CHAPTER 3

MAIN RESULTS

3.1 Method of Taylor Polynomial Solutions

3.1.1 First Method

In this section, we study the integro-differential equations

$$P(x)y'' + Q(x)y' + R(x)y = f(x) + \int_a^b K(x,\xi)d\xi \tag{3.1}$$

and its solution $y(x)$ in the form

$$y(x) = \sum_{n=0}^{N} \frac{1}{n!} y^{(n)}(c)(x-c)^n, \quad a \le c \le b. \tag{3.2}$$

By differentiating (3.1) n times with respect to x, we have

$$\Big[P(x)y''(x)\Big]^{(n)} + \Big[Q(x)y'(x)\Big]^{(n)} + \Big[R(x)y(x)\Big]^{(n)} = f^{(n)}(x) + \int_a^b \frac{\partial^{(n)} K(x,\xi)}{\partial x^{(n)}} d\xi$$

where $n = 0, 1, \ldots, N$, hence

$$\sum_{m=0}^{n} \binom{n}{m} \Big\{ P^{(n-m)}(x)y^{(m+2)}(x) + Q^{(n-m)}(x)y^{(m+1)}(x) + R^{(n-m)}(x)y^{(m)}(x) \Big\}$$

$$= f^{(n)}(x) + \int_a^b \frac{\partial^{(n)} K(x,\xi)}{\partial x^{(n)}} d\xi. \tag{3.3}$$

By substituting $x = c$, we get

$$\sum_{m=0}^{n} \binom{n}{m} \Big\{ P^{(n-m)}(c)y^{(m+2)}(c) + Q^{(n-m)}(c)y^{(m+1)}(c) + R^{(n-m)}(c)y^{(m)}(c) \Big\}$$

$$= f^{(n)}(c) + \int_a^b \frac{\partial^{(n)} K(x,\xi)}{\partial x^{(n)}}\Big|_{x=c} d\xi \tag{3.4}$$

where $n = 0, 1, \ldots, N$. This is a system of $(N+1)$ equations for the $N+1$ unknown coefficients $y^{(0)}(c),\ y^{(1)}(c), \ldots,\ y^{(N)}(c)$. Here $P^{(i)}(c),\ Q^{(i)}(c),\ R^{(i)}(c)$,

$f^{(i)}(c)$ and $\frac{\partial^{(i)}K(x,\xi)}{\partial x^{(i)}}|_{x=c}$, respectively, denote the values of the i^{th} derivatives of the known functions $P,\ Q,\ R,\ f$ and $K(x,\xi)$ at $x=c$. We can write the matrix form of this system is

$$W_1 Y \;=\; G \tag{3.5}$$

where

$$Y \;=\; \begin{bmatrix} y^{(0)}(c) & y^{(1)}(c) & \ldots & y^{(N)}(c) \end{bmatrix}^T,$$

$$G \;=\; \begin{bmatrix} f^{(0)}(c)+\int_a^b K(x,\xi)|_{x=c}d\xi \\ f^{(1)}(c)+\int_a^b \frac{\partial^{(1)}K(x,\xi)}{\partial x^{(1)}}|_{x=c}d\xi \\ \vdots \\ f^{(N)}(c)+\int_a^b \frac{\partial^{(N)}K(x,\xi)}{\partial x^{(N)}}|_{x=c}d\xi \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_N \end{bmatrix},$$

and

$$W_1 = \begin{bmatrix} w_{nm} \end{bmatrix}; \quad n,m = 0,1,\ldots,N.$$

The elements w_{nm} of W_1 are defined by

$$w_{nm} = \binom{n}{m-2}P^{(n-m+2)}(c) + \binom{n}{m-1}Q^{(n-m+1)}(c) + \binom{n}{m}R^{(n-m)}(c).$$

Note that for $k<0$; $P^{(k)}(c)=0,\ Q^{(k)}(c)=0,\ R^{(k)}(c)=0$ and for $j<0$ and $j>i$; $\binom{i}{j}=0$, where $i,\ j$ and k are integers. Hence, the augmented matrix of the system (3.4) becomes

$$\begin{bmatrix} W_1 & ; & G \end{bmatrix}. \tag{3.6}$$

Next, we consider integro-differential equation

$$P(x)y'' + Q(x)y' + R(x)y \;=\; f(x) + \int_a^b K(x,\xi)y(\xi)d\xi \tag{3.7}$$

and its solution $y(x)$ in the form of expression (3.2). By differentiating (3.7) n times with respect to x, we have

$$\Big[P(x)y''(x)\Big]^{(n)} + \Big[Q(x)y'(x)\Big]^{(n)} + \Big[R(x)y(x)\Big]^{(n)} = f^{(n)}(x) + \int_a^b \frac{\partial^{(n)}K(x,\xi)}{\partial x^{(n)}}y(\xi)d\xi$$

where $n = 0, 1, \ldots, N$. Thus,

$$\sum_{m=0}^{n} \binom{n}{m} \Big\{ P^{(n-m)}(x) y^{(m+2)}(x) + Q^{(n-m)}(x) y^{(m+1)}(x) + R^{(n-m)}(x) y^{(m)}(x) \Big\} = f^{(n)}(x) + \int_a^b \frac{\partial^{(n)} K(x,\xi)}{\partial x^{(n)}} y(\xi) d\xi. \quad (3.8)$$

By substituting $x = c$, we have

$$\sum_{m=0}^{n} \binom{n}{m} \Big\{ P^{(n-m)}(c) y^{(m+2)}(c) + Q^{(n-m)}(c) y^{(m+1)}(c) + R^{(n-m)}(c) y^{(m)}(c) \Big\} = f^{(n)}(c) + \int_a^b \frac{\partial^{(n)} K(x,\xi)}{\partial x^{(n)}} \Big|_{x=c} y(\xi) d\xi \quad (3.9)$$

$n = 0, 1, \ldots, N$. Next, we expand $y(\xi)$ in Taylor series at $\xi = c$, namely,

$$y(\xi) = \sum_{m=0}^{n} \frac{1}{m!} y^{(m)}(c) (\xi - c)^m \quad (3.10)$$

and substitute it in (3.9), we get

$$\sum_{m=0}^{n} \binom{n}{m} \Big\{ P^{(n-m)}(c) y^{(m+2)}(c) + Q^{(n-m)}(c) y^{(m+1)}(c) + R^{(n-m)}(c) y^{(m)}(c) \Big\} = f^{(n)}(c) + \int_a^b \frac{\partial^{(n)} K(x,\xi)}{\partial x^{(n)}} \Big|_{x=c} \Big[\sum_{m=0}^{n} \frac{1}{m!} y^{(m)}(c) (\xi - c)^m \Big] d\xi$$

or

$$\sum_{m=0}^{n} \binom{n}{m} \Big\{ P^{(n-m)}(c) y^{(m+2)}(c) + Q^{(n-m)}(c) y^{(m+1)}(c) + R^{(n-m)}(c) y^{(m)}(c) \Big\} = f^{(n)}(c) + \sum_{m=0}^{n} T_{nm} y^{(m)}(c)$$

or

$$\sum_{m=0}^{n} \Big[\binom{n}{m} \Big\{ P^{(n-m)}(c) y^{(m+2)}(c) + Q^{(n-m)}(c) y^{(m+1)}(c) + R^{(n-m)}(c) y^{m}(c) \Big\} - T_{nm} y^{(m)}(c) \Big] = f^{(n)}(c) \quad (3.11)$$

where

$$T_{nm} = \frac{1}{m!} \int_a^b \frac{\partial^{(n)} K(x,\xi)}{\partial x^{(n)}} \Big|_{x=c} (\xi - c)^m d\xi, \quad n = 0, 1, \ldots, N.$$

This is a system of $(N+1)$ equations for the $N+1$ unknown coefficients $y^{(0)}(c),\ y^{(1)}(c),\ldots,\ y^{(N)}(c)$. The matrix form of system (3.11) can be written as

$$W_2 Y = F \tag{3.12}$$

where

$$\begin{aligned} Y &= \left[\ y^{(0)}(c)\quad y^{(1)}(c)\quad \ldots\quad y^{(N)}(c)\ \right]^T, \\ F &= \left[\ f^{(0)}(c)\quad f^{(1)}(c)\quad \ldots\quad f^{(N)}(c)\ \right]^T = \left[\ f_0\quad f_1\quad \ldots\quad f_N\ \right]^T, \end{aligned}$$

and

$$W_2 = \left[\ w_{nm}\ \right];\quad n,m = 0,1,\ldots,N.$$

The elements w_{nm} of W_2 which are defined by

$$w_{nm} = \binom{n}{m-2}P^{(n-m+2)}(c) + \binom{n}{m-1}Q^{(n-m+1)}(c) + \binom{n}{m}R^{(n-m)}(c) - T_{nm}.$$

Note that for $k<0$; $P^{(k)}(c)=0,\ Q^{(k)}(c)=0,\ R^{(k)}(c)=0$ and for $j<0$ and $j>i$; $\binom{i}{j}=0$, where $i,\ j$ and k are integers. Thus, the augmented matrix of the system (3.11) becomes

$$\left[\ W_2\ \ ;\ \ F\ \right]. \tag{3.13}$$

Since the solution $y(x)$ of (3.1) and (3.7) satisfies the conditions (1.5) and (1.6), we have the following matrix equations

$$y^{(0)}(x) = \left[\ \tfrac{1}{0!}\quad \tfrac{(x-c)}{1!}\quad \tfrac{(x-c)^2}{2!}\quad \ldots\quad \tfrac{(x-c)^N}{N!}\ \right] Y$$

and

$$y^{(1)}(x) = \left[\ 0\quad \tfrac{1}{0!}\quad \tfrac{(x-c)}{1!}\quad \ldots\quad \tfrac{(x-c)^{N-1}}{(N-1)!}\ \right] Y$$

where

$$Y = \left[\ y^{(0)}(c)\quad y^{(1)}(c)\quad \ldots\quad y^{(N)}(c)\ \right]^T.$$

As a results, $y^{(i)}(a),\ y^{(i)}(b)$ and $y^{(i)}(c),\ i=0,1$ can be written in the matrix forms as follow

$$\begin{aligned}
y^{(0)}(c) &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} Y \\
y^{(0)}(a) &= \begin{bmatrix} \frac{1}{0!} & \frac{h}{1!} & \frac{h^2}{2!} & \dots & \frac{h^N}{N!} \end{bmatrix} Y \\
y^{(0)}(b) &= \begin{bmatrix} \frac{1}{0!} & \frac{k}{1!} & \frac{k^2}{2!} & \dots & \frac{k^N}{N!} \end{bmatrix} Y \\
y^{(1)}(c) &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix} Y \\
y^{(1)}(a) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{h}{1!} & \dots & \frac{h^{N-1}}{(N-1)!} \end{bmatrix} Y \\
y^{(1)}(b) &= \begin{bmatrix} 0 & \frac{1}{0!} & \frac{k}{1!} & \dots & \frac{k^{N-1}}{(N-1)!} \end{bmatrix} Y
\end{aligned} \tag{3.14}$$

where $h=a-c$ and $k=b-c$. By substituting quantities (3.14) into (1.5) and (1.6) and after simplifying, we have

$$\sum_{i=0}^{1}[a_i y^{(i)}(a) + b_i y^{(i)}(b) + c_i y^{(i)}(c)] = \lambda$$

or in the matrix form

$$\begin{aligned}
& a_0 \begin{bmatrix} \frac{1}{0!} & \frac{h}{1!} & \frac{h^2}{2!} & \dots & \frac{h^N}{N!} \end{bmatrix} Y + b_0 \begin{bmatrix} \frac{1}{0!} & \frac{k}{1!} & \frac{k^2}{2!} & \dots & \frac{k^N}{N!} \end{bmatrix} Y \\
& + c_0 \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} Y + a_1 \begin{bmatrix} 0 & \frac{1}{0!} & \frac{h}{1!} & \dots & \frac{h^{N-1}}{(N-1)!} \end{bmatrix} Y \\
& + b_1 \begin{bmatrix} 0 & \frac{1}{0!} & \frac{k}{1!} & \dots & \frac{k^{N-1}}{(N-1)!} \end{bmatrix} Y + c_1 \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix} Y = \begin{bmatrix} \lambda \end{bmatrix}
\end{aligned}$$

or

$$\begin{bmatrix} u_0 & u_1 & \dots & u_N \end{bmatrix} Y = \begin{bmatrix} \lambda \end{bmatrix}.$$

Also,

$$\sum_{i=0}^{1}[\alpha_i y^{(i)}(a) + \beta_i y^{(i)}(b) + \gamma_i y^{(i)}(c)] = \mu$$

or in the matrix form

$$\begin{aligned}&\alpha_0\begin{bmatrix}\frac{1}{0!} & \frac{h}{1!} & \frac{h^2}{2!} & \dots & \frac{h^N}{N!}\end{bmatrix}Y+\beta_0\begin{bmatrix}\frac{1}{0!} & \frac{k}{1!} & \frac{k^2}{2!} & \dots & \frac{k^N}{N!}\end{bmatrix}Y\\&+\gamma_0\begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}Y+\alpha_1\begin{bmatrix}0 & \frac{1}{0!} & \frac{h}{1!} & \dots & \frac{h^{N-1}}{(N-1)!}\end{bmatrix}Y\\&+\beta_1\begin{bmatrix}0 & \frac{1}{0!} & \frac{k}{1!} & \dots & \frac{k^{N-1}}{(N-1)!}\end{bmatrix}Y+\gamma_1\begin{bmatrix}0 & 1 & 0 & \dots & 0\end{bmatrix}Y = \begin{bmatrix}\mu\end{bmatrix}\end{aligned}$$

or

$$\begin{bmatrix}v_0 & v_1 & \dots & v_N\end{bmatrix}Y = \begin{bmatrix}\mu\end{bmatrix}$$

where u_j and $v_j,\ j=0,1,\dots,N$ are constants. We have the augmented matrices

$$\begin{bmatrix}u_0 & u_1 & \dots & u_N & ; & \lambda\end{bmatrix} \text{ and } \begin{bmatrix}v_0 & v_1 & \dots & v_N & ; & \mu\end{bmatrix}. \tag{3.15}$$

