CHAPTER 3

MAIN RESULTS

This chapter is divided into 3 sections. We obtain a theory of fixed point of selfmapping in a complete metric space in Section 3.1. This result generalizes the result in [1]. In Section 3.2, common fixed point of two and four mappings are studied and we obtain many results which generalize those in [1],[3] and [4]. In the last section, Section 3.3, some fixed points theory of composition of mappings are studied there results generalize those in [5].

3.1 Fixed Point of Selfmappings in Metric Spaces

Theorem 3.1.1 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), c_1 d(x, Ty) + c_2 d(y, Tx)\}, \forall x \neq y \in X,$

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 = 1$. Then T has a unique fixed point.

Proof. Choose any $x_0 \in X$ and define the sequence (x_n) by $x_n = Tx_{n-1}, n \in \mathbb{N}$. Then

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq \Phi(x_n) - \Phi(Tx_n) = \Phi(x_n) - \Phi(x_{n+1}).$$

Define $a_n = \Phi(x_n), n = 1, 2, \ldots$. It is easy to see that the sequence (a_n) is nonnegative real sequence and nonincreasing. Thus (a_n) is a convergente sequence, so it is Cauchy.

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq (\Phi(x_n) - \Phi(x_{n+1})) + (\Phi(x_{n+1}) - \Phi(x_{n+2})) + \ldots + (\Phi(x_{m-1}) - \Phi(x_m)) \\ &= \Phi(x_n) - \Phi(x_m) = a_n - a_m. \end{aligned}$$

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Since (a_n) is Cauchy, it implies that (x_n) is Cauchy in X. Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T

CaseI. There exists $m \in \mathbb{N}$ such that $x_n = x$ for all $n > m$. Then
$0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.

CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \neq x, \forall k \in \mathbb{N}$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), c_1 d(x, Tx_{n_k}) + c_2 d(x_{n_k}, Tx)\}$$

take $k \to \infty$, we have

$$\begin{aligned} d(Tx, x) &\leq \max\{d(x,x), c_1 d(x,x) + c_2 d(x, Tx)\} \\ &\leq c_2 d(Tx, x), \end{aligned}$$

so $d(Tx, x) = 0$ and hence $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$. Suppose that $u \neq v$. Then

$$\begin{aligned} d(u,v) = d(Tu, Tv) &< \max\{d(u,v), c_1 d(u, Tv) + c_2 d(v, Tu)\} \\ &\leq \max\{d(u,v), c_1 d(u,v) + c_2 d(v,u)\} \\ &\leq \max\{d(u,v), (c_1 + c_2) d(u,v)\} \\ &= d(u,v) \qquad \text{(since } c_1 + c_2 = 1\text{)}, \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. $\square$

Example 3.1.2 Let $X = [1, \infty)$ with the usual metric $d(x,y) = |x - y|$. Define $T : X \to X$ by $Tx = \frac{1}{2}(x+1), \forall x \in X$, and define $\Phi : X \to \mathbb{R}^+$ by $\Phi(x) = 3x + 1, \forall x \in X$. Then $d(x, Tx) = |x - Tx| = |x - \frac{1}{2}x - \frac{1}{2}| = \frac{1}{2}|x - 1|$ and

$$\begin{aligned} \Phi(x) - \Phi(Tx) &= (3x+1) - [3(Tx) + 1] \\ &= 3x + 1 - 3(\frac{1}{2}(x+1)) - 1 \\ &= 3x - \frac{3}{2}x - \frac{3}{2} \\ &= \frac{3}{2}x - \frac{3}{2} \\ &= \frac{3}{2}|x - 1| \end{aligned}$$

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so $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$. And for $x \neq y \in X$ we have

$$d(Tx, Ty) = |Tx - Ty| = |\frac{1}{2}(x+1) - \frac{1}{2}(y+1)| = |\frac{1}{2}x + \frac{1}{2} - \frac{1}{2}y - \frac{1}{2}| = \frac{1}{2}|x - y|$$

and $d(x, y) = |x - y|$ so $d(Tx, Ty) < d(x, y)$. Thus T satisfies the condition (2) of Theorem 3.1.1 and $T(1) = \frac{1}{2}(1+1) = 1$. □

Corollary 3.1.3 (cf. [1]) *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx, Ty) < \max\{d(x, y), [d(x, Ty) + d(y, Tx)]/2\}, \forall x \neq y \in X$.

Then T has a unique fixed point.

