CHAPTER 2

PRELIMINARIES

The aim of this chapter is to give some definitions, notations and results of metric spaces, fixed point of selfmappings and common fixed point of selfmappings in metric spaces which will be used in the later chapters.

2.1 Metric Spaces

Definition 2.1.1 (cf. [11])A *metric space* is a pair (X, d), where X is a set and d is a metric on X(or distance function on X),that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Definition 2.1.2 (cf. [11]) A sequence (x_n) in a metric space $X = (X, d)$ is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x is called the *limit* of (x_n) and we write

$$\lim_{n\to\infty} x_n = x$$

or, simple, $x_n \to x$

we say that (x_n) *converges to* x .If (x_n) is not convergent, it is said to be *divergent*.

Proposition 2.1.3 (cf. [11]) *Let $X = (X, d)$ be a metric space. Then:*

(a) *A convergent sequence in X is bounded and its limit is unique.*

(b) *If $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.*

Definition 2.1.4 (cf. [11]) A sequence (x_n) in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N(\epsilon)$. The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

Theorem 2.1.5 (cf. [11]) *Every convergent sequence in a metric space is a Cauchy sequence.*

2.2 Fixed Point of Selfmappings in Metric Spaces

In this section, we give definitions and some results of fixed point in metric spaces.

Definition 2.2.1 (cf. [11]) A *fixed point* of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself (is"kept fixed"by T), that is, $Tx = x$, the image Tx coincides with x.

Definition 2.2.2 (cf. [11]) Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a *contraction* on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Theorem 2.2.3 (Banach Fixed Point Theorem)*Suppose that X is a complete metric space and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.*

Theorem 2.2.4 (Brouwer's Fixed Point Theorem)*Let B be closed ball in $\mathbb{R}^n$. Then any continuous mapping $f : B \to B$ has at least one fixed point.*

Theorem 2.2.5 (cf. [9]) *Let M be a complete metric space and suppose $f : M \to M$ satisfies $d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y)$ for each $x, y \in M$,where $\alpha : [0, \infty) \to [0, 1)$ is monotonically decreasing. Then f has a unique fixed point $\overline{x}$,and $\{f^n(x)\}$ converges to $\overline{x}$ for each $x \in M$.*

Theorem 2.2.6 (cf. [9]) *Let M be a complete metric space and suppose $f : M \to M$ satisfies $d(f(x), f(y)) \leq \psi(d(x, y))$ for each $x, y \in M$, where $\psi : [0, \infty) \to [0, \infty)$ is upper semicontinuous (that is $\{x \in [0, \infty)|\psi(x) \geq a\}$ is closed for each $a \in \mathbb{R}$) from the right and satisfies $0 \leq \psi(t) < t$ for $t > 0$. Then f has a unique fixed point $\overline{x}$,and $\{f^n(x)\}$ converges to $\overline{x}$ for each $x \in M$.*

Theorem 2.2.7 (cf. [2]) *Let (M, d) be a complete metric space and $f : M \to M$. If there exists a lower semicontinuous function ψ mapping M into the nonnegative numbers such that*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)), x \in M,$$

then f has a fixed point.

2.3 Common Fixed Point of Selfmappings in Metric Spaces

In this section, we give some definitions and theorems concerning common fixed point of selfmappings in metric spaces.

Definition 2.3.1 (cf. [1]) Two selfmappings S and T of a metric space (X, d) are said to be *weakly commuting* if

$$d(STx, TSx) \leq d(Sx, Tx), \forall x \in X.$$

It is clear that two commuting mappings are weakly commuting.

Definition 2.3.2 (cf. [1]) Let T and S be two selfmapping of a metric space (X, d). Then S and T are said to be *compatible* if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Obviously, two weakly commuting mappings are compatible.

Definition 2.3.3 (cf. [1]) Two selfmapping T and S of a metric space (X, d) are said to be *weakly compatible* if they commute at there coincidence point; i.e, if $Tu = Su$ for some $u \in X$, then $STu = TSu$.

It is easy to see that two compatible maps are weakly compatible.

Definition 2.3.4 (cf. [1]) Let S and T be two selfmappings of a metric space (X, d). We say that T and S *satisfy property (E.A)* if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Example 2.3.5

(1) Let $X = [0, \infty)$. Define $T, S : X \to X$ by $Tx = \frac{x}{4}$ and $Sx = \frac{3x}{4}, \forall x \in X$. Consider the sequence $x_n = \frac{1}{n}$, clearly

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0.$$

Then S and T satisfy property(E.A).

(2) Let $X = [2, \infty)$. Define $T, S : X \to X$ by $Tx = x + 1$ and $Sx = 2x + 1, \forall x \in X$. Suppose that S and T satisfy property (E.A). Then there exists in X a sequence (x_n) satisfying

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$. Therefore $\lim_{n\to\infty} x_n = t - 1$ and $\lim_{n\to\infty} x_n = \frac{t-1}{2}$. Then $t = 1$, which is a contradiction since $1 \notin X$. Hence T and S do not satisfy property(E.A). □

Theorem 2.3.6 (cf. [1]) *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2)

$$\begin{aligned} d(Tx,Ty) &< \max\{d(Sx,Sy),[d(Tx,Sx)+d(Ty,Sy)]/2, \\ &\qquad [d(Ty,Sx)+d(Tx,Sy)]/2\}, \forall x \neq y \in X, \end{aligned}$$

(3) *$TX \subset SX$.*

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Theorem 2.3.7 (cf. [1]) *Let A,B,T and S be selfmappings of a metric space (X,d) such that*

(1) *$d(Ax,By) \leq F(\max\{d(Sx,Ty),d(Sx,By),d(Ty,By)\}), \forall (x,y) \in X^2$, where $F:\mathbb{R}^+ \to \mathbb{R}^+$ and F is nondecreasing on $\mathbb{R}^+$ such that $0 < F(t) < t$, for each $t \in (0,\infty)$,*

(2) *(A,S) or (B,T) are weakly compatible,*

(3) *(A,S) or (B,T) satisfies the property (E.A),*

(4) *$AX \subset TX$ and $BX \subset SX$.*

If the range of the one of the mappings A,B,T or S is a complete subspace of X, then A,B,T and S have a unique common fixed point.

Theorem 2.3.8 (cf. [3]) *Let S and T be two commuting mappings of a complete metric space (X,d) into itself satisfying the inequality*

$$d(Sx,Sy) \leq c \cdot \max\{d(Tx,Ty),d(Tx,Sx),d(Ty,Sy),d(Tx,Sy),d(Ty,Sx)\}$$

for all $x,y \in X$, where $0 \leq c < 1$. If the range of T contains the range of S and if T is continuous, then S and T have a unique common fixed point.

Theorem 2.3.9 (cf. [4]) *Let S and I be commuting mappings and T and J be commuting mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Sx, Ty) \leq c \cdot \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\}$$

for all $x, y \in X$, where $0 \leq c < 1$. If the range of I contains the range of T and the range of J contains the range of S and if one of S, T, I and J is continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Theorem 2.3.10 (cf. [4]) *Let S and I be commuting mappings and T and J be commuting mappings of a compact metric space (X, d) into itself satisfying the inequality*

$$d(Sx, Ty) < \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \frac{1}{2}d(Ix, Ty), \frac{1}{2}d(Jy, Sx)\}$$

for all $x, y \in X$ for which the right hand side of the inequality is positive. If the range of I contains the range of T and the range of J contains the range of S and if S, T, I and J are continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.