Consequently, by replacing the matrices (3.15) into the last two rows of the augmented matrices (3.6) and (3.13), we have the new augmented matrices of (3.4) and (3.11) as

$$W_1^* = \begin{bmatrix} w_{00} & w_{01} & w_{02} & 0 & \dots & 0 & ; & g_0\\ w_{10} & w_{11} & w_{12} & w_{13} & \dots & 0 & ; & g_1\\ w_{20} & w_{21} & w_{22} & w_{23} & \dots & 0 & ; & g_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ w_{N-2,0} & w_{N-2,1} & w_{N-2,2} & w_{N-2,3} & \dots & w_{N-2,N} & ; & g_{N-2}\\ u_0 & u_1 & u_2 & u_3 & \dots & u_N & ; & \lambda\\ v_0 & v_1 & v_2 & v_3 & \dots & v_N & ; & \mu \end{bmatrix} \tag{3.16}$$

and

$$W_2^* = \begin{bmatrix} w_{00} & w_{01} & w_{02} & 0 & \dots & 0 & ; & f_0\\ w_{10} & w_{11} & w_{12} & w_{13} & \dots & 0 & ; & f_1\\ w_{20} & w_{21} & w_{22} & w_{23} & \dots & 0 & ; & f_2\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ w_{N-2,0} & w_{N-2,1} & w_{N-2,2} & w_{N-2,3} & \dots & w_{N-2,N} & ; & f_{N-2}\\ u_0 & u_1 & u_2 & u_3 & \dots & u_N & ; & \lambda\\ v_0 & v_1 & v_2 & v_3 & \dots & v_N & ; & \mu \end{bmatrix}. \tag{3.17}$$

Let

$$W_1^{**} = W_2^{**} = \begin{bmatrix} w_{00} & w_{01} & w_{02} & 0 & \dots & 0 \\ w_{10} & w_{11} & w_{12} & w_{13} & \dots & 0 \\ w_{20} & w_{21} & w_{22} & w_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ w_{N-2,0} & w_{N-2,1} & w_{N-2,2} & w_{N-2,3} & \dots & w_{N-2,N} \\ u_0 & u_1 & u_2 & u_3 & \dots & u_N \\ v_0 & v_1 & v_2 & v_3 & \dots & v_N \end{bmatrix}.$$

If $detW_1^{**} = detW_2^{**} \neq 0$, we can write

$$Y = (W_1^{**})^{-1}G \qquad (3.18)$$

$$Y = (W_2^{**})^{-1}F \qquad (3.19)$$

where

$$Y = \left[\begin{array}{cccc} y^{(0)}(c) & y^{(1)}(c) & \dots & y^{(N)}(c) \end{array} \right]^T.$$

Thus the coefficients $y^{(n)}(c),\ n = 0, 1, \dots, N$ are uniquely determined. These solutions are given by the Taylor polynomial in the form of (3.2). We can easily check the accuracy of this solution. Since the Taylor polynomial (3.2) is an approximate solution of (3.1) and (3.7). By substituting $y(x)$ and its derivatives $y^{(1)}(x)$ and $y^{(2)}(x)$ in (3.1) and (3.7), the resulting equations will be satisfied approximately for a given value of x in the interval $a \leq x \leq b$. $x = x_i \in [a, b],\ i = 0, 1, \dots, M$, we define the relative error of the obtained solutions of (3.1) and (3.7) as follow

$$D_1(x_i) = \frac{\left| P(x_i)y^{(2)}(x_i) + Q(x_i)y^{(1)}(x_i) + R(x_i)y(x_i) - f(x_i) - \int_a^b K(x_i, \xi)d\xi \right|}{\left| f(x_i) + \int_a^b K(x_i, \xi)d\xi \right|}$$

and

$$D_2(x_i) = \frac{\left| P(x_i)y^{(2)}(x_i) + Q(x_i)y^{(1)}(x_i) + R(x_i)y(x_i) - f(x_i) - \int_a^b K(x_i, \xi)y(\xi)d\xi \right|}{\left| f(x_i) + \int_a^b K(x_i, \xi)y(\xi)d\xi \right|}.$$

If max $\{D_1(x_i), D_2(x_i)\} \leq 10^{-k}$ where k is a prescribed integer, then the truncation limit N is increased until the difference $D_1(x_i)$ or $D_2(x_i)$ at each the points x_i becomes smaller than the prescribed tolerance (10^{-k}).

Example 3.1

Let us consider the problem

$$y'' + xy' + xy = 1 + x + x^2 + \int_{-1}^{1} \sin(x(\xi+1))d\xi$$

$$y(0) = 1, \qquad y'(0) + 2y(1) - y(-1) = -1$$

and approximate the solution $y(x)$ by the Taylor polynomial.
Since $P(x) = 1,\ Q(x) = x,\ R(x) = x,\ f(x) = 1+x+x^2,\ K(x,\xi) = \sin(x(\xi+1))$ and $a = -1,\ b = 1,\ c = 0$, we have

$$\begin{aligned}
P^{(0)}(0) &= 1,\ P^{(1)}(0) = P^{(2)}(0) = \ldots = 0\\
Q^{(0)}(0) &= 0,\ Q^{(1)}(0) = 1,\ Q^{(2)}(0) = Q^{(3)}(0) = \ldots = 0\\
R^{(0)}(0) &= 0,\ R^{(1)}(0) = 1,\ R^{(2)}(0) = R^{(3)}(0) = \ldots = 0\\
f^{(0)}(0) &= 1,\ f^{(1)}(0) = 1,\ f^{(2)}(0) = 2,\ f^{(3)}(0) = f^{(4)}(0 = \ldots = 0.
\end{aligned}$$

Then, by using these quantities and relation (3.4) for $N = 9$, we have the augmented matrix as

$$W = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1\\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 3\\
0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 2\\
0 & 0 & 3 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & ; & -4\\
0 & 0 & 0 & 4 & 4 & 0 & 1 & 0 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 5 & 5 & 0 & 1 & 0 & 0 & ; & \frac{32}{3}\\
0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 1 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 0 & 1 & ; & -32\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & ; & \frac{512}{5}
\end{bmatrix}.$$

For the conditions, by taking $c = 0,\ a = -1,\ b = 1$ and using relations (3.14), we have

$$y^{(0)}(0) \ = \ 1$$

or in the matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} Y \ = \ \begin{bmatrix} 1 \end{bmatrix}$$

or in the augmented matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \end{bmatrix}.$$

Also,

$$y^{(1)}(0) + 2y(1) - y(-1) \ = \ -1$$

or in the matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} Y + 2\begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{8!} & \frac{1}{9!} \end{bmatrix} Y$$
$$-\begin{bmatrix} 1 & -1 & \frac{1}{2} & -\frac{1}{3!} & \frac{1}{4!} & -\frac{1}{5!} & \frac{1}{6!} & -\frac{1}{7!} & \frac{1}{8!} & -\frac{1}{9!} \end{bmatrix} Y = \begin{bmatrix} -1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{24} & \frac{1}{40} & \frac{1}{720} & \frac{1}{1680} & \frac{1}{40320} & \frac{1}{120960} \end{bmatrix} Y \ = \ \begin{bmatrix} -1 \end{bmatrix}$$

or in the augmented matrix form

$$\begin{bmatrix} 1 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{24} & \frac{1}{40} & \frac{1}{720} & \frac{1}{1680} & \frac{1}{40320} & \frac{1}{120960} & ; & -1 \end{bmatrix}.$$

Thus, the new augmented matrix for this problem is

$$W^* = \left[\begin{array}{cccccccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 3 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 2 \\ 0 & 0 & 3 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & ; & -4 \\ 0 & 0 & 0 & 4 & 4 & 0 & 1 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 5 & 5 & 0 & 1 & 0 & 0 & ; & \frac{32}{3} \\ 0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 1 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 0 & 1 & ; & -32 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 1 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{24} & \frac{1}{40} & \frac{1}{720} & \frac{1}{1680} & \frac{1}{40320} & \frac{1}{120960} & ; & -1 \end{array}\right]$$

and has the solution

$$Y = \left[\begin{array}{cccccccccc} 1 & -\frac{2263}{2471} & 1 & \frac{866}{297} & \frac{1991}{1087} & -\frac{1559}{99} & -\frac{1880}{99} & \frac{9469}{118} & \frac{6878}{33} & -\frac{11059}{24} \end{array}\right]^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$y(x) = 1 - \frac{2263}{2471}x + \frac{1}{2!}x^2 + \frac{866}{297 \cdot 3!}x^3 + \frac{1991}{1087 \cdot 4!}x^4$$
$$-\frac{1559}{99 \cdot 5!}x^5 - \frac{1880}{99 \cdot 6!}x^6 + \frac{9469}{118 \cdot 7!}x^7 + \frac{6878}{33 \cdot 8!}x^8 - \frac{11059}{24 \cdot 9!}x^9.$$

Also, by taking $N = 14,\ c = 0$ and then, following the same procedure, we have the solution

$$y(x) = 1 - \frac{239}{261}x + \frac{1}{2!}x^2 + \frac{761}{261 \cdot 3!}x^3 + \frac{478}{261 \cdot 4!}x^4 - \frac{1370}{87 \cdot 5!}x^5$$
$$-\frac{1652}{87 \cdot 6!}x^6 + \frac{4253}{53 \cdot 7!}x^7 + \frac{6044}{29 \cdot 8!}x^8 - \frac{29491}{64 \cdot 9!}x^9 - \frac{25402}{11 \cdot 10!}x^{10}$$
$$+\frac{47477}{20 \cdot 11!}x^{11} + \frac{83102}{3 \cdot 12!}x^{12} - \frac{13672}{13 \cdot 13!}x^{13} - \frac{1082683}{3 \cdot 14!}x^{14}.$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_1(x_i)$ at any point x_i in

$[-1, 1]$. Also, by taking $N = 9$ and $N = 14$

i	x_i	$y(x_i), N = 9$	$D(x_i)$	$y(x_i), N = 14$	$D(x_i)$
0	-1.0	2.10154274	1.50281×10^{-1}	2.10079470	2.62972×10^{-4}
1	-0.8	1.86888763	2.31807×10^{-2}	1.86873173	1.22263×10^{-5}
2	-0.6	1.64304306	3.37000×10^{-3}	1.64297308	3.83226×10^{-7}
3	-0.4	1.41839443	2.21406×10^{-2}	1.41834944	2.94890×10^{-7}
4	-0.2	1.19943917	3.36499×10^{-7}	1.19941636	2.53070×10^{-12}
5	0	1.00000000	0.00000	1.00000000	0.00000
6	0.2	0.84080170	8.75680×10^{-8}	0.84082448	4.54495×10^{-13}
7	0.4	0.74530379	1.53948×10^{-5}	0.74534818	9.08167×10^{-11}
8	0.6	0.73445128	2.93920×10^{-4}	0.73451151	8.24481×10^{-9}
9	0.8	0.82153907	2.30664×10^{-3}	0.82155900	9.29268×10^{-8}
10	1.0	1.00868328	1.12028×10^{-2}	1.00825021	1.54324×10^{-6}.

Example 3.2

Let us consider the problem

$$y'' + xy' + xy = 1 + x + x^2 + \int_{-1}^{1} (1 - 3x\xi) y(\xi) d\xi$$

$$y(0) = 1, \qquad y'(0) + 2y(1) - y(-1) = -1$$

and approximate the solution $y(x)$ by the Taylor polynomial.