3.2 Common Fixed Point of Selfmappings in Metric Spaces

Theorem 3.2.1 *Let S and T be two weakly compatible selfmapping of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1 d(Tx, Sx) + c_2 d(Ty, Sy), d(Tx, Sy)\}$, $\forall x \neq y \in X$, *where* $0 \leq c_1 < 1$ *and* $0 \leq c_2 < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Proof. Since T and S satisfy the property(E.A), there exists a sequence (x_n) in X satisfying $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Suppose SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$, so $\lim_{n\to\infty} Tx_n = Sa$.
We show that $Ta = Sa$.

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If there exists $n_0 \in \mathbb{N}$ such that $x_n = a \ \forall n \geq n_0$, we obtain that $Ta = Sa$.
If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq a \ \forall k \in \mathbb{N}$. By (2), we have

$$d(Tx_{n_k}, Ta) < \max\{d(Sx_{n_k}, Sa), c_1 d(Tx_{n_k}, Sx_{n_k}) + c_2 d(Ta, Sa), d(Tx_{n_k}, Sa)\}.$$

Take $k \to \infty$, we have

$$\begin{aligned} d(Sa, Ta) &\leq \max\{d(Sa, Sa), c_1 d(Sa, Sa) + c_2 d(Ta, Sa), d(Sa, Sa)\} \\ &= c_2 d(Ta, Sa). \end{aligned}$$

Since $c_2 < 1$, it implies that $d(Ta, Sa) = 0$, hence $Ta = Sa$.

Since T and S are weakly compatible, $TSa = STa$ and $TTa = TSa = STa = SSa$.
If $Ta \neq a$, by(2), we have

$$\begin{aligned} d(Ta, TTa) &< \max\{d(Sa, STa), c_1 d(Ta, Sa) + c_2 d(TTa, STa), d(Ta, STa)\} \\ &\leq \max\{d(Ta, TTa), c_1 d(Ta, Ta) + c_2 d(TTa, TTa), d(Ta, TTa)\} \\ &= d(Ta, TTa), \end{aligned}$$

which is a contradiction. Thus $Ta = a$, hence $Ta = Sa = a$, so a is a common fixed point of S and T. The prove is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.

Finally, we show common fixed point is unique. Let $Tv = Sv = v$ and $Tu = Su = u$. Suppose $u \neq v$. By(2), we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(Su, Sv), c_1 d(Tu, Su) + c_2 d(Tv, Sv), d(Tu, Sv)\} \\ &\leq \max\{d(Tu, Tv), d(Tu, Tv)\} \\ &= d(Tu, Tv), \end{aligned}$$

which is a contradiction, hence $u = v$.
Therefore T and S have a unique common fixed point. □

Taking $c_1 = c_2$ in Theorem 3.2.1, we get the following result:

Corollary 3.2.2 *Let S and T be two weakly compatible selfmapping of a metric space (X, d) such that*

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(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c[d(Tx, Sx) + d(Ty, Sy)], d(Tx, Sy)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX *or* TX *is a complete subspace of* X, *then* T *and* S *have a unique commom fixed point.*

Taking $c_2 = 0$ in Theorem 3.2.1, we have the following result:

Corollary 3.2.3 *Let* S *and* T *be two weakly compatible selfmapping of a metric space* (X, d) *such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Tx, Sx), d(Tx, Sy)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX *or* TX *is a complete subspace of* X, *then* T *and* S *have a unique commom fixed point.*

Taking $c_1 = 0$ in Theorem 3.2.1, we have the following result:

Corollary 3.2.4 *Let* S *and* T *be two weakly compatible selfmapping of a metric space* (X, d) *such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Ty, Sy), d(Tx, Sy)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX *or* TX *is a complete subspace of* X, *then* T *and* S *have a unique commom fixed point.*

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Theorem 3.2.5 *Let (X, d) be a complete metric space and let $S, T : X \to X$ are commuting mappings satisfying the inequality*

$$d(Sx, Sy) \leq F(\max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Ty, Sx)\}), \forall x, y \in X \quad (1)$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function such that $F(t) < t$ for each $t > 0$.
If $SX \subset TX$ and T is continuous then S and T have a unique common fixed point. Moreover, if x is the common fixed point of S and T, then for any $x_0 \in X$, $Sx_n \to x$ and $Tx_n \to x$ where (x_n) is the sequence given by $Sx_n = Tx_{n+1}$, $n = 0, 1, 2, \ldots$

