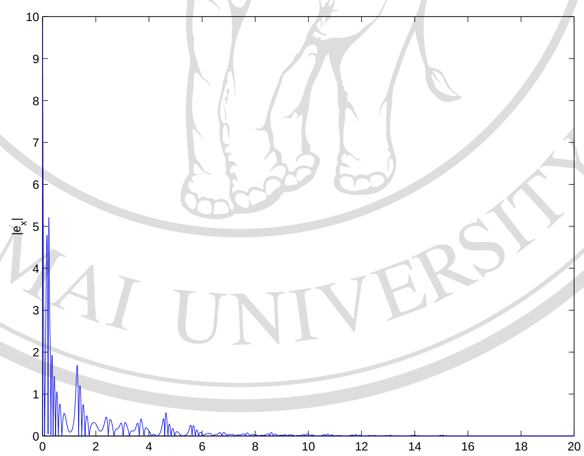

Figure 3.25: Synchronization errors: $| \, e_x \, |$.

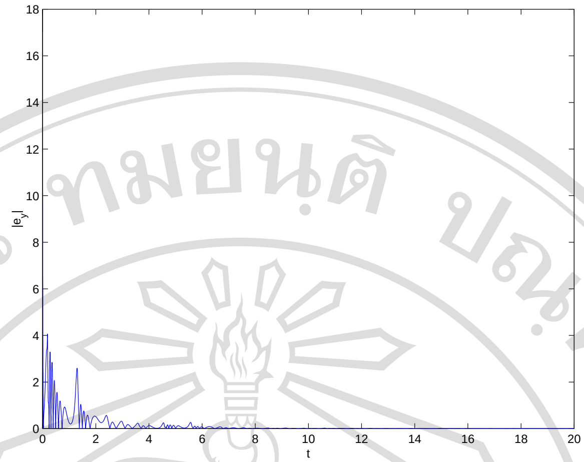

Figure 3.26: Synchronization errors: $|\, e_y \,|$.

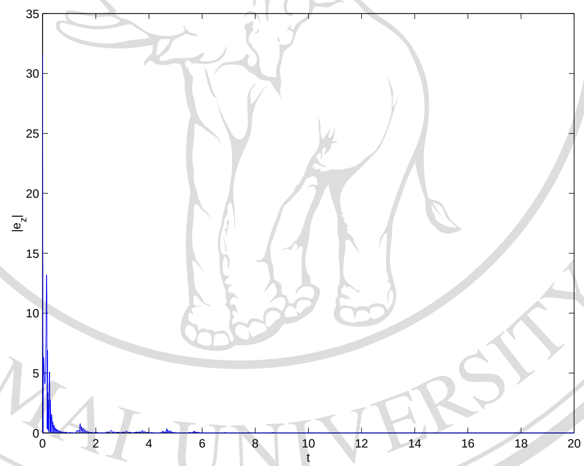

Figure 3.27: Synchronization errors: $|\, e_z \,|$.

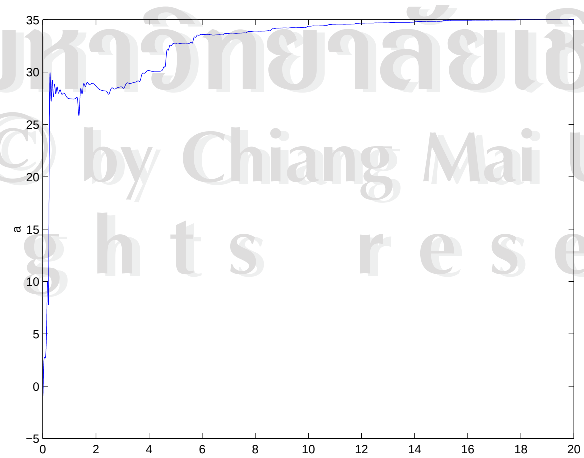

Figure 3.28: Changing parameters: $\hat{a}$.

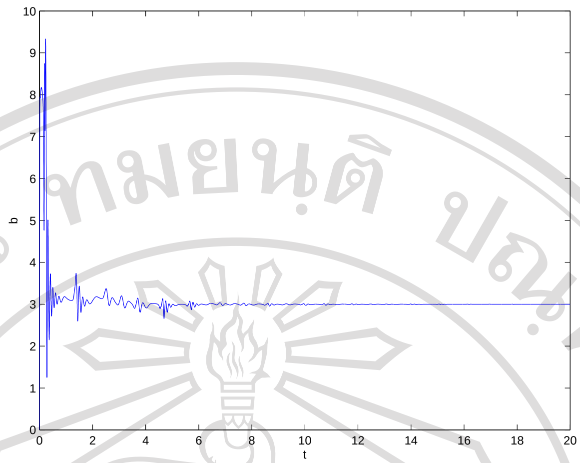

Figure 3.29: Changing parameters: $\hat{b}$.

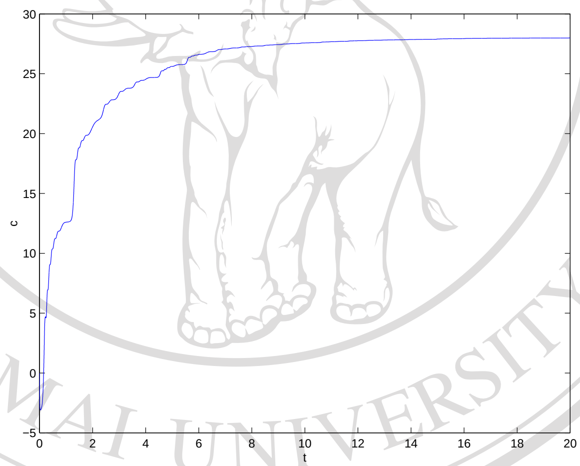

Figure 3.30: Changing parameters: $\hat{c}$.

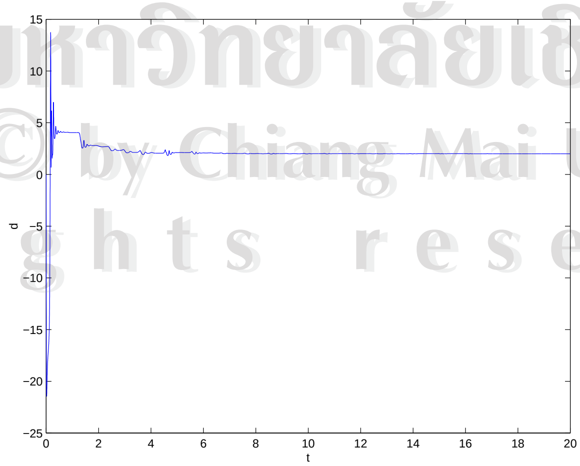

Figure 3.31: Changing parameters: $\hat{d}$.