Since $P(x) = 1,\ Q(x) = x,\ R(x) = x,\ f(x) = 1 + x + x^2,\ K(x, \xi) = 1 - 3x\xi$ and $a = -1,\ b = 1,\ c = 0$, by using relation (3.11) for $N = 9$, we have the augmented matrix is

$$W=\begin{bmatrix}
-2 & 0 & \frac{2}{3} & 0 & \frac{-1}{60} & 0 & \frac{-1}{2520} & 0 & \frac{-1}{181440} & 0 & ; & 1\\
1 & 3 & 0 & \frac{6}{5} & 0 & \frac{1}{140} & 0 & \frac{1}{7560} & 0 & \frac{1}{665280} & ; & 1\\
0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 2\\
0 & 0 & 3 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 4 & 4 & 0 & 1 & 0 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 5 & 5 & 0 & 1 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 1 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 0 & 1 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 & ; & 0
\end{bmatrix}$$

By using relation (3.14), the augmented matrices corresponding to the conditions are obtained as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{24} & \frac{1}{40} & \frac{1}{720} & \frac{1}{1680} & \frac{1}{40320} & \frac{1}{120960} & ; & -1 \end{bmatrix}.$$

Thus, we have the new augmented matrix for this problem is

$$W^{*}=\begin{bmatrix}
-2 & 0 & \frac{2}{3} & 0 & \frac{-1}{60} & 0 & \frac{-1}{2520} & 0 & \frac{-1}{181440} & 0 & ; & 1\\
1 & 3 & 0 & \frac{6}{5} & 0 & \frac{1}{140} & 0 & \frac{1}{7560} & 0 & \frac{1}{665280} & ; & 1\\
0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 2\\
0 & 0 & 3 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 4 & 4 & 0 & 1 & 0 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 5 & 5 & 0 & 1 & 0 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 1 & 0 & ; & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 0 & 1 & ; & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1\\
1 & 4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{24} & \frac{1}{40} & \frac{1}{720} & \frac{1}{1680} & \frac{1}{40320} & \frac{1}{120960} & ; & -1
\end{bmatrix}$$

and has the solution

$$Y = \begin{bmatrix} 1 & -\frac{408}{313} & \frac{2419}{550} & \frac{1296}{383} & -\frac{2279}{544} & -\frac{4926}{211} & \frac{3206}{995} & \frac{4681}{34} & \frac{8935}{74} & -\frac{30575}{31} \end{bmatrix}^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$y(x) = 1 - \frac{408}{313}x + \frac{2419}{550 \cdot 2!}x^2 + \frac{1296}{383 \cdot 3!}x^3 - \frac{2279}{544 \cdot 4!}x^4 - \frac{4926}{211 \cdot 5!}x^5 + \frac{3206}{995 \cdot 6!}x^6 + \frac{4681}{34 \cdot 7!}x^7 + \frac{8935}{74 \cdot 8!}x^8 - \frac{30575}{31 \cdot 9!}x^9.$$

Also, by taking $N = 19,\ c = 0$ and then, following the same procedure, we have the solution

$$\begin{aligned} y(x) = 1 &- \frac{335}{257}x + \frac{3602}{819 \cdot 2!}x^2 + \frac{1993}{589 \cdot 3!}x^3 - \frac{1152}{275 \cdot 4!}x^4 - \frac{1961}{84 \cdot 5!}x^5 \\ &+ \frac{538}{167 \cdot 6!}x^6 + \frac{9224}{67 \cdot 7!}x^7 + \frac{19198}{159 \cdot 8!}x^8 - \frac{93694}{95 \cdot 9!}x^9 - \frac{59952}{29 \cdot 10!}x^{10} \\ &+ \frac{38948}{5 \cdot 11!}x^{11} + \frac{244285}{8 \cdot 12!}x^{12} - \frac{503561}{8 \cdot 13!}x^{13} - \frac{459903}{14!}x^{14} + \frac{842647}{2 \cdot 15!}x^{15} \\ &+ \frac{7319870}{16!}x^{16} + \frac{578689}{17!}x^{17} - \frac{123859091}{18!}x^{18} - \frac{134275491}{19!}x^{19}. \end{aligned}$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_2(x_i)$ at any point x_i in $[-1, 1]$. Also, by taking $N = 9$ and $N = 19$

i	x_i	$y(x_i), N=9$	$D(x_i)$	$y(x_i), N=19$	$D(x_i)$
0	-1.0	3.94150159	3.02097×10^{-2}	3.94074792	1.35617×10^{-7}
1	-0.8	3.15004136	4.34808×10^{-3}	3.14992650	4.42276×10^{-9}
2	-0.6	2.44399167	3.64640×10^{-4}	2.44396067	1.63963×10^{-9}
3	-0.4	1.83466611	1.17091×10^{-5}	1.83465162	1.22635×10^{-9}
4	-0.2	1.34393767	3.71452×10^{-8}	1.34393264	9.88551×10^{-10}
5	0	1.00000000	4.59418×10^{-15}	1.00000000	8.87467×10^{-16}
6	0.2	0.83143161	2.37407×10^{-8}	0.83143126	8.97629×10^{-10}
7	0.4	0.86014631	4.75219×10^{-6}	0.86013954	9.98092×10^{-10}
8	0.6	1.09462711	9.51251×10^{-5}	1.09460534	1.17839×10^{-9}
9	0.8	1.52515002	7.23663×10^{-4}	1.52507222	2.32085×10^{-9}
10	1.0	2.12250826	3.19501×10^{-3}	2.12212375	4.74664×10^{-8}.

Example 3.3

Let us consider the equation

$$y'' + xy' + y = 1 + x + \int_0^1 (x\xi + \xi^2 + x^2\xi^2) y(\xi) d\xi, \qquad 0 \le x \le 1$$

with conditions

$$y(0) = 0, \quad y(1) = 0.$$

We want to find a Taylor polynomial solution of the problem above. We first take $N = 9,\ c = 0$, and then proceed as before. Note that

$P(x) = 1,\ Q(x) = x,\ R(x) = 1,\ f(x) = x+1,\ K(x,\xi) = x\xi + \xi^2 + x^2\xi^2;\ a = 0,\ b = 1.$

Then, we have the augmented matrix

$$W_1^* = \left[\begin{array}{cccccccccccc}
\frac{2}{3} & -\frac{1}{4} & \frac{9}{10} & -\frac{1}{36} & -\frac{1}{168} & -\frac{1}{960} & -\frac{1}{6480} & -\frac{1}{50400} & -\frac{1}{443520} & -\frac{1}{4354560} & ; & 1 \\
-\frac{1}{2} & \frac{5}{3} & -\frac{1}{8} & \frac{29}{30} & -\frac{1}{144} & -\frac{1}{840} & -\frac{1}{5760} & -\frac{1}{45360} & -\frac{1}{403200} & -\frac{1}{3991680} & ; & 1 \\
-\frac{2}{3} & -\frac{1}{2} & \frac{14}{5} & -\frac{1}{18} & \frac{83}{84} & -\frac{1}{480} & -\frac{1}{3240} & -\frac{1}{25200} & -\frac{1}{221760} & -\frac{1}{2177280} & ; & 0 \\
0 & 0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 & ; & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 1 & 0 & 0 & 0 & ; & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 1 & 0 & 0 & ; & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 1 & 0 & ; & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 1 & ; & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\
1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} & \frac{1}{5040} & \frac{1}{40320} & \frac{1}{362880} & ; & 0
\end{array}\right]$$

and has the solution

$$Y = \left[\begin{array}{cccccccccc} 0 & -\frac{374}{537} & \frac{1261}{1310} & \frac{557}{239} & -\frac{1188}{401} & -\frac{2228}{239} & \frac{3881}{262} & \frac{12529}{224} & -\frac{5703}{55} & -\frac{12529}{28} \end{array}\right]^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$y(x) = -\frac{374}{537}x + \frac{1261}{1310 \cdot 2!}x^2 + \frac{557}{239 \cdot 3!}x^3 - \frac{1188}{401 \cdot 4!}x^4$$
$$-\frac{2228}{239 \cdot 5!}x^5 + \frac{3881}{262 \cdot 6!}x^6 + \frac{12529}{224 \cdot 7!}x^7 - \frac{5703}{55 \cdot 8!}x^8 - \frac{12529}{28 \cdot 9!}x^9.$$

Also, by taking $N = 9,\ c = \frac{1}{2}$, then, following the same procedure, we have the augmented matrix

$W_2^* =$

$$\left[\begin{array}{cccccccccccc} \frac{1}{3} & \frac{17}{48} & \frac{31}{32} & -\frac{7}{1920} & -\frac{11}{26880} & -\frac{1}{30720} & -\frac{1}{400419} & -\frac{1}{6635520} & -\frac{1}{113541120} & -\frac{1}{2335703040} & ; & \frac{3}{2} \\ -\frac{5}{6} & \frac{11}{16} & \frac{37}{80} & \frac{239}{240} & -\frac{13}{26880} & -\frac{1}{26880} & -\frac{1}{341534} & -\frac{1}{5806080} & -\frac{1}{97320960} & -\frac{1}{2043740160} & ; & 1 \\ -\frac{2}{3} & -\frac{1}{6} & \frac{89}{30} & \frac{119}{240} & \frac{2239}{2240} & -\frac{1}{26880} & -\frac{1}{362880} & -\frac{1}{5806080} & -\frac{1}{102187008} & -\frac{1}{2043740160} & ; & 0 \\ 0 & 0 & 0 & 4 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 5 & \frac{1}{2} & 1 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & \frac{1}{2} & 1 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & \frac{1}{2} & 1 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & \frac{1}{2} & 1 & ; & 0 \\ 1 & -\frac{1}{2} & \frac{1}{2^2 \cdot 2!} & -\frac{1}{2^3 \cdot 3!} & \frac{1}{2^4 \cdot 4!} & -\frac{1}{2^5 \cdot 5!} & \frac{1}{2^6 \cdot 6!} & -\frac{1}{2^7 \cdot 7!} & \frac{1}{2^8 \cdot 8!} & -\frac{1}{2^9 \cdot 9!} & ; & 0 \\ 1 & \frac{1}{2} & \frac{1}{2^2 \cdot 2!} & \frac{1}{2^3 \cdot 3!} & \frac{1}{2^4 \cdot 4!} & \frac{1}{2^5 \cdot 5!} & \frac{1}{2^6 \cdot 6!} & \frac{1}{2^7 \cdot 7!} & \frac{1}{2^8 \cdot 8!} & \frac{1}{2^9 \cdot 9!} & ; & 0 \end{array}\right]$$

and has the solution

$$Y = \left[\begin{array}{cccccccccc} -\frac{513}{2710} & -\frac{163}{30588} & \frac{581}{360} & \frac{168}{1619} & -\frac{2365}{476} & \frac{1376}{665} & \frac{5452}{229} & -\frac{2821}{116} & -\frac{17149}{111} & \frac{94858}{349} \end{array}\right]^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$\begin{aligned} y(x) = -\frac{513}{2710} - \frac{163}{30588}(x-1/2) + \frac{581}{360 \cdot 2!}(x-1/2)^2 + \frac{168}{1619 \cdot 3!}(x-1/2)^3 \\ -\frac{2365}{476 \cdot 4!}(x-1/2)^4 + \frac{1376}{665 \cdot 5!}(x-1/2)^5 + \frac{5452}{229 \cdot 6!}(x-1/2)^6 \\ -\frac{2821}{116 \cdot 7!}(x-1/2)^7 - \frac{17149}{111 \cdot 8!}(x-1/2)^8 + \frac{94858}{349 \cdot 9!}(x-1/2)^9. \end{aligned}$$

Also, by taking $N = 9,\ c = 1$, then, following the same procedure, we have the augmented matrix

$$W_3^* = \left[\begin{array}{cccccccccc|c}
-\frac{1}{6} & \frac{4}{3} & \frac{37}{40} & \frac{1}{72} & -\frac{11}{5040} & \frac{1}{3360} & -\frac{6}{167483} & \frac{1}{259200} & -\frac{1}{2661120} & \frac{1}{29937600} & ;\ 2 \\
-\frac{7}{6} & \frac{7}{3} & \frac{37}{40} & \frac{73}{72} & -\frac{11}{5040} & \frac{1}{3360} & -\frac{6}{167483} & \frac{1}{259200} & -\frac{1}{2661120} & \frac{1}{29937600} & ;\ 1 \\
-\frac{2}{3} & \frac{1}{6} & \frac{89}{30} & \frac{181}{180} & \frac{1259}{1260} & \frac{1}{10080} & -\frac{1}{90720} & \frac{1}{907200} & -\frac{1}{9979200} & \frac{1}{119750400} & ;\ 0 \\
0 & 0 & 0 & 4 & 4 & 20 & 0 & 0 & 0 & 0 & ;\ 0 \\
0 & 0 & 0 & 0 & 5 & 5 & 30 & 0 & 0 & 0 & ;\ 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 6 & 42 & 0 & 0 & ;\ 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 56 & 0 & ;\ 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 72 & ;\ 0 \\
1 & -1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{24} & -\frac{1}{120} & \frac{1}{720} & -\frac{1}{5040} & \frac{1}{40320} & -\frac{1}{362880} & ;\ 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ;\ 0
\end{array}\right]$$

and has the solution

$$Y = \left[\begin{array}{cccccccccc} 0 & \frac{626}{867} & \frac{1599}{1402} & -\frac{1493}{867} & -\frac{2683}{1512} & \frac{3543}{409} & \frac{971}{4629} & -\frac{14925}{286} & \frac{2688}{53} & \frac{23473}{64} \end{array}\right]^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$\begin{aligned} y(x) &= \frac{626}{867}(x-1) + \frac{1599}{1402 \cdot 2!}(x-1)^2 - \frac{1493}{867 \cdot 3!}(x-1)^3 \\ &\quad - \frac{2683}{1512 \cdot 4!}(x-1)^4 + \frac{3543}{409 \cdot 5!}(x-1)^5 + \frac{971}{4629 \cdot 6!}(x-1)^6 \\ &\quad - \frac{14925}{286 \cdot 7!}(x-1)^7 + \frac{2688}{53 \cdot 8!}(x-1)^8 + \frac{23473}{64 \cdot 9!}(x-1)^9. \end{aligned}$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_2(x_i)$ at any point x_i in $[0, 1]$. Also, by taking $N = 9$

i	x_i	$y(x_i), c = 0$	$y(x_i), c = 1/2$	$y(x_i), c = 1$
0	0	0.00000000	1.46861×10^{-7}	-4.32699×10^{-8}
1	0.2	-0.11715398	-0.11723605	-0.11732886
2	0.4	-0.18057311	-0.18073470	-0.18081359
3	0.6	-0.18153418	-0.18176554	-0.18181585
4	0.8	-0.11915953	-0.11941775	-0.11944093
5	1.0	2.93636×10^{-7}	1.42718×10^{-7}	0.00000000

i	x_i	$D(x_i), c = 0$	$D(x_i), c = 1/2$	$D(x_i), c = 1$
0	0	2.30673×10^{-16}	1.41473×10^{-4}	1.07115×10^{-2}
1	0.2	5.70817×10^{-8}	1.90580×10^{-6}	1.80134×10^{-3}
2	0.4	1.38181×10^{-5}	2.39302×10^{-10}	1.81050×10^{-4}
3	0.6	3.39365×10^{-4}	2.00893×10^{-10}	7.09947×10^{-6}
4	0.8	3.27944×10^{-3}	1.12168×10^{-6}	2.78898×10^{-8}
5	1.0	1.90444×10^{-2}	5.74010×10^{-5}	2.38432×10^{-16}.