Proof. Let $x_0 \in X$, chose $x_1 \in X$ such that $Sx_0 = Tx_1$. This can be done since $SX \subset TX$. In general, having chosen x_n choose x_{n+1} such that $Sx_n = Tx_{n+1}$.
We shall show that

$$d(Sx_n, Sx_{n+1}) \leq F(Sx_{n-1}, Sx_n) \quad (2)$$

$$d(Sx_n, Sx_{n+1}) \leq d(Sx_{n-1}, Sx_n) \quad (3)$$

By (1) we have

$$\begin{aligned} d(Sx_n, Sx_{n+1}) &\leq F(\max\{d(Tx_n, Tx_{n+1}), d(Tx_n, Sx_n), d(Tx_{n+1}, Sx_{n+1}), d(Tx_{n+1}, Sx_n)\}) \\ &\leq F(\max\{d(Sx_{n-1}, Sx_n), d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1}), d(Sx_n, Sx_n)\}) \\ &\leq F(\max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}). \end{aligned}$$

If $0 \leq d(Sx_{n-1}, Sx_n) < d(Sx_n, Sx_{n+1})$, then $d(Sx_n, Sx_{n+1}) \leq F(d(Sx_n, Sx_{n+1})) < d(Sx_n, Sx_{n+1})$ which is a contradiction. Hence $d(Sx_{n-1}, Sx_n) \geq d(Sx_n, Sx_{n+1})$ and $d(Sx_n, Sx_{n+1}) \leq F(d(Sx_{n-1}, Sx_n))$. Thus (2) and (3) are satisfied. Thus the sequence $(d(Sx_n, Sx_{n+1}))_{n=0}^{\infty}$ is a nonincreasing sequence of positive real number and therefore has a limit $L \geq 0$. We claim that $L = 0$. Suppose $L > 0$, by taking $n \to \infty$ in (2) and continuity of F, we have

$$L = \lim_{n\to\infty} d(Sx_n, Sx_{n+1}) \leq \lim_{n\to\infty} F(d(Sx_{n-1}, Sx_n)) = F(L) < L,$$

which is a contradiction, hence $L = 0$. Thus $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$.
Next, we show that $(Sx_n)_{n=0}^{\infty}$ is a Cauchy sequence in X.

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Suppose not. Then there exist $\epsilon > 0$ and strictly increasing sequences of positive integer (m_k) and (n_k) with $m_k > n_k \geq k$ such that

$$d(Sx_{m_k}, Sx_{n_k}) \geq \epsilon \tag{4}$$

Assume that for each k, m_k is the smallest number greater than n_k for which (4) holds. By (3) and (4)

$$\begin{aligned} \epsilon \leq d(Sx_{m_k}, Sx_{n_k}) &\leq d(Sx_{m_k}, Sx_{m_k-1}) + d(Sx_{m_k-1}, Sx_{n_k}) \\ &\leq d(Sx_{m_k}, Sx_{m_k-1}) + \epsilon \\ &\leq d(Sx_k, Sx_{k-1}) + \epsilon \end{aligned}$$

This implies $\lim_{n\to\infty} d(Sx_{m_k}, Sx_{n_k}) = \epsilon$.
By triangle inequality and (3), we have

$$\begin{aligned} d(Sx_{m_k}, Sx_{n_k}) &\leq d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) \\ &\leq d(Sx_{m_k}, Sx_{m_k-1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k-1}, Sx_{n_k}) \\ &\leq 2d(Sx_k, Sx_{k-1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) \end{aligned} \tag{5}$$

Consider for $n_k, m_k \in N$ with $m_k > n_k$, by (1) and (3), we have

$$\begin{aligned} d(Sx_{m_k+1}, Sx_{n_k+1}) &\leq F(\max\{d(Tx_{m_k+1}, Tx_{n_k+1}), d(Tx_{m_k+1}, Sx_{m_k+1}), \\ &\quad d(Tx_{n_k+1}, Sx_{n_k+1}), d(Tx_{n_k+1}, Sx_{m_k+1})\}) \\ &\leq F(\max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{m_k}, Sx_{m_k+1}), d(Sx_{n_k}, Sx_{n_k+1}), \\ &\quad d(Sx_{n_k}, Sx_{m_k+1})\}) \\ &\leq F(\max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{n_k}, Sx_{n_k+1}), d(Sx_{n_k}, Sx_{m_k+1})\}) \end{aligned}$$

Since $d(Sx_{n_k}, Sx_{m_k+1}) \leq d(Sx_{m_k+1}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k})$, so by (1) and (3), we have

$$\begin{aligned} d(Sx_{m_k+1}, Sx_{n_k+1}) &\leq F(\max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{n_k}, Sx_{n_k+1}), d(Sx_{m_k+1}, Sx_{m_k}) + \\ &\quad d(Sx_{m_k}, Sx_{n_k})\}) \\ &\leq F(\max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{n_k}, Sx_{n_k+1}), d(Sx_{n_k+1}, Sx_{n_k}) + \\ &\quad d(Sx_{m_k}, Sx_{n_k})\}) \\ &\leq F(d(Sx_{m_k}, Sx_{n_k}) + d(Sx_{n_k+1}, Sx_{n_k})) \end{aligned} \tag{6}$$

