CHAPTER 2

PRELIMINARIES

In this chapter, we give some notations that will be used in the later chapters.

2.1 Taylor Series

Let f be a function analytic at a point z_0. Denoted by C the largest circle centered at z_0 such that f is analytic at all the points interior to C, and let R be its radius. Then there exists a power series

$$\sum_{k\geq 0} a_k(z-z_0)^k$$

which converges to $f(z)$ in C. This series is unique; it is calls the Taylor series of f at z_0. The coefficients of this series are determined by the following formula:

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

The Taylor series of a function about 0 is called the Maclaurin series of f.

Generalizations

Taylor series can also be extended to functions of more than one variable. The two-variable Taylor series of $f(x,y)$ is

$$T(x,y) = \sum_{i=0}^{\infty}\sum_{i=0}^{\infty} \frac{f^{(i,j)}(x_0,y_0)}{i!j!}(x-x_0)^i(y-y_0)^j.$$

Where $f^{(i,j)}$ is the partial derivative of f taken with respect to x for i times and with respect to y for j times. We can generalize this to n variables, or functions $f(X)$, $X \in \mathbb{R}^{n\times 1}$. The Taylor series of this function of a vector is then

$$T(X) = \sum_{i_1=0}^{\infty}\cdots\sum_{i_n=0}^{\infty} \frac{f^{(i_1,i_2,\ldots,i_n)}(x_{01},x_{02},\ldots,x_{0n})}{i_1!i_2!\cdots i_n!}(x_1-x_{01})^{i_1}(x_2-x_{02})^{i_2}\cdots(x_n-x_{0n})^{i_n}.$$

There are some generalization to Taylor series.

2.2 Chebyshev Polynomials

Chebyshev polynomials are used in many parts of numerical analysis, and more generally, in applications of mathematics. For an integer $n \geq 0$, we define the function

$$T_n(x) = \cos(n \cos^{-1} x), \qquad -1 \leq x \leq 1. \tag{2.1}$$

This may not appear to be a polynomial, but we will show that it is a polynomial of degree n. To simplify the manipulation of (2.1), we introduce

$$\theta = \cos^{-1}(x) \qquad \text{or} \qquad x = \cos(\theta), \qquad 0 \leq \theta \leq \pi. \tag{2.2}$$

Then

$$T_n(x) = \cos(n\theta). \tag{2.3}$$

Example

$$\begin{aligned}
&n = 0; \\
&\quad T_0(x) \;=\; \cos(0 \cdot 1) = 1 \\
&n = 1; \\
&\quad T_1(x) \;=\; \cos(\theta) = x \\
&n = 2; \\
&\quad T_2(x) \;=\; \cos(2\theta) = 2\cos^2(\theta) - 1 = 2x^2 - 1 \\
&n = 3; \\
&\quad T_3(x) \;=\; \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) = 4x^3 - 3x.
\end{aligned}$$

The triple recursion relation

Recall the trigonometric addition formulas

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta).$$

Let $n \geq 1$, and apply these identities to get

$$\begin{aligned} T_{n+1}(x) &= \cos[(n+1)\theta] = \cos(n\theta + \theta) \\ &= \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta) \\ T_{n-1}(x) &= \cos[(n-1)\theta] = \cos(n\theta - \theta) \\ &= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta). \end{aligned}$$

Add these two equations, and then use (2.1) and (2.3) to obtain

$$\begin{aligned} T_{n+1}(x) + T_{n-1}(x) &= 2\cos(n\theta)\cos(\theta) = 2xT_n(x) \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1. \end{aligned} \tag{2.4}$$

This is called the triple recursion relation for the Chebyshev polynomials.

Chebyshev approximation

The Chebyshev polynomials are orthogonal in the interval $[-1, 1]$ over a weight $(1 - x^2)^{-1/2}$. In particular,

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & , i \neq j \\ \pi/2 & , i = j \neq 0 \\ \pi & , i = j = 0. \end{cases} \tag{2.5}$$

The polynomial $T_n(x)$ has n zeros in the interval $[-1, 1]$, and they are located at the points

$$x = \cos\Big(\frac{\pi(k - 1/2)}{n}\Big), \qquad k = 1, 2, \ldots, n. \tag{2.6}$$

In this same interval there are $n + 1$ extrema(maxima and minima), located at

$$x = \cos\Big(\frac{\pi k}{n}\Big), \qquad k = 0, 1, \ldots, n. \tag{2.7}$$

At all of maxima $T_n(x) = 1$, while at all of the minima $T_n(x) = -1$; it is precisely this property that makes the Chebyshev polynomials so useful in polynomial approximation of functions. The Chebyshev polynomials satisfy a discrete orthogonality relation as well as the continuous one (2.5). If x_k $(k = 1, 2, \ldots, m)$ are the m zeros of $T_m(x)$ given by (2.6), and if $i, j < m$, then

$$\sum_{k=1}^{m} T_i(x_k)T_j(x_k) = \begin{cases} 0 & , i \neq j \\ m/2 & , i = j \neq 0 \\ m & , i = j = 0. \end{cases} \tag{2.8}$$

If $f(x)$ is an arbitrary function in the interval $[-1, 1]$, and if N coefficients $c_j,\ j = 0, 1, \ldots, N-1$, are defined by

$$\begin{aligned} c_j &= \frac{2}{N}\sum_{k=1}^{N} f(x_k)T_j(x_k) \\ &= \frac{2}{N}\sum_{k=1}^{N} f\Big[\cos(\frac{\pi(k-1/2)}{N})\Big]\cos\Big(\frac{\pi j(k-1/2)}{N}\Big) \end{aligned} \tag{2.9}$$

then the approximation formula

$$f(x) \approx \Big[\sum_{k=0}^{N-1} c_k T_k(x)\Big] - \frac{1}{2}c_0 \tag{2.10}$$

is exact for x equal to all of the N zeros of $T_N(x)$.