3.1.2 Second Method

We study (3.1) and will find a solution of (3.1) in the form of Taylor series

$$y(x) = \sum_{r=0}^{N} a_r (x-c)^r, \quad a \le c \le b$$

where

$$a_r = \frac{y^{(r)}(c)}{r!},$$

and

$$y^{(n)}(x) = \sum_{r=0}^{N} a_r^{(n)} (x-c)^r, \quad a \le c \le b$$

where

$$a_r^{(n)} = \frac{y^{(r+n)}(c)}{r!}.$$

The recurrence relation between the Taylor coefficients $a_r^{(n)}$ and $a_r^{(n+1)}$ of $y^{(n)}(x)$ and $y^{(n+1)}(x)$, is given by

$$a_r^{(n+1)} = (r+1)a_{r+1}^{(n)}. \tag{3.20}$$

Take $r = 0, 1, \ldots, N$ and assume $a_r^{(n)} = 0$ for $r > N$, we get

$$\begin{aligned} a_0^{(n+1)} &= (1)a_1^{(n)} \\ a_1^{(n+1)} &= (2)a_2^{(n)} \\ a_2^{(n+1)} &= (3)a_3^{(n)} \\ &\vdots \\ a_{N-1}^{(n+1)} &= (N)a_N^{(n)} \\ a_N^{(n+1)} &= (N+1)a_{N+1}^{(n)} = 0. \end{aligned}$$

Hence, the system can be written in the matrix form as

$$\begin{bmatrix} a_0^{(n+1)} \\ a_1^{(n+1)} \\ a_2^{(n+1)} \\ \vdots \\ a_N^{(n+1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 2 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 3 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & N \\ 0 & 0 & . & . & . & . & . & 0 \end{bmatrix} \begin{bmatrix} a_0^{(n)} \\ a_1^{(n)} \\ a_2^{(n)} \\ \vdots \\ a_N^{(n)} \end{bmatrix}$$

or

$$A^{(n+1)} = MA^{(n)}; \quad n = 0, 1, 2, \ldots \tag{3.21}$$

where

$$A^{(n)} = \begin{bmatrix} a_0^{(n)} & a_1^{(n)} & \ldots & a_N^{(n)} \end{bmatrix}^T,$$

$$M = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 2 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 3 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & N \\ 0 & 0 & . & . & . & . & . & 0 \end{bmatrix}.$$

For $n = 0, 1, 2, \ldots$, it follows from relation (3.21) that

$$\begin{aligned} A^{(1)} &= MA^{(0)} = MA \\ A^{(2)} &= MA^{(1)} = M^2A \\ A^{(3)} &= MA^{(2)} = M^3A \\ &\vdots \\ A^{n} &= MA^{(n-1)} = M^nA \end{aligned} \tag{3.22}$$

where

$$A^{(0)} = A = \left[\begin{array}{cccc} a_0 & a_1 & \ldots & a_N \end{array} \right]^T$$

which are matrices for relations between the Taylor coefficient matrix A of $y(x)$ and the Taylor coefficient matrix $A^{(n)}$ of the n^{th} derivative of $y(x)$. In (3.1), we assume that the function $P(x)$, $Q(x)$ and $R(x)$ can be expressed in Taylor polynomial of degree I, namely,

$$P(x) = \sum_{i=0}^{I} p_i(x-c)^i, \quad Q(x) = \sum_{i=0}^{I} q_i(x-c)^i, \quad R(x) = \sum_{i=0}^{I} r_i(x-c)^i. \tag{3.23}$$

Substituting the expressions (3.23) in (3.1), we obtain

$$\sum_{i=0}^{I} \left\{ p_i(x-c)^i y'' + q_i(x-c)^i y' + r_i(x-c)^i y \right\} = f(x) + \int_a^b K(x,\xi)d\xi. \tag{3.24}$$

The matrix representation of Taylor expansions $\left[\begin{array}{c} (x-c)^p y^{(s)}(x) \end{array} \right]$ was be written as

$$\left[\begin{array}{c} (x-c)^p y^{(s)}(x) \end{array} \right] = XC_pA^{(s)}.$$

Since $A^{(s)} = M^s A$ by (3.22), we have

$$\left[\ (x-c)^p y^{(s)}(x)\ \right] = XC_pM^sA, \quad s=0,1,2; \quad p=0,1,\ldots,I \tag{3.25}$$

where

$$X = \left[\ 1 \quad (x-c) \quad (x-c)^2 \quad \ldots \quad (x-c)^N\ \right].$$

For $p = 0$

$$C_0 = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{bmatrix}_{(N+1)\times(N+1)}.$$

For $p = 1$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 & 0 \end{bmatrix}_{(N+1)\times(N+1)}.$$

For $p > 1$, C_p can be defined as follows

$$C_p = \left[\ c_{ij}\ \right] = \begin{cases} 1 & \text{, for } i-j=p \\ 0 & \text{, otherwise.} \end{cases}$$

Also we assume that the function $f(x)$ and $K(x,\xi)$ can be expanded as

$$f(x) = \sum_{r=0}^{N} f_r (x-c)^r$$

or in the matrix form as

$$\left[\ f(x)\ \right] \ = \ XF \tag{3.26}$$

where

$$F = \left[\begin{array}{cccc} f_0 & f_1 & \dots & f_N \end{array}\right]^T,$$

and

$$K(x,\xi) = \sum_{i=0}^{N}\sum_{j=0}^{N} \frac{K^{(i,j)}(c,0)}{i!j!}(x-c)^i\xi^j$$

where $K^{(i,j)}$ is the partial derivative of $K(x,\xi)$ taken with respect to x for i times and with respect to ξ for j times. In the matrix form, we have

$$\left[\begin{array}{c} K(x,\xi) \end{array}\right] = XK\Psi \tag{3.27}$$

where

$$\begin{aligned} X &= \left[\begin{array}{ccccc} 1 & (x-c) & (x-c)^2 & \dots & (x-c)^N \end{array}\right], \\ \Psi &= \left[\begin{array}{cccc} 1 & \xi & \dots & \xi^N \end{array}\right]^T, \end{aligned}$$

and

$$K = \left[\begin{array}{c} k_{nm} \end{array}\right]; \quad n,m = 0,1,\dots,N$$

the elements of which are defined by

$$k_{nm} = \frac{K^{(n,m)}(c,0)}{n!m!}.$$

Note that for $m+n > N$, $k_{nm} = 0$ where n and m are integers and we have

$$\int_a^b \left[\begin{array}{c} K(x,\xi) \end{array}\right] d\xi = XK\int_a^b \Psi d\xi = XK\bar{\Psi} \tag{3.28}$$

where

$$\bar{\Psi} = \int_a^b \Psi d\xi.$$

Substituting the expansions (3.25), (3.26) and (3.28) into (3.24), we obtain

$$\sum_{i=0}^{I}\Big\{p_iXC_iM^2A + q_iXC_iMA + r_iXC_iA\Big\} = XF + XK\bar{\Psi}$$

or

$$\sum_{i=0}^{I}\Big\{p_iC_iM^2 + q_iC_iM + r_iC_i\Big\}A = F + K\bar{\Psi} \tag{3.29}$$

which corresponds to a system of $(N+1)$ equations for the unknown Taylor coefficients $a_r,\ r=0,1,\ldots,N$. We can write (3.29) in the form

$$W_3 A = \bar{G}$$

where

$$W_3 = \left[\ w_{nm}\ \right] = \sum_{i=0}^{I}\left\{p_i C_i M^2 + q_i C_i M + r_i C_i\right\};\quad n,m=0,1,\ldots,N;\quad I=0,1,\ldots$$

and

$$\bar{G} = F + K\bar{\Psi} = \left[\ \bar{g_0}\quad \bar{g_1}\quad \ldots\quad \bar{g_N}\ \right]^T.$$

Then the augmented matrix of (3.1) becomes

$$\left[\ W_3\ \ ;\ \ \bar{G}\ \right]. \tag{3.30}$$

Next, we consider integro-differential equation (3.7). and solution in the form of expression (3.2). We assume that the function $P(x)$, $Q(x)$ and $R(x)$ can be expressed in Taylor polynomial of degree I, it follows from (3.23) and by substituting them into (3.7), that

$$\sum_{i=0}^{I}\left\{p_i(x-c)^i y'' + q_i(x-c)^i y' + r_i(x-c)^i y\right\}$$
$$= f(x) + \int_a^b K(x,\xi)y(\xi)d\xi. \tag{3.31}$$

Also we assume that the function $f(x)$ and $K(x,\xi)$ can be expanded in term of Taylor polynomial. Next, we expand $y(\xi)$ at $\xi = c$, namely,

$$y(\xi)\ \ =\ \ \sum_{i=0}^{I} a_i(\xi-c)^i$$

or in the matrix form as

$$\left[\ y(\xi)\ \right]\ \ =\ \ \hat{\Psi}A \tag{3.32}$$

where

$$\hat{\Psi} = \left[\ 1\quad (\xi-c)\quad (\xi-c)^2\quad \ldots\quad (\xi-c)^N\ \right].$$

Substituting the expansion (3.25), (3.26), (3.27) and (3.32) into (3.31), we obtain

$$X\sum_{i=0}^{I}\Big\{p_iC_iM^2+q_iC_iM+r_iC_i\Big\}A=XF+XK\int_a^b\Psi\hat{\Psi}d\xi A$$

or

$$\sum_{i=0}^{I}\Big\{p_iC_iM^2+q_iC_iM+r_iC_i-\Psi^*\Big\}A=F \qquad (3.33)$$

where

$$\Psi^*=K\int_a^b\Psi\hat{\Psi}d\xi.$$

(3.33) can be written in the form

$$W_4A=F$$

where

$$W_4=\Big[\ w_{nm}\ \Big]=\sum_{i=0}^{I}\Big\{p_iC_iM^2+q_iC_iM+r_iC_i-\Psi^*\Big\},\quad n,m=0,1,\ldots,N;\quad I=0,1,\ldots$$

and

$$F=\Big[\ f_0\quad f_1\quad\ldots\quad f_N\ \Big]^T.$$

Then the augmented matrix of (3.7) becomes

$$\Big[\ W_4\quad;\quad F\ \Big]. \qquad (3.34)$$

Since, (3.1) and (3.7) satisfy conditions (1.5) and (1.6) and they are equivalent to the matrix (3.25), we have

$$y^{(0)}(x)\ =\ \Big[\ 1\quad(x-c)\quad(x-c)^2\quad\ldots\quad(x-c)^N\ \Big]A$$

and

$$y^{(1)}(x)\ =\ \Big[\ 0\quad 1\quad 2(x-c)\quad\ldots\quad N(x-c)^{N-1}\ \Big]A$$

where

$$A=\Big[\ a_0\quad a_1\quad\ldots\quad a_N\ \Big]^T.$$

By using these equations, the quantities $y^{(i)}(a),\ y^{(i)}(b)$ and $y^{(i)}(c),\ i=0,1$ can be written as

$$
\begin{aligned}
y^{(0)}(c) &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} A \\
y^{(0)}(a) &= \begin{bmatrix} 1 & h & h^2 & \dots & h^N \end{bmatrix} A \\
y^{(0)}(b) &= \begin{bmatrix} 1 & k & k^2 & \dots & k^N \end{bmatrix} A \\
y^{(1)}(c) &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix} A \\
y^{(1)}(a) &= \begin{bmatrix} 0 & 1 & 2h & \dots & Nh^{N-1} \end{bmatrix} A \\
y^{(1)}(b) &= \begin{bmatrix} 0 & 1 & 2k & \dots & Nk^{N-1} \end{bmatrix} A
\end{aligned} \tag{3.35}
$$

where $h = a - c$ and $k = b - c$. By substituting quantities (3.35) into (1.5) and (1.6), and then by simplifying, we obtain

$$\sum_{i=0}^{1}[a_i y^{(i)}(a) + b_i y^{(i)}(b) + c_i y^{(i)}(c)] = \lambda$$

or in the matrix form

$$
\begin{aligned}
&a_0 \begin{bmatrix} 1 & h & h^2 & \dots & h^N \end{bmatrix} A + b_0 \begin{bmatrix} 1 & k & k^2 & \dots & k^N \end{bmatrix} A \\
&+c_0 \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} A + a_1 \begin{bmatrix} 0 & 1 & 2h & \dots & Nh^{N-1} \end{bmatrix} A \\
&+b_1 \begin{bmatrix} 0 & 1 & 2k & \dots & Nk^{N-1} \end{bmatrix} A + c_1 \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix} A = \begin{bmatrix} \lambda \end{bmatrix}
\end{aligned}
$$

or

$$\begin{bmatrix} u_0 & u_1 & \dots & u_N \end{bmatrix} A = \begin{bmatrix} \lambda \end{bmatrix}.$$