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Hence by (3),(5) and (6), we have

$$\begin{aligned} d(Sx_{m_k}, Sx_{n_k}) &\leq 2d(Sx_k, Sx_{k-1}) + F(d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k})) \\ &\leq 2d(Sx_k, Sx_{k-1}) + F(d(Sx_{n_k-1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k})) \\ &\leq 2d(Sx_k, Sx_{k-1}) + F(d(Sx_{k-1}, Sx_k) + d(Sx_{m_k}, Sx_{n_k})) \end{aligned}$$

By taking $k \to \infty$ in above inequality, we have $\epsilon \leq F(\epsilon) < \epsilon$ which is a contradiction. Hence $(Sx_n)_{n=0}^{\infty}$ is a Cauchy sequence in X. Since X is a complete metric space, there exist $t \in X$ such that $\lim_{n\to\infty} Sx_n = t$. Also $\lim_{n\to\infty} Tx_n = t$.

Since T is continuous, we have $\lim_{n\to\infty} T^2x_n = Tt$ and $\lim_{n\to\infty} TSx_n = Tt$. So $\lim_{n\to\infty} STx_n = Tt$ because T and S are commute. We now have

$$d(STx_n, Sx_n) \leq F(\max\{d(T^2x_n, Tx_n), d(T^2x_n, STx_n), d(Tx_n, Sx_n), d(Tx_n, STx_n)\}).$$

By taking $n \to \infty$, we have

$$\begin{aligned} d(Tt, t) &\leq F(\max\{d(Tt, t), d(Tt, Tt), d(t, t), d(t, Tt)\}) \\ &\leq F(d(Tt, t)). \end{aligned}$$

This implies $d(Tt, t) = 0$, hence $Tt = t$.
By (1), we have

$$d(St, Sx_n) \leq F(\max\{d(Tt, Tx_n), d(Tt, St), d(Tx_n, Sx_n), d(Tx_n, St)\}).$$

By taking $n \to \infty$, we have

$$\begin{aligned} d(St, t) &\leq F(\max\{d(Tt, t), d(Tt, St), d(t, t), d(t, St)\}) \\ &\leq F(d(t, St)). \end{aligned}$$

This implies $St = t$. Hence t is a commom fixed point of S and T.

Finally, we show that common fixed point of T and S is unique.
Let $Sw = Tw = w$ and $Sv = Tv = v$, then by (1)

$$\begin{aligned} d(w, v) = d(Sw, Sv) &\leq F(\max\{d(Tw, Tv), d(Tw, Sw), d(Tv, Sv), d(Tv, Sw)\}) \\ &\leq F(d(w, v)). \end{aligned}$$

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This implies $w = v$. Therefore S and T have a unique common fixed point. $\square$

Example 3.2.6 Let $X = [1, \infty)$ with the usual metric $d(x,y) = |x - y|$. Define $S, T : X \to X$ by $Sx = x$ and $Tx = x^2, \forall x \in X$ and define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(t) = \frac{t}{2}, \forall t \in \mathbb{R}^+$. Then

(1) S and T are commute.

(2) We see that

$$\begin{aligned} d(Sx, Sy) &= |Sx - Sy| \\ &= |x - y| \end{aligned}$$

and

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| \\ &= |x^2 - y^2| \\ &= |x + y||x - y| \end{aligned}$$

since $|x - y| \leq |x + y||x - y|, \forall x, y \in X$ and $|x - y| \leq \frac{|x+y|}{2}|x - y|$, so we have $d(Sx, Sy) \leq F(d(Tx, Ty))$.
Thus $d(Sx, Sy) \leq F(\max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Ty, Sx)\})$.

(3) $T1 = S1 = 1$. $\square$

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Corollary 3.2.7 *Let (X, d) be a complete metric space and let $S, T : X \to X$ be commuting mappings satisfying the inequality*

$$d(Sx, Sy) \leq c \cdot \max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Ty, Sx)\}, \forall x, y \in X,$$

where $0 \leq c < 1$. If $SX \subset TX$ and T is continuous then S and T have a unique common fixed point.

Proof. Define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(t) = ct$ for all $t \in \mathbb{R}^+$. Then F is satisfied the condition in Theorem 3.2.5. Hence the corollary is obtained directly by Theorem 3.2.5. $\square$

Corollary 3.2.8 *Let S be selfmapping of a complete metric space (X,d) satisfying the inequality*

$$d(Sx,Sy) \le F(\max\{d(x,y), d(x,Sx), d(y,Sy), d(y,Sx)\}), \forall x,y \in X$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function such that $F(t) < t$ for each $t > 0$. Then S has a unique fixed point. Moreover, for any $x_0 \in X$, (Sx_n) converges to the fixed point of S where $x_{n+1} = Sx_n, n = 0,1,2,\ldots$.

Proof. Let T be the identity mapping in Theorem 3.2.5. Then all conditions of Theorem 3.2.5 are satisfied and the Corollary is obtained. □

The next result give some sufficient conditions to guarantee that four self-mappings have a unique common fixed point.

Theorem 3.2.9 *Let (X,d) be a complete metric space and let $S,T,I,J : X \to X$ and S and I be commuting mappings and T and J be commuting mappings satisfying the inequality*

$$d(Sx,Ty) \le F(\max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)\}), \forall x,y \in X, \tag{7}$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous function such that $F(t) < t$ for each $t > 0$. If $TX \subset IX$ and $SX \subset JX$ and if one of S,T,I and J is continuous, then S,T,I and J have a unique common fixed point.

Proof. Let $x_0 \in X$ and choose $x_1 \in X$ such that $Sx_0 = Jx_1$, this can be done since $SX \subset JX$. Next, choose $x_2 \in X$ such that $Tx_1 = Ix_2$, which can be done since $TX \subset IX$. In general, having chosen $x_{2n} \in X$ choose $x_{2n+1} \in X$ such that $Sx_{2n} = Jx_{2n+1}$ and choose $x_{2n+2} \in X$ such that $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0,1,2,\ldots$ Then

$$\begin{aligned} d(Sx_{2n},Tx_{2n+1}) &\le F(\max\{d(Ix_{2n},Jx_{2n+1}), d(Ix_{2n},Sx_{2n}), d(Jx_{2n+1},Tx_{2n+1})\}) \\ &\le F(\max\{d(Tx_{2n-1},Sx_{2n}), d(Tx_{2n-1},Sx_{2n}), d(Sx_{2n},Tx_{2n+1})\}) \\ &= F(\max\{d(Tx_{2n-1},Sx_{2n}), d(Sx_{2n},Tx_{2n+1})\}) \end{aligned}$$

If $0 \le d(Tx_{2n-1},Sx_{2n}) < d(Sx_{2n},Tx_{2n+1})$, then $d(Sx_{2n},Tx_{2n+1}) \le F(d(Sx_{2n},Tx_{2n+1})) < d(Sx_{2n},Tx_{2n+1})$ which is a contradiction. Hence for $n = 1,2,\ldots$

$$d(Sx_{2n},Tx_{2n+1}) \le d(Sx_{2n},Tx_{2n-1}) \tag{8}$$

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and

$$d(Sx_{2n}, Tx_{2n+1}) \leq F(d(Sx_{2n}, Tx_{2n-1})) \tag{9}$$

By (7), we have

$$\begin{aligned} d(Sx_{2n}, Tx_{2n-1}) &\leq F(\max\{d(Ix_{2n}, Jx_{2n-1}), d(Ix_{2n}, Sx_{2n}), d(Jx_{2n-1}, Tx_{2n-1})\}) \\ &\leq F(\max\{d(Tx_{2n-1}, Sx_{2n-2}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n-2}, Tx_{2n-1})\}) \\ &\leq F(\max\{d(Tx_{2n-1}, Sx_{2n-2}), d(Tx_{2n-1}, Sx_{2n})\}) \end{aligned}$$

If $0 \leq d(Tx_{2n-1}, Sx_{2n-2}) < d(Tx_{2n-1}, Sx_{2n})$, then $d(Sx_{2n}, Tx_{2n-1}) \leq F(d(Tx_{2n-1}, Sx_{2n}))$ $< d(Tx_{2n-1}, Sx_{2n})$ which is a contradiction. Hence for $n = 1, 2, 3, \ldots$

$$d(Sx_{2n}, Tx_{2n-1}) \leq d(Tx_{2n-1}, Sx_{2n-2}) \tag{10}$$

and

$$d(Sx_{2n}, Tx_{2n-1}) \leq F(d(Tx_{2n-1}, Sx_{2n-2})) \tag{11}$$

By (8) and (10), we have

$$d(Sx_{2n-2}, Tx_{2n-1}) \geq d(Sx_{2n}, Tx_{2n-1}) \geq d(Sx_{2n}, Tx_{2n+1}) \tag{12}$$

for all $n \in \mathbb{N}$.