Also,

$$\sum_{i=0}^{1}[\alpha_i y^{(i)}(a) + \beta_i y^{(i)}(b) + \gamma_i y^{(i)}(c)] = \mu$$

or in the matrix form

$$
\begin{aligned}
&\alpha_0 \begin{bmatrix} 1 & h & h^2 & \dots & h^N \end{bmatrix} A + \beta_0 \begin{bmatrix} 1 & k & k^2 & \dots & k^N \end{bmatrix} A \\
&+\gamma_0 \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix} A + \alpha_1 \begin{bmatrix} 0 & 1 & 2h & \dots & Nh^{N-1} \end{bmatrix} A \\
&+\beta_1 \begin{bmatrix} 0 & 1 & 2k & \dots & Nk^{N-1} \end{bmatrix} A + \gamma_1 \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix} A = \begin{bmatrix} \mu \end{bmatrix}
\end{aligned}
$$

or

$$\begin{bmatrix} v_0 & v_1 & \dots & v_N \end{bmatrix} A \; = \; \begin{bmatrix} \mu \end{bmatrix}$$

where u_j and v_j, $j = 0, 1, \dots, N$ are constants. We have the augmented matrices

$$\begin{bmatrix} u_0 & u_1 & \dots & u_N & ; & \lambda \end{bmatrix} \text{ and } \begin{bmatrix} v_0 & v_1 & \dots & v_N & ; & \mu \end{bmatrix}. \quad (3.36)$$

Consequently, by replacing the matrices (3.36) into the last two rows of the augmented matrices (3.30) and (3.34), we have the required augmented matrices as follows

$$W_3^* = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & \bar{g}_0 \\ w_{10} & w_{11} & \dots & w_{1N} & ; & \bar{g}_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \dots & w_{N-2,N} & ; & \bar{g}_{N-2} \\ u_0 & u_1 & \dots & u_N & ; & \lambda \\ v_0 & v_1 & \dots & v_N & ; & \mu \end{bmatrix} \quad (3.37)$$

and

$$W_4^* = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & f_0 \\ w_{10} & w_{11} & \dots & w_{1N} & ; & f_1 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \dots & w_{N-2,N} & ; & f_{N-2} \\ u_0 & u_1 & \dots & u_N & ; & \lambda \\ v_0 & v_1 & \dots & v_N & ; & \mu \end{bmatrix}. \quad (3.38)$$

Let

$$W_3^{**} = W_4^{**} = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} \\ w_{10} & w_{11} & \dots & w_{1N} \\ \vdots & \vdots & \dots & \vdots \\ w_{N-2,0} & w_{N-2,1} & \dots & w_{N-2,N} \\ u_0 & u_1 & \dots & u_N \\ v_0 & v_1 & \dots & v_N \end{bmatrix}.$$

If $detW_3^{**} = detW_4^{**} \neq 0$, we can write

$$A = (W_3^{**})^{-1}\bar{G} \quad (3.39)$$

$$A = (W_4^{**})^{-1}F \quad (3.40)$$

where

$$A = \left[\begin{array}{cccc} a_0 & a_1 & \dots & a_N \end{array}\right]^T.$$

Thus, the matrix A are uniquely determined. These solutions are given by the Taylor polynomials in the form of (3.2). We can easily check the accuracy of this solution. Since the Taylor polynomial (3.2) is an approximate solution of (3.1) and (3.7). Substituting $y(x)$ and its derivatives $y^{(1)}(x)$ and $y^{(2)}(x)$ in (3.1) and (3.7), the resulting equations will be satisfied approximately for a given value of x in the interval $a \leq x \leq b$. $x = x_i \in [a, b],\ i = 0, 1, \dots, M$, we define the relative error of the obtained solutions of (3.1) and (3.7)as follow

$$D_3(x_i) = \frac{\left|P(x_i)y^{(2)}(x_i) + Q(x_i)y^{(1)}(x_i) + R(x_i)y(x_i) - f(x_i) - \int_a^b K(x_i,\xi)d\xi\right|}{\left|f(x_i) + \int_a^b K(x_i,\xi)d\xi\right|}$$

and

$$D_4(x_i) = \frac{\left|P(x_i)y^{(2)}(x_i) + Q(x_i)y^{(1)}(x_i) + R(x_i)y(x_i) - f(x_i) - \int_a^b K(x_i,\xi)y(\xi)d\xi\right|}{\left|f(x_i) - \int_a^b K(x_i,\xi)y(\xi)d\xi\right|}.$$

If max $\{D_3(x_i), D_4(x_i)\} \leq 10^{-k}$ where k is a prescribed integer, then the truncation limit N is increased until the difference $D_3(x_i)$ or $D_4(x_i)$ at each the points x_i becomes smaller than the prescribed tolerance (10^{-k}).

Example 3.4

Let us consider the problem

$$y'' + xy' + xy = 1 + x + x^2 + \int_{-1}^{1} \sin(x+\xi)^2 d\xi$$

$$y(0) = 1, \qquad y'(0) + 2y(1) - y(-1) = -1$$

and approximate the solution $y(x)$ by the Taylor polynomial.
Since $P(x) = 1,\ Q(x) = x,\ R(x) = x,\ f(x) = 1 + x + x^2,\ K(x,\xi) = \sin(x+\xi)^2$ and $a = -1,\ b = 1,\ c = 0$, we have

$$\begin{aligned} p_0 &= 1,\ p_1 = p_2 = \ldots = 0 \\ q_0 &= 0,\ q_1 = 1,\ q_2 = q_3 = \ldots = 0 \\ r_0 &= 0,\ r_1 = 1,\ r_2 = r_3 = \ldots = 0 \\ f_0 &= 1,\ f_1 = 1,\ f_2 = 1,\ f_3 = f_4 = 0 = \ldots = 0. \end{aligned}$$

Thus, the matrix of (3.29) for $N = 5$ is

$$\Big\{C_0 M^2 + C_1 M + C_1\Big\} A = WA = \bar{G}$$

where

$$W = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 6 & 0 & 0 \\ 0 & 1 & 2 & 0 & 12 & 0 \\ 0 & 0 & 1 & 3 & 0 & 20 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

and

$$\bar{G} = \begin{bmatrix} \frac{5}{3} & 1 & 2 & 0 & -\frac{5}{3} & 0 \end{bmatrix}^T.$$

For the conditions, by taking $c = 0,\ a = -1,\ b = 1$ and using relations (3.35), we have

$$y^{(0)}(0) = 1$$

or in the matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 \end{bmatrix}$$

or in the augmented matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ; & 1 \end{bmatrix}.$$

Also,

$$y^{(1)}(0) + 2y(1) - y(-1) = -1$$

or in the matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} A + 2 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} A - \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} A = \begin{bmatrix} -1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 1 & 3 \end{bmatrix} A = \begin{bmatrix} -1 \end{bmatrix}$$

or in the augmented matrix form

$$\begin{bmatrix} 1 & 4 & 1 & 3 & 1 & 3 & ; & -1 \end{bmatrix}.$$

Thus, the augmented matrix for this problem as

$$W^* = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & ; & \frac{5}{3} \\ 1 & 1 & 0 & 6 & 0 & 0 & ; & 1 \\ 0 & 1 & 2 & 0 & 12 & 0 & ; & 2 \\ 0 & 0 & 1 & 3 & 0 & 20 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 1 & 4 & 1 & 3 & 1 & 3 & ; & -1 \end{bmatrix}$$

and has the solution

$$A = \begin{bmatrix} 1 & -\frac{985}{1257} & \frac{5}{6} & \frac{985}{7542} & \frac{39}{419} & -\frac{77}{1257} \end{bmatrix}^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$y(x) = 1 - \frac{985}{1257}x + \frac{5}{6}x^2 + \frac{985}{7542}x^3 + \frac{39}{419}x^4 - \frac{77}{1257}x^5.$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_3(x_i)$ at any point x_i in $[-1, 1]$. Note that $\int_{-1}^{1} \sin(x_i + \xi)^2 d\xi$ is approximated by Simpson rule when $h = \frac{1}{2}$. Also, by taking $N = 5$

i	x_i	$y(x_i), N = 5$	$D(x_i)$
0	-1.0	2.64067886	4.36707×10^{-1}
1	-0.8	2.15155229	2.28609×10^{-1}
2	-0.6	1.75878340	8.16975×10^{-2}
3	-0.4	1.44142960	4.89035×10^{-2}
4	-0.2	1.18917940	2.02843×10^{-1}
5	0	1.00000000	3.79022×10^{-1}
6	0.2	1.18917940	5.35999×10^{-1}
7	0.4	0.83000269	6.41552×10^{-1}
8	0.6	0.86534262	6.93787×10^{-1}
9	0.8	0.99136450	7.04177×10^{-1}
10	1.0	1.21214532	6.84173×10^{-1}.

Example 3.5

Let us consider the problem

$$y'' + xy' + xy = 1 + x + x^2 + \int_{-1}^{1} (1 - 3x\xi)y(\xi)d\xi$$

$$y(0) = 1, \qquad y'(0) + 2y(1) - y(-1) = -1$$

and approximate the solution $y(x)$ by the Taylor polynomial. By using relation (3.33), we have the matrix of equation for $N = 9$ is

$$\Big\{C_0M^2 + C_1M + C_1 - \Psi^*\Big\}A = WA = F$$

where

$$W=\begin{bmatrix} -2 & 0 & \frac{4}{3} & 0 & -\frac{2}{5} & 0 & -\frac{2}{7} & 0 & -\frac{2}{9} & 0 \\ 1 & 3 & 0 & \frac{36}{5} & 0 & \frac{6}{7} & 0 & \frac{6}{9} & 0 & \frac{6}{11} \\ 0 & 1 & 2 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & 42 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 0 & 56 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 0 & 72 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 \end{bmatrix},$$

and

$$F=\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

The augmented matrices corresponding to the conditions as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 4 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & ; & -1 \end{bmatrix}.$$

Thus, we have the required augmented matrix for this problem as

$$W^*=\begin{bmatrix} -2 & 0 & \frac{4}{3} & 0 & -\frac{2}{5} & 0 & -\frac{2}{7} & 0 & -\frac{2}{9} & 0 & ; & 1 \\ 1 & 3 & 0 & \frac{36}{5} & 0 & \frac{6}{7} & 0 & \frac{6}{9} & 0 & \frac{6}{11} & ; & 1 \\ 0 & 1 & 2 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 0 & 0 & 1 & 3 & 0 & 20 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 30 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 & 42 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 0 & 56 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 0 & 72 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 1 \\ 1 & 4 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 3 & ; & -1 \end{bmatrix}$$

and has the solution

$$A = \begin{bmatrix} 1 & -\frac{408}{313} & \frac{2905}{1321} & \frac{216}{383} & -\frac{992}{5693} & -\frac{564}{2899} & -\frac{57}{12737} & \frac{186}{6809} & \frac{44}{14693} & -\frac{92}{33849} \end{bmatrix}^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$\begin{aligned} y(x) = 1 &- \frac{408}{313}x + \frac{2905}{1321}x^2 + \frac{216}{383}x^3 - \frac{992}{5683}x^4 - \frac{564}{2899}x^5 \\ &- \frac{57}{12737}x^6 + \frac{186}{6809}x^7 + \frac{44}{14693}x^8 - \frac{92}{33849}x^9. \end{aligned}$$

Also, by taking $N = 19$, $c = 0$, and then, following the same procedure, we have the solution

$$\begin{aligned} y(x) = 1 &- \frac{335}{257}x + \frac{1801}{819}x^2 + \frac{1045}{1853}x^3 - \frac{48}{275}x^4 - \frac{599}{3079}x^5 \\ &+ \frac{89}{19891}x^6 + \frac{317}{11605}x^7 + \frac{46}{15361}x^8 - \frac{97}{35690}x^9 - \frac{37}{64947}x^{10} \\ &+ \frac{8}{40995}x^{11} + \frac{14}{219613}x^{12} - \frac{4}{395711}x^{13} - \frac{1}{189558}x^{14} + \frac{1}{3103730}x^{15} \\ &+ \frac{1}{2858356}x^{16} + \frac{1}{614643930}x^{17} - \frac{1}{51690786}x^{18} - \frac{1}{905936737}x^{19}. \end{aligned}$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_4(x_i)$ at any point x_i in $[-1, 1]$. Also, by taking $N = 9$ and $N = 19$

i	x_i	$y(x_i), N=9$	$D(x_i)$	$y(x_i), N=19$	$D(x_i)$
0	-1.0	3.94150240	3.02097×10^{-2}	3.94074779	1.32616×10^{-7}
1	-0.8	3.15004185	4.34808×10^{-3}	3.14992643	2.20500×10^{-9}
2	-0.6	2.44399193	3.64640×10^{-4}	2.44396064	1.12663×10^{-11}
3	-0.4	1.83466622	1.17091×10^{-5}	1.83465161	6.87637×10^{-15}
4	-0.2	1.34393770	3.71452×10^{-8}	1.34393264	8.71445×10^{-17}
5	0	1.00000000	2.52428×10^{-16}	1.00000000	0.00000
6	0.2	0.83141643	2.37407×10^{-8}	0.83143125	1.75735×10^{-16}
7	0.4	0.86014642	4.78219×10^{-6}	0.86013954	4.21837×10^{-15}
8	0.6	1.09462736	9.51251×10^{-5}	1.09460537	5.70613×10^{-12}
9	0.8	1.52515047	7.23663×10^{-4}	1.52507228	9.11402×10^{-10}
10	1.0	2.12250896	3.19501×10^{-3}	2.12212388	4.57665×10^{-8}.