Define

$$a_n = \begin{cases} d(Sx_{n-1}, Tx_n) \text{ if } n \text{ is odd,} \\ d(Sx_n, Tx_{n-1}) \text{ if } n \text{ is even.} \end{cases}$$

Then $(a_n)_{n=1}^{\infty}$ is a nonincreasing sequence of positive real numbers and therefore has a limit $L \geq 0$. Suppose $L > 0$, then $\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} d(Sx_{2k}, Tx_{2k-1}) = L$ and $\lim_{k\to\infty} a_{2k-1} = \lim_{k\to\infty} d(Sx_{2k-2}, Tx_{2k-1}) = L$.
By (11), we have

$$d(Sx_{2k}, Tx_{2k-1}) \leq F(d(Tx_{2k-1}, Sx_{2k-2})) \tag{13}$$

by taking $k \to \infty$ in (13) and by continuity of F, we have $L \leq F(L) < L$, which is a contradiction, hence $L = 0$ and $\lim_{n\to\infty} a_n = 0$.

Define

$$b_n = \begin{cases} Tx_n \text{ if } n \text{ is odd,} \\ Sx_n \text{ if } n \text{ is even.} \end{cases}$$

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We shall show that the sequence $(b_n)_{n=0}^{\infty}$ is a Cauchy sequence in X.

To show this, suppose not. Then there exists $\epsilon > 0$ and strictly increasing sequence of positive integers (m_k) and (n_k) with $m_k > n_k \geq k$ such that

$$d(b_{m_k}, b_{n_k}) \geq \epsilon \tag{14}$$

Assume that for each k, m_k is the smallest positive integer greater than n_k for which (14) holds.

CaseI. $b_{m_k} = Sx_{m_k}$ and $b_{n_k} = Sx_{n_k}$. Then by (8) and (10), we have

$$\begin{aligned}\epsilon \leq d(b_{m_k}, b_{n_k}) = d(Sx_{m_k}, Sx_{n_k}) &\leq d(Sx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Sx_{n_k}) \\ &\leq a_{m_k} + \epsilon \end{aligned} \tag{15}$$

and

$$\begin{aligned} d(b_{m_k}, b_{n_k}) = d(Sx_{m_k}, Sx_{n_k}) &\leq d(Sx_{m_k}, Tx_{m_k+1}) + d(Tx_{m_k+1}, Tx_{n_k+1}) + d(Tx_{n_k+1}, Sx_{n_k}) \\ &\leq a_{m_k+1} + d(Tx_{m_k+1}, Tx_{n_k+1}) + a_{n_k+1} \\ &\leq 2a_{n_k+1} + d(Tx_{m_k+1}, Tx_{n_k+1}) \\ &\leq 2a_{n_k+1} + d(Tx_{m_k+1}, Sx_{m_k}) + d(Sx_{m_k}, Tx_{n_k+1}) \\ &\leq 2a_{n_k+1} + a_{m_k+1} + d(Sx_{m_k}, Tx_{n_k+1}) \\ &\leq 3a_{n_k+1} + d(Sx_{m_k}, Tx_{n_k+1}) \\ &\leq 3a_{n_k} + d(Sx_{m_k}, Tx_{n_k+1}) \end{aligned} \tag{16}$$

by (7) and (13), we have

$$\begin{aligned} d(Sx_{m_k}, Tx_{n_k+1}) &\leq F(\max\{d(Ix_{m_k}, Jx_{n_k+1}), d(Ix_{m_k}, Sx_{m_k}), d(Jx_{n_k+1}, Tx_{n_k+1})\}) \\ &\leq F(\max\{d(Tx_{m_k-1}, Sx_{n_k}), d(Tx_{m_k-1}, Sx_{m_k}), d(Sx_{n_k}, Tx_{n_k+1})\}) \\ &\leq F(\max\{d(Tx_{m_k-1}, Sx_{n_k}), d(Sx_{n_k}, Tx_{n_k+1})\}) \\ &\leq F(\max\{d(Tx_{m_k-1}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k}), d(Sx_{n_k}, Tx_{n_k+1})\}) \\ &\leq F(\max\{d(Tx_{n_k-1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k}), d(Sx_{n_k}, Tx_{n_k+1})\}) \\ &\leq F(\max\{d(Tx_{n_k-1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k})\}) \end{aligned} \tag{17}$$

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By (16) and (17), we have

$$\begin{aligned} d(b_{m_k}, b_{n_k}) = d(Sx_{m_k}, Sx_{n_k}) &\leq 3a_{n_k} + F(\max\{d(Tx_{n_k-1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k})\}) \\ &\leq 3a_{n_k} + F(\max\{d(Tx_{n_k-1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k})\}) \\ &\leq 3a_{n_k} + F(\max\{a_{n_k} + d(Sx_{m_k}, Sx_{n_k})\}) \qquad (18) \end{aligned}$$

CaseII. $b_{m_k} = Sx_{m_k}$ and $b_{n_k} = Tx_{n_k}$, Then by (12)

$$\begin{aligned} \epsilon \leq d(b_{m_k}, b_{n_k}) = d(Sx_{m_k}, Tx_{n_k}) &\leq d(Sx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{n_k}) \\ &\leq a_{m_k} + \epsilon \qquad (19) \end{aligned}$$

$$\begin{aligned} d(b_{m_k}, b_{n_k}) = d(Sx_{m_k}, Tx_{n_k}) &\leq d(Sx_{m_k}, Tx_{m_k+1}) + d(Tx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Tx_{n_k}) \\ &\leq a_{m_k+1} + d(Tx_{m_k+1}, Sx_{n_k+1}) + a_{n_k} \\ &\leq 2a_{n_k} + d(Tx_{m_k+1}, Sx_{n_k+1}) \qquad (20) \end{aligned}$$

by (7), we have

$$\begin{aligned} d(Sx_{n_k+1}, Tx_{m_k+1}) &\leq F(\max\{d(Ix_{n_k+1}, Jx_{m_k+1}), d(Ix_{n_k+1}, Sx_{n_k+1}), d(Jx_{m_k+1}, Tx_{m_k+1})\}) \\ &\leq F(\max\{d(Tx_{n_k}, Sx_{m_k}), d(Tx_{n_k}, Sx_{n_k+1}), d(Sx_{m_k}, Tx_{m_k+1})\}) \\ &\leq F(\max\{d(Sx_{m_k}, Tx_{n_k}), a_{n_k+1}, a_{m_k+1}\}) \\ &\leq F(\max\{d(Sx_{m_k}, Tx_{n_k}), a_{n_k}\}). \qquad (21) \end{aligned}$$

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From (20) and (21), we have

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$$\begin{aligned} d(b_{m_k}, b_{n_k}) = d(Sx_{m_k}, Tx_{n_k}) &\leq 2a_{n_k} + F(\max\{d(Sx_{m_k}, Tx_{n_k}), a_{n_k}\}) \\ &\leq 2a_{n_k} + F(\max\{d(Sx_{m_k}, Tx_{n_k}), a_{n_k}\}). \qquad (22) \end{aligned}$$

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CaseIII. $b_{m_k} = Tx_{m_k}$ and $b_{n_k} = Tx_{n_k}$. Then by (12)

$$\begin{aligned} \epsilon \leq d(b_{m_k}, b_{n_k}) = d(Tx_{m_k}, Tx_{n_k}) &\leq d(Tx_{m_k}, Sx_{m_k-1}) + d(Sx_{m_k-1}, Tx_{n_k}) \\ &\leq a_{m_k} + \epsilon \qquad (23) \end{aligned}$$

and

$$\begin{aligned} d(b_{m_k}, b_{n_k}) = d(Tx_{m_k}, Tx_{n_k}) &\le d(Tx_{m_k}, Sx_{m_k+1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Tx_{n_k}) \\ &\le a_{m_k+1} + d(Sx_{m_k+1}, Sx_{n_k+1}) + a_{n_k+1} \\ &\le 2a_{n_k} + d(Sx_{m_k+1}, Sx_{n_k+1}) \\ &\le 2a_{n_k} + d(Sx_{m_k+1}, Tx_{m_k}) + d(Tx_{m_k}, Sx_{n_k+1}) \\ &\le 2a_{n_k} + a_{m_k+1} + d(Tx_{m_k}, Sx_{n_k+1}) \\ &\le 3a_{n_k} + d(Tx_{m_k}, Sx_{n_k+1}). \end{aligned} \tag{24}$$