Example 3.6

Let us consider the equation

$$y'' + xy' + y = 1 + x + \int_0^1 (x\xi + \xi^2 + x^2\xi^2)y(\xi)d\xi, \qquad 0 \le x \le 1$$

with conditions

$$y(0) = 0, \quad y(1) = 0$$

We want to find a Taylor polynomial solution of the problem above. We first take $N = 9,\ c = 0$, and then proceed as before.

$P(x) = 1,\ Q(x) = x,\ R(x) = 1,\ f(x) = x+1,\ K(x,\xi) = x\xi+\xi^2+x^2\xi^2;\ a = 0,\ b = 1.$

Then we obtain the matrix of this equation is

$$\Big\{C_0M^2 + C_1M + C_0 - \Psi^*\Big\}A = W_1A = F$$

Thus, we have the augmented matrix

$$W_1^* = \left[\begin{array}{cccccccccccc} \frac{2}{3} & -\frac{1}{4} & \frac{9}{5} & -\frac{1}{6} & -\frac{1}{7} & -\frac{1}{8} & -\frac{1}{9} & -\frac{1}{10} & -\frac{1}{11} & -\frac{1}{12} & ; & 1 \\ -\frac{1}{2} & \frac{5}{3} & -\frac{1}{4} & \frac{14}{5} & -\frac{1}{6} & -\frac{83}{7} & -\frac{1}{8} & -\frac{1}{9} & -\frac{1}{10} & -\frac{1}{11} & ; & 1 \\ -\frac{1}{3} & -\frac{1}{4} & \frac{14}{5} & -\frac{1}{6} & \frac{83}{7} & -\frac{1}{8} & -\frac{1}{9} & -\frac{1}{10} & -\frac{1}{11} & -\frac{1}{12} & ; & 0 \\ 0 & 0 & 0 & 4 & 0 & 20 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 30 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 42 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 56 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 72 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & ; & 0 \end{array}\right]$$

and has the solution

$$A = \left[\begin{array}{cccccccccc} 0 & -\frac{374}{537} & \frac{1261}{2620} & \frac{557}{1434} & -\frac{99}{802} & -\frac{659}{8483} & \frac{33}{1604} & \frac{65}{5857} & -\frac{33}{12832} & -\frac{33}{26762} \end{array}\right]^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$y(x) = -\frac{374}{537}x + \frac{1261}{2620}x^2 + \frac{557}{1434}x^3 - \frac{99}{802}x^4$$
$$-\frac{659}{8483}x^5 + \frac{33}{1604}x^6 + \frac{65}{5857}x^7 - \frac{33}{12832}x^8 - \frac{33}{26762}x^9.$$

Also, by taking $N = 9,\ c = \frac{1}{2}$, then, following the same procedure, the matrix of this equation is

$$\left\{C_0M^2 + \frac{1}{2}M + C_1M + C_0 - \Psi^*\right\}A = W_2A = F.$$

Thus, we have the augmented matrix

$$W_2^* = \begin{bmatrix} \frac{1}{3} & \frac{17}{48} & \frac{31}{16} & -\frac{7}{320} & -\frac{11}{1120} & -\frac{1}{256} & -\frac{29}{16128} & -\frac{7}{9216} & -\frac{1}{2816} & -\frac{7}{45056} & ; & \frac{3}{2} \\ -\frac{5}{6} & \frac{11}{6} & \frac{37}{40} & \frac{239}{40} & -\frac{13}{1120} & -\frac{1}{224} & -\frac{17}{8064} & -\frac{1}{1152} & -\frac{7}{16896} & -\frac{1}{5632} & ; & 1 \\ -\frac{1}{3} & -\frac{1}{12} & \frac{89}{30} & \frac{119}{80} & \frac{2243}{187} & -\frac{1}{448} & -\frac{1}{1008} & -\frac{1}{2304} & -\frac{5}{25344} & -\frac{1}{11264} & ; & 0 \\ 0 & 0 & 0 & 4 & 2 & 20 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 5 & \frac{5}{2} & 30 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & \frac{1}{2} & 42 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & \frac{7}{2} & 56 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 4 & 72 & ; & 0 \\ 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{16} & -\frac{1}{32} & \frac{1}{64} & -\frac{1}{128} & \frac{1}{256} & -\frac{1}{512} & ; & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} & \frac{1}{256} & \frac{1}{512} & ; & 0 \end{bmatrix}$$

and has the solution

$$A = \begin{bmatrix} -\frac{513}{2710} & -\frac{163}{30588} & \frac{581}{720} & \frac{28}{1619} & -\frac{519}{2507} & \frac{172}{9975} & \frac{95}{2873} & -\frac{57}{11813} & -\frac{47}{12266} & \frac{49}{65420} \end{bmatrix}^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$\begin{aligned} y(x) = & -\frac{513}{2710} - \frac{163}{30588}(x-1/2) + \frac{581}{720}(x-1/2)^2 + \frac{28}{1619}(x-1/2)^3 \\ & -\frac{519}{2507}(x-1/2)^4 + \frac{172}{9975}(x-1/2)^5 + \frac{95}{2873}(x-1/2)^6 \\ & -\frac{57}{11813}(x-1/2)^7 - \frac{47}{12266}(x-1/2)^8 + \frac{49}{65420}(x-1/2)^9. \end{aligned}$$

Also, by taking $N = 9$, $c = 1$, then, following the same procedure, the matrix of this equation is

$$\left\{ C_0 M^2 + M + C_1 M + C_0 - \Psi^* \right\} A = W_3 A = F.$$

Thus, we have the augmented matrix

$$W_3^* = \begin{bmatrix} -\frac{1}{6} & \frac{4}{3} & \frac{37}{20} & \frac{1}{12} & -\frac{11}{210} & \frac{1}{28} & -\frac{13}{504} & \frac{7}{360} & -\frac{1}{66} & \frac{2}{165} & ; & 2 \\ -\frac{7}{6} & \frac{7}{3} & \frac{37}{20} & \frac{73}{12} & -\frac{11}{210} & \frac{1}{28} & -\frac{13}{504} & \frac{7}{360} & -\frac{1}{66} & \frac{2}{165} & ; & 1 \\ -\frac{1}{3} & \frac{1}{12} & \frac{89}{30} & \frac{181}{60} & \frac{1259}{105} & \frac{1}{168} & -\frac{1}{252} & \frac{1}{360} & -\frac{1}{495} & \frac{1}{660} & ; & 0 \\ 0 & 0 & 0 & 4 & 4 & 20 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 5 & 5 & 30 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 6 & 42 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 56 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 72 & ; & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \end{bmatrix}$$

and has the solution

$$A = \begin{bmatrix} 0 & \frac{626}{867} & \frac{556}{975} & -\frac{413}{1439} & -\frac{318}{4301} & \frac{319}{4419} & \frac{19}{65216} & -\frac{19}{1835} & \frac{1}{795} & \frac{32}{31661} \end{bmatrix}^T.$$

Substituting the elements of column matrix in to the Taylor polynomial (3.2), we get

$$\begin{aligned} y(x) = \frac{626}{867}(x-1) + \frac{556}{975}(x-1)^2 - \frac{413}{1439}(x-1)^3 \\ -\frac{318}{4301}(x-1)^4 + \frac{319}{4419}(x-1)^5 + \frac{19}{65216}(x-1)^6 \\ -\frac{19}{1835}(x-1)^7 + \frac{1}{795}(x-1)^8 + \frac{32}{31661}(x-1)^9. \end{aligned}$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_4(x_i)$ at any point x_i in $[0, 1]$. Also, by taking $N = 9$

i	x_i	$y(x_i), c = 0$	$y(x_i), c = 1/2$	$y(x_i), c = 1$
0	0	0.00000000	1.44833×10^{-7}	-5.68058×10^{-7}
1	0.2	-0.11715398	-0.11723605	-0.11732917
2	0.4	-0.18057311	-0.18073470	-0.18081375
3	0.6	-0.18153418	-0.18176554	-0.18181592
4	0.8	-0.11915953	-0.11941775	-0.11944095
5	1.0	3.06258×10^{-7}	1.40673×10^{-7}	0.00000000

i	x_i	$D(x_i), c = 0$	$D(x_i), c = 1/2$	$D(x_i), c = 1$
0	0	0.00000	1.41473×10^{-4}	1.07115×10^{-2}
1	0.2	5.70817×10^{-8}	1.90580×10^{-6}	1.80134×10^{-3}
2	0.4	1.38181×10^{-5}	2.39302×10^{-10}	1.81050×10^{-4}
3	0.6	3.39365×10^{-4}	2.00893×10^{-10}	7.09947×10^{-6}
4	0.8	3.27944×10^{-3}	1.12168×10^{-6}	2.78898×10^{-8}
5	1.0	1.90444×10^{-2}	5.74010×10^{-5}	1.19216×10^{-16}.

3.2 Method of Chebyshev Polynomial Solutions

We consider the integro-differential equations

$$P(x)y'' + Q(x)y' + R(x)y = f(x) + \int_{-1}^{1} K(x,\xi)d\xi \qquad (3.41)$$

and

$$P(x)y'' + Q(x)y' + R(x)y = f(x) + \int_{-1}^{1} K(x,\xi)y(\xi)d\xi \qquad (3.42)$$

under the prescribed conditions. We will find approximate solution $y(x)$ in the truncated Chebyshev series as

$$y(x) = \sum_{r=0}^{N}{}' a_r T_r(x), \quad -1 \le x \le 1 \qquad (3.43)$$

or in the matrix form

$$\left[\ y(x)\ \right] \ = \ TA \tag{3.44}$$

where

$$T = \left[\ T_0(x) \ \ T_1(x) \ \ \ldots \ \ T_N(x)\ \right],$$

and

$$A = \left[\ \tfrac{1}{2}a_0 \ \ a_1 \ \ \ldots \ \ a_N\ \right]^T$$

Here $\sum{}'$ denotes a sum whose first term is halved, $T_r(x)$ denotes the Chebyshev polynomial of the first kind of degree r and $a_r,\ r = 0, 1, \ldots, N$ are the Chebyshev coefficients to be determined. In (3.41) and (3.42), we assume that the functions $P,\ Q,\ R,\ f$ and K are defined in the range $-1 \leq x, \xi \leq 1$, and its derivative with respect to x can be expanded in Chebyshev series.

For

$$y(x) = \sum_{r=0}^{N}{}' a_r T_r(x)$$

and

$$y^{(n)}(x) = \sum_{r=0}^{N}{}' a_r^{(n)} T_r(x)$$

where $a_r^{(n)}$ and a_r are Chebyshev coefficients, we have $a_r^{(0)} = a_r$ and $y^{(0)}(x) = y(x)$; and the recurrence relation between the Chebyshev coefficients $a_r^{(n)}$ and $a_r^{(n+1)}$ of $y^{(n)}(x)$ and $y^{(n+1)}(x)$, is given by

$$2r a_r^{(n)} = a_{r-1}^{(n+1)} - a_{r+1}^{(n+1)}, \quad r \geq 1. \tag{3.45}$$

From this, we can deduce the relations

$$\begin{aligned} 2(r+1)a_{r+1}^{(n)} &= a_r^{(n+1)} - a_{r+2}^{(n+1)} \\ 2(r+3)a_{r+3}^{(n)} &= a_{r+2}^{(n+1)} - a_{r+4}^{(n+1)} \\ 2(r+5)a_{r+5}^{(n)} &= a_{r+4}^{(n+1)} - a_{r+6}^{(n+1)} \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

By adding both side, we get

$$
\begin{aligned}
a_r^{(n+1)} &= 2(r+1)a_{r+1}^{(n)} + a_{r+2}^{(n+1)} \\
a_r^{(n+1)} &= 2(r+1)a_{r+1}^{(n)} + 2(r+3)a_{r+3}^{(n)} + a_{r+4}^{(n+1)} \\
a_r^{(n+1)} &= 2(r+1)a_{r+1}^{(n)} + 2(r+3)a_{r+3}^{(n)} + 2(r+5)a_{r+5}^{(n)} + a_{r+6}^{(n+1)} \\
&\dots \\
a_r^{(n+1)} &= 2\sum_{i=0}^{\infty}(r+2i+1)a_{r+2i+1}^{(n)}. \qquad (3.46)
\end{aligned}
$$