By (7), we have

$$\begin{aligned} d(Sx_{n_k+1}, Tx_{m_k}) &\le F(\max\{d(Ix_{n_k+1}, Jx_{m_k}), d(Ix_{n_k+1}, Sx_{n_k+1}), d(Jx_{m_k}, Tx_{m_k})\}) \\ &\le F(\max\{d(Tx_{n_k}, Sx_{m_k-1}), d(Tx_{n_k}, Sx_{n_k+1}), d(Sx_{m_k-1}, Tx_{m_k})\}) \\ &\le F(\max\{d(Tx_{n_k}, Tx_{m_k}) + d(Tx_{m_k}, Sx_{m_k-1}), d(Tx_{n_k}, Sx_{n_k+1}), \\ &\quad d(Sx_{m_k-1}, Tx_{m_k})\}) \\ &\le F(\max\{d(Tx_{n_k}, Tx_{m_k}) + d(Tx_{m_k}, Sx_{m_k-1}), d(Tx_{n_k}, Sx_{n_k+1})\}) \\ &\le F(\max\{d(Tx_{n_k}, Tx_{m_k}) + a_{m_k}, a_{n_k+1}\}) \\ &\le F(d(Tx_{n_k}, Tx_{m_k}) + a_{n_k}) \end{aligned} \tag{25}$$

by (24) and (25), we have

$$d(b_{m_k}, b_{n_k}) = d(Tx_{m_k}, Tx_{n_k}) \le 3a_{n_k} + F((d(Tx_{n_k}, Tx_{m_k}) + a_{n_k}). \tag{26}$$

By (15),(19) and (23) and $\lim_{k\to\infty} a_{n_k} = 0$, we obtain that

$$\lim_{k\to\infty} d(b_{m_k}, b_{n_k}) = \epsilon. \tag{27}$$

By (18),(22) and (26) , we have $\epsilon \le F(\epsilon) < \epsilon$ which is a contradiction. Hence $(b_n)_{n=0}^{\infty}$ is a Cauchy sequence in the complete metric space X and so has a limit v in X. Thus the sequences $(Sx_{2n})_{n=0}^{\infty} = (Jx_{2n+1})_{n=0}^{\infty}$ and $(Tx_{2n-1})_{n=1}^{\infty} = (Ix_{2n})_{n=1}^{\infty}$ converges to the point v.

Now, suppose I is continuous, we have $\lim_{n\to\infty} I^2x_{2n} = Iv$ and $\lim_{n\to\infty} ISx_{2n} = Iv$. Since I and S are commute, $\lim_{n\to\infty} SIx_{2n} = Iv$.

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By(7), we have

$$d(SIx_{2n}, Tx_{2n+1}) \leq F(\max\{d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1})\}).$$

By taking $n \to \infty$, we have

$$\begin{aligned} d(Iv, v) &\leq F(\max\{d(Iv, v), d(Iv, Iv), d(v, v)\}) \\ &\leq F(d(Iv, v)). \end{aligned}$$

Thus $Iv = v$. Again by (7),

$$d(Sv, Tx_{2n+1}) \leq F(\max\{d(Iv, Jx_{2n+1}), d(Iv, Sv), d(Jx_{2n+1}, Tx_{2n+1})\}).$$

By taking $n \to \infty$, we get that

$$\begin{aligned} d(Sv, v) &\leq F(\max\{d(Iv, v), d(Iv, Sv), d(v, v)\}) \\ &\leq F(d(v, Sv)) \end{aligned}$$

so $Sv = v$.

Since $SX \subset JX$, there exist $t \in X$ such that $Jt = Sv = v$ and so

$$TJt = Tv = JTt \tag{28}$$

since T and J are commute. Thus

$$\begin{aligned} d(v, Tt) = d(Sv, Tt) &\leq F(\max\{d(Iv, Jt), d(Iv, Sv), d(Jt, Tt)\}) \\ &\leq F(d(v, Tt)) \end{aligned}$$

so $Tt = v$ and from (28), we have $Jv = Tv$.

By (7), we have

$$\begin{aligned} d(v, Tv) = d(Sv, Tv) &\leq F(\max\{d(Iv, Jv), d(Iv, Sv), d(Jv, Tv)\}) \\ &\leq F(d(v, Tv)) \end{aligned}$$

so $Tv = v$ and $Tv = Jv = v$. Thus v is a common fixed point of S, T, I and J.

If the mapping J is continuous instead of I, then the proof that v is again a common fixed point of S, T, I and J is of course similar.

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Now suppose that S is continuous. Then $\lim_{n\to\infty} S^2x_{2n} = Sv$ and $\lim_{n\to\infty} SIx_{2n}$ $= Sv$. Since I and S are commute, $\lim_{n\to\infty} ISx_{2n} = Sv$.

We now have

$$d(S^2x_{2n}, Tx_{2n+1}) \leq F(\max\{d(ISx_{2n}, Jx_{2n+1}), d(ISx_{2n}, S^2x_{2n}), d(Jx_{2n+1}, Tx_