Let us take $r = 0, 1, \dots, N$ and assume $a_r^{(n)} = 0$ for $r > N$

$$
\begin{aligned}
r = 0; \quad a_0^{(n+1)} &= 2\Big((1)a_1^{(n)} + (3)a_3^{(n)} + (5)a_5^{(n)} + \dots\Big) \\
\frac{1}{2}a_0^{(n+1)} &= 2\Big(\frac{1}{2}a_1^{(n)} + \frac{3}{2}a_3^{(n)} + \frac{5}{2}a_5^{(n)} + \dots\Big) \\
r = 1; \quad a_1^{(n+1)} &= 2\Big((2)a_2^{(n)} + (4)a_4^{(n)} + (6)a_6^{(n)} + \dots\Big). \\
r = 2; \quad a_2^{(n+1)} &= 2\Big((3)a_3^{(n)} + (5)a_5^{(n)} + (7)a_7^{(n)} + \dots\Big) \\
r = 3; \quad a_3^{(n+1)} &= 2\Big((4)a_4^{(n)} + (6)a_6^{(n)} + (8)a_8^{(n)} + \dots\Big) \\
&\vdots
\end{aligned}
$$

Then the system (3.46) can be transformed into the matrix form

$$
A^{(n+1)} = 2MA^{(n)} \quad , n = 0, 1, 2, \dots \qquad (3.47)
$$

where

$$
A^{(n)} = \left[\begin{array}{cccc} \frac{1}{2}a_0^{(n)} & a_1^{(n)} & \dots & a_N^{(n)} \end{array}\right]^T.
$$

When N is odd, we have

$$
M = \begin{bmatrix}
0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & \frac{N}{2} \\
0 & 0 & 2 & 0 & 4 & 0 & \dots & 0 \\
0 & 0 & 0 & 3 & 0 & 5 & \dots & N \\
. & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & N \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0
\end{bmatrix}_{(N+1)\times(N+1)}
$$

and when N is even, we have

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & \frac{5}{2} & \dots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \dots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \dots & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Consequently, the matrix (3.47) gives a relation between the Chebyshev coefficient matrix A of $y(x)$ and the Chebyshev coefficient matrix $A^{(n)}$ of the n^{th} derivative of $y(x)$. For $n = 0, 1, 2, \dots$, it follows from relation (3.47) that

$$\begin{aligned} n = 0; \quad A^{(1)} &= 2MA \\ n = 1; \quad A^{(2)} &= 2MA^{(1)} = 2^2M^2A \\ n = 2; \quad A^{(3)} &= 2MA^{(2)} = 2^3M^3A \\ &\vdots \\ A^{(n)} &= 2MA^{(n-1)} = 2^nM^nA \end{aligned} \tag{3.48}$$

where

$$A^{(0)} = A = \begin{bmatrix} \frac{1}{2}a_0 & a_1 & \dots & a_N \end{bmatrix}^T.$$

To obtain the solution of (3.41) in the form of expression (3.43), we first reduce (3.41) into a differential equation whose coefficients are polynomials. Thus, we assume that the functions $P(x)$, $Q(x)$ and $R(x)$ can be expanded in the forms

$$P(x) = \sum_{i=0}^{I} p_i x^i, \quad Q(x) = \sum_{i=0}^{I} q_i x^i, \quad R(x) = \sum_{i=0}^{I} r_i x^i \tag{3.49}$$

which are Taylor polynomials of degree I, at $x = 0$. By using the expansions (3.49) in (3.41), we get

$$\sum_{i=0}^{I} \left\{ p_i x^i y'' + q_i x^i y' + r_i x^i y \right\} = f(x) + \int_{-1}^{1} K(x, \xi) d\xi. \tag{3.50}$$

The Chebyshev expansions of terms $x^p y^{(s)}(x),\ s=0,1;\ p=0,1,\ldots,I$ in (3.50) are obtained by means of the formula

$$x^p y^{(s)}(x) \;=\; \sum_{r=0}^{n}\sum_{j=0}^{p} 2^{-p}\binom{p}{j} a^{(s)}_{|r-p+2j|} T_r(x). \qquad (3.51)$$

The matrix representation of formula (3.51) can be given by

$$\left[\ x^p y^{(s)}(x)\ \right] = TM_pA^{(s)}$$

or, since $A^{(s)} = 2^s M^s A$ from relation (3.48), by

$$\left[\ x^p y^{(s)}(x)\ \right] \;=\; 2^s TM_pM^sA, \quad s=0,1,2; \quad p=0,1,\ldots,I \qquad (3.52)$$

where

$$T \;=\; \left[\ T_0(x)\ \ T_1(x)\ \ \ldots\ \ T_N(x)\ \right],$$

$$M_0 \;=\; \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$M_1 \;=\; \frac{1}{2^1}\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 2 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$M_2 = \frac{1}{2^2}\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 2 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$M_3 = \frac{1}{2^3}\begin{bmatrix} 0 & 3 & 0 & 1 & 0 & \dots & & & & & & \\ 6 & 0 & 4 & 0 & 1 & \dots & & & & & & \\ 0 & 4 & 0 & 3 & 0 & \dots & & & & & & \\ 2 & 0 & 3 & 0 & 3 & \dots & & & & & & \\ 0 & 1 & 0 & 3 & 0 & \dots & & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & \dots & 1 & 0 & 3 & 0 & 3 & 0 & 1 \\ & & & & \dots & 0 & 1 & 0 & 3 & 0 & 3 & 0 \\ & & & & \dots & 0 & 0 & 1 & 0 & 3 & 0 & 3 \\ & & & & \dots & 0 & 0 & 0 & 1 & 0 & 3 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

and so on.

Also we assume that the function $f(x)$ can be expanded as

$$f(x) = \sum_{r=0}^{N}{}' f_r T_r(x)$$

or in the matrix form as

$$\left[\, f(x) \,\right] = TF \tag{3.53}$$

where

$$F = \left[\, \tfrac{1}{2}f_0 \quad f_1 \quad \dots \quad f_N \,\right]^T.$$

We consider the function $K(x,\xi)$, if the function $K(x,\xi)$ can be approximated by a truncated double Chebyshev series of degree N in both x and ξ by

$$K(x,\xi) \;=\; \sum_{r=0}^{N}{}' \sum_{s=0}^{N}{}' k_{r,s} T_r(x) T_s(\xi) \tag{3.54}$$

then we can write in the matrix form

$$\Big[\; K(x,\xi) \;\Big] \;=\; T_x K T_\xi^T \tag{3.55}$$

where

$$T_x = \Big[\; T_0(x) \;\; T_1(x) \;\; \ldots \;\; T_N(x) \;\Big],$$

$$T_\xi = \Big[\; T_0(\xi) \;\; T_1(\xi) \;\; \ldots \;\; T_N(\xi) \;\Big],$$

and

$$K = \begin{bmatrix} \frac{1}{4}k_{00} & \frac{1}{2}k_{01} & \ldots & \frac{1}{2}k_{0N} \\ \frac{1}{2}k_{10} & k_{11} & \ldots & k_{1N} \\ \vdots & & & \\ \frac{1}{2}k_{N0} & k_{N1} & \ldots & k_{NN} \end{bmatrix}.$$

By substituting the expansions (3.52), (3.53) and (3.55) into (3.50) and then by simplifying the result, we have the matrix equation

$$\sum_{i=0}^{I} \Big\{ p_i 2^2 T M_i M^2 + q_i 2 T M_i M + r_i T M_i \Big\} A \;=\; TF + T_x K \int_{-1}^{1} T_\xi^T \, d\xi$$

or

$$\sum_{i=0}^{I} \Big\{ 4p_i M_i M^2 + 2q_i M_i M + r_i M_i \Big\} A \;=\; F + K \int_{-1}^{1} T_\xi^T \, d\xi \tag{3.56}$$

which corresponds to a system of $(N+1)$ equations for the unknown Chebyshev coefficients $a_r,\; r = 0, 1, \ldots, N$. Briefly, we can write this equation in the form

$$WA \;=\; G \tag{3.57}$$

where

$$W = \Big[\; w_{nm} \;\Big] = \sum_{i=0}^{I} \Big\{ 4p_i M_i M^2 + 2q_i M_i M + r_i M_i \Big\}, \quad n, m = 0, 1, \ldots, N; \quad I = 0, 1, \ldots$$

and

$$G = F + K\int_{-1}^{1} T_\xi^T d\xi = \left[\begin{array}{cccc} g_0 & g_1 & \dots & g_N \end{array} \right]^T.$$

Then the augmented matrix becomes

$$\left[\begin{array}{ccc} W & ; & G \end{array} \right]. \tag{3.58}$$

Next, we consider (3.42) and look for the solution $y(x)$ in the form of (3.43). We assume that the functions $P(x)$, $Q(x)$ and $R(x)$ can be expanded in the Taylor polynomials of degree I, at $x = 0$, following from (3.49) and substituting them into (3.42), we have

$$\sum_{i=0}^{I}\left\{p_i x^i y'' + q_i x^i y' + r_i x^i y\right\} \;=\; f(x) + \int_{-1}^{1} K(x,\xi)y(\xi)d\xi \tag{3.59}$$

If the function $K(x,\xi)$ can be approximated by a truncated double Chebyshev series of degree N in both x and ξ, its follow form (3.55). On the other hand, for the unknown function $y(\xi)$ in integrand, we write from expansions (3.44)

$$\left[\begin{array}{c} y(\xi) \end{array} \right] \;=\; T_\xi A \tag{3.60}$$

where

$$T_\xi = \left[\begin{array}{cccc} T_0(\xi) & T_1(\xi) & \dots & T_N(\xi) \end{array} \right]$$

Substituting the matrix forms (3.52), (3.53), (3.55) and (3.60) into (3.59), and simplifying the equation, we have

$$\sum_{i=0}^{I}\left\{p_i 2^2 T M_i M^2 + q_i 2 T M_i M + r_i T M_i\right\}A \;=\; TF + T_x K\int_{-1}^{1} T_\xi^T T_\xi d\xi A$$

or

$$\sum_{i=0}^{I}\left\{4p_i M_i M^2 + 2q_i M_i M + r_i M_i\right\}A \;=\; F + K\int_{-1}^{1} T_\xi^T T_\xi d\xi A \tag{3.61}$$

or briefly

$$\left(W - KQ\right)A \;=\; F$$

where

$$W = \left[\ w_{nm}\ \right] = \sum_{i=0}^{I} \left\{4p_iM_iM^2+2q_iM_iM+r_iM_i\right\}, \quad n,m=0,1,\ldots,N; \quad I=0,1,\ldots$$

and

$$Q = \int_{-1}^{1} T_\xi^T T_\xi d\xi = \left[\ q_{ij}\ \right], \quad i,j=0,1,\ldots,N.$$

The elements of the fixed matrix Q are given by

$$q_{ij} = \int_{-1}^{1} T_i(t)T_j(t)dt = \begin{cases} \frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2} & \text{, for even } (i+j) \\ 0 & \text{, for odd } (i+j). \end{cases}$$

Then the augmented matrix becomes

$$\left[\ \bar{W}\ \ ;\ \ F\ \right] \tag{3.62}$$

where

$$\bar{W} = W - KQ.$$

If the conditions are given as

$$y(a) = \lambda \quad \text{and} \quad y'(a) = \mu,$$

then it is seen form (3.52) that

$$UA = \left[\ \lambda\ \right] \quad \text{and} \quad VA = \left[\ \mu\ \right] \tag{3.63}$$

where

$$\begin{aligned} A &= \left[\ \tfrac{1}{2}a_0\ \ a_1\ \ \ldots\ \ a_N\ \right]^T, \\ U &= \left[\ T_0(a)\ \ T_1(a)\ \ \ldots\ \ T_N(a)\ \right], \\ V &= 2\left[\ T_0(a)\ \ T_1(a)\ \ \ldots\ \ T_N(a)\ \right]M. \end{aligned}$$

Then the augmented matrices of them become

$$\left[\ U\ \ ;\ \ \lambda\ \right] \quad \text{and} \quad \left[\ V\ \ ;\ \ \mu\ \right]$$

or

$$\begin{bmatrix} u_0 & u_1 & \dots & u_N & ; & \lambda \end{bmatrix} \text{ and } \begin{bmatrix} v_0 & v_1 & \dots & v_N & ; & \mu \end{bmatrix}. \tag{3.64}$$

Consequently, by replacing the matrices (3.64) into the last two rows of augmented matrices (3.58) and (3.62), we have the new augmented matrices

$$W^* = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & \dots & w_{0N} & ; & g_0 \\ w_{10} & w_{11} & w_{12} & w_{13} & \dots & w_{1N} & ; & g_1 \\ w_{20} & w_{21} & w_{22} & w_{23} & \dots & w_{2N} & ; & g_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & w_{N-2,2} & w_{N-2,3} & \dots & w_{N-2,N} & ; & g_{N-2} \\ u_0 & u_1 & u_2 & u_3 & \dots & u_N & ; & \lambda \\ v_0 & v_1 & v_2 & v_3 & \dots & v_N & ; & \mu \end{bmatrix} \tag{3.65}$$

and

$$\bar{W}^* = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & \dots & w_{0N} & ; & f_0 \\ w_{10} & w_{11} & w_{12} & w_{13} & \dots & w_{1N} & ; & f_1 \\ w_{20} & w_{21} & w_{22} & w_{23} & \dots & w_{2N} & ; & f_2 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ w_{N-2,0} & w_{N-2,1} & w_{N-2,2} & w_{N-2,3} & \dots & w_{N-2,N} & ; & f_{N-2} \\ u_0 & u_1 & u_2 & u_3 & \dots & u_N & ; & \lambda \\ v_0 & v_1 & v_2 & v_3 & \dots & v_N & ; & \mu \end{bmatrix}. \tag{3.66}$$

Let

$$W^{**} = \bar{W}^{**} = \begin{bmatrix} w_{00} & w_{01} & w_{02} & w_{03} & \dots & w_{0N} \\ w_{10} & w_{11} & w_{12} & w_{13} & \dots & w_{1N} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ w_{N-2,0} & w_{N-2,1} & w_{N-2,2} & w_{N-2,3} & \dots & w_{N-2,N} \\ u_0 & u_1 & u_2 & u_3 & \dots & u_N \\ v_0 & v_1 & v_2 & v_3 & \dots & v_N \end{bmatrix}.$$

If $detW^{**} = \bar{W}^{**} \neq 0$, we can write

$$A = (W^{**})^{-1}G \quad (3.67)$$

$$A = (\bar{W}^{**})^{-1}F. \quad (3.68)$$

Thus the matrix A (Chebyshev coefficients a_r) is uniquely determined. We can easily check the accuracy of this solution. Since the Chebyshev polynomial (3.43) is an approximate solution of (3.41) and (3.42). Substituting $y(x)$ and its derivatives $y^{(1)}(x)$ and $y^{(2)}(x)$ in (3.41) and (3.42), the resulting equations will be satisfied approximately for a given value of x in the interval $a \leq x \leq b$. $x = x_i \in [a,b]$, $i = 0, 1, \ldots, M$, we define the relative error of the obtained solutions of (3.41) and (3.42) as follow

$$D_1(x_i) = \frac{\left|P(x_i)y^{(2)}(x_i) + Q(x_i)y^{(1)}(x_i) + R(x_i)y(x_i) - f(x_i) - \int_{-1}^{1} K(x_i,\xi)d\xi\right|}{\left|f(x_i) + \int_{-1}^{1} K(x_i,\xi)d\xi\right|}$$

and

$$D_2(x_i) = \frac{\left|P(x_i)y^{(2)}(x_i) + Q(x_i)y^{(1)}(x_i) + R(x_i)y(x_i) - f(x_i) - \int_{-1}^{1} K(x_i,\xi)y(\xi)d\xi\right|}{\left|f(x_i) - \int_{-1}^{1} K(x_i,\xi)y(\xi)d\xi\right|}.$$

If max $\{D_1(x_i), D_2(x_i)\} \leq 10^{-k}$ where k is a prescribed integer, then the truncation limit N is increased until the difference $D_1(x_i)$ or $D_2(x_i)$ at each the points x_i becomes smaller than the prescribed tolerance (10^{-k}).

Example 3.7

Let us consider the problem

$$y'' + xy' + xy = 1 + x + x^2 + \int_{-1}^{1} (1 - 3x\xi)y(\xi)d\xi$$

$$y(0) = 1, \qquad y'(0) + 2y(1) - y(-1) = -1$$

and approximate the solution $y(x)$ by the truncated Chebyshev polynomials. Thus, the matrix of this equation for $N = 9$ is

$$\left\{4M_0M^2 + 2M_1M + M_1 - KQ\right\}A = F$$

where

$$KQ = \begin{bmatrix} 2 & 0 & -\frac{2}{3} & 0 & -\frac{2}{15} & 0 & -\frac{2}{35} & 0 & -\frac{2}{63} & 0 \\ 0 & -2 & 0 & \frac{6}{5} & 0 & \frac{2}{7} & 0 & \frac{2}{15} & 0 & \frac{6}{77} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$F = \begin{bmatrix} \frac{3}{2} & 2 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

We obtain (3.61), we have the augmented matrix

$$\left[\begin{array}{cccccccccccc} -2 & \frac{1}{2} & \frac{20}{3} & 0 & \frac{542}{15} & 0 & \frac{3992}{35} & 0 & \frac{16634}{63} & 0 & ; & \frac{3}{2} \\ 1 & 3 & \frac{1}{2} & \frac{144}{5} & 0 & \frac{908}{7} & 0 & \frac{5248}{15} & 0 & \frac{56820}{77} & ; & 1 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} & 56 & 0 & 204 & 0 & 496 & 0 & ; & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 3 & \frac{1}{2} & 90 & 0 & 294 & 0 & 666 & ; & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 4 & \frac{1}{2} & 132 & 0 & 400 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 5 & \frac{1}{2} & 182 & 0 & 522 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 6 & \frac{1}{2} & 240 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 7 & \frac{1}{2} & 306 & ; & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & ; & 1 \\ 1 & 4 & 1 & 0 & 1 & 8 & 1 & -4 & 1 & 12 & ; & -1 \end{array}\right]$$

and has the solution

$$A = \begin{bmatrix} \frac{2478}{1217} & -\frac{599}{606} & \frac{1447}{1425} & \frac{205}{2321} & -\frac{201}{9832} & -\frac{8817}{925786} & \frac{10}{35137} & \frac{75}{220088} & \frac{4}{332881} & -\frac{7}{845398} \end{bmatrix}^T.$$

By substituting the elements of column matrix into the Chebyshev polynomials (3.43), we get

$$y(x) = 9.99999848 - 1.30349977x + 2.19916308x^2 + 0.56384887x^3 - 0.17528580x^4 - 0.19412444x^5 + 0.00603103x^6 + 0.02657881x^7 + 0.00153809x^8 - 0.00211971x^9.$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_2(x_i)$ at any point x_i in $[-1, 1]$. Also, by taking $N = 9$

i	x_i	$y(x_i), N = 9$	$D(x_i)$
0	-1.0	3.94076239	9.84824×10^{-5}
1	-0.8	3.14993658	4.88772×10^{-5}
2	-0.6	2.44396720	2.22169×10^{-5}
3	-0.4	1.83466276	7.28456×10^{-5}
4	-0.2	1.34393725	1.17971×10^{-5}
5	0	0.99999985	4.94803×10^{-5}
6	0.2	0.83143536	1.24346×10^{-5}
7	0.4	0.86014591	3.05685×10^{-5}
8	0.6	1.09460779	1.72723×10^{-5}
9	0.8	1.52507582	4.60307×10^{-5}
10	1.0	2.12213009	1.66696×10^{-5}.

Example 3.8

Let us consider the problem

$$y'' + 8x^2y = x + 1 + \int_{-1}^{1}(x\xi + x^2\xi^2)d\xi$$

$$y(0) = 1, \qquad y(1) = 0$$

and approximate the solution $y(x)$ by the truncated Chebyshev polynomials. Since $P(x) = 1,\ R(x) = 8x^2$, we have

$$\begin{aligned} p_0 &= 1,\ p_1 = p_2 = \ldots = 0 \\ r_0 &= r_1 = 0,\ r_2 = 8,\ r_3 = r_4 = \ldots = 0 \end{aligned}$$

and $f(x) = x + 1,\ K(x,\xi) = x\xi + x^2\xi^2$ by using the expansions for the powers x^r in terms of the Chebyshev polynomials $T_r(x)$, we find the representations

$$f(x) = x + 1 = T_0(x) + T_1(x)$$

and

$$\begin{aligned} K(x,\xi) &= x\xi + x^2\xi^2 \\ &= \frac{1}{4}T_0(x)T_0(\xi) + \frac{1}{4}T_0(x)T_2(\xi) + T_1(x)T_1(\xi) + \frac{1}{4}T_2(x)T_0(\xi) + \frac{1}{4}T_2(x)T_2(\xi) \end{aligned}$$

We first take $N = 9$, and then proceed as before. Then the matrix (3.56) becomes

$$\left\{4M_0M^2 + 8M_2\right\}A = F + K\int_{-1}^{1} T_\xi^T d\xi.$$

Thus, we have the augmented matrix

$$\begin{bmatrix} 4 & 0 & 6 & 0 & 32 & 0 & 108 & 0 & 256 & 0 & ; & \frac{4}{3} \\ 0 & 6 & 0 & 26 & 0 & 120 & 0 & 336 & 0 & 720 & ; & 1 \\ 4 & 0 & 4 & 0 & 50 & 0 & 192 & 0 & 480 & 0 & ; & \frac{4}{3} \\ 0 & 2 & 0 & 4 & 0 & 82 & 0 & 280 & 0 & 648 & ; & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 122 & 0 & 384 & 0 & ; & 0 \\ 0 & 0 & 0 & 2 & 0 & 4 & 0 & 170 & 0 & 504 & ; & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 226 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 290 & ; & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & ; & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & ; & 0 \end{bmatrix}$$

and has the solution

$$A = \begin{bmatrix} \frac{845}{841} & -\frac{857}{797} & -\frac{385}{4987} & \frac{389}{1920} & -\frac{140}{1731} & \frac{104}{3959} & \frac{36}{20423} & -\frac{19}{7394} & \frac{19}{27756} & -\frac{10}{68623} \end{bmatrix}^T.$$

By substituting the elements of column matrix into the Chebyshev polynomials (3.43), we get

$$\begin{aligned} y(x) = {} & 1.00000068 - 1.53507246x + 0.50244716x^2 + 0.15861786x^3 - 0.62210946x^4 \\ & +0.64515642x^5 - 0.11883440x^6 - 0.08052080x^7 + 0.08762069x^8 - 0.03730528x^9. \end{aligned}$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_1(x_i)$ at any point x_i in $[-1, 1]$. Also, by taking $N = 9$

i	x_i	$y(x_i), N = 9$	$D(x_i)$
0	-1.0	1.69824893	2.04754×10^{-2}
1	-0.8	2.00763365	5.08312×10^{-1}
2	-0.6	1.93542830	2.52844×10^{-3}
3	-0.4	1.66144964	1.15840×10^{-2}
4	-0.2	1.32463596	6.84454×10^{-3}
5	0	1.00000068	4.89431×10^{-3}
6	0.2	0.71356576	4.06527×10^{-3}
7	0.4	0.46662415	1.76414×10^{-3}
8	0.6	0.25693894	2.28411×10^{-3}
9	0.8	0.09296517	5.60362×10^{-4}
10	1.0	4.13848×10^{-7}	6.10006×10^{-4}.

Example 3.9

Let us consider the problem

$$y'' + 8x^2 y = x + 1 + \int_{-1}^{1} (x\xi + x^2\xi^2) y(\xi) d\xi$$

$$y(0) = 1, \qquad y(1) = 0$$

and approximate the solution $y(x)$ by the truncated Chebyshev polynomials. We first take $N = 9$, and then proceed as before. Then the matrix (3.61) becomes

$$\Big\{4M_0M^2 + 8M_2 - KQ\Big\}A \; = \; F.$$

Thus, we have the augmented matrix

$$\begin{bmatrix} \frac{11}{3} & 0 & \frac{89}{15} & 0 & \frac{3373}{105} & 0 & \frac{9291}{86} & 0 & \frac{14593}{57} & 0 & ; & 1 \\ 0 & \frac{16}{3} & 0 & \frac{132}{5} & 0 & \frac{2522}{21} & 0 & \frac{15122}{45} & 0 & \frac{27361}{38} & ; & 1 \\ \frac{11}{3} & 0 & \frac{59}{15} & 0 & \frac{5263}{105} & 0 & \frac{16515}{86} & 0 & \frac{27361}{57} & 0 & ; & 0 \\ 0 & 2 & 0 & 4 & 0 & 82 & 0 & 280 & 0 & 648 & ; & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 122 & 0 & 384 & 0 & ; & 0 \\ 0 & 0 & 0 & 2 & 0 & 4 & 0 & 170 & 0 & 504 & ; & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 226 & 0 & ; & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 4 & 0 & 290 & ; & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & ; & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & ; & 0 \end{bmatrix}$$

and has the solution

$$A = \begin{bmatrix} \frac{981}{976} & -\frac{5157}{4987} & -\frac{293}{3819} & \frac{379}{2359} & -\frac{192}{2377} & \frac{17}{665} & \frac{35}{19956} & -\frac{43}{20946} & \frac{19}{27787} & -\frac{14}{94603} \end{bmatrix}^T.$$

By substituting the elements of column matrix into the Chebyshev polynomials (3.43), we get

$$y(x) = 1.00000044 - 1.37521457x + 0.50243809x^2 + 0.03416313x^3 - 0.62097421x^4$$
$$+0.57501680x^5 - 0.11892241x^6 - 0.04614504x^7 + 0.087523294x^8 - 0.03788463x^9.$$

The solutions are tabulated together with Taylor solution and the accuracy of solution can be checked by means of the difference $D_2(x_i)$ at any point x_i in $[-1, 1]$. Also, by taking $N = 9$

i	x_i	$y(x_i), N = 9$	$D(x_i)$
0	-1.0	1.70012915	8.69870×10^{-3}
1	-0.8	1.95973962	5.75383×10^{-3}
2	-0.6	1.87103125	7.65766×10^{-4}
3	-0.4	1.60616059	4.74881×10^{-3}
4	-0.2	1.29368323	2.61478×10^{-3}
5	0	1.00000044	2.43216×10^{-3}
6	0.2	0.74450800	1.75544×10^{-3}
7	0.4	0.52196709	1.22547×10^{-3}
8	0.6	0.32161174	1.23989×10^{-3}
9	0.8	0.14169816	1.04884×10^{-4}
10	1.0	5.34148×10^{-7}	6.67687×10^{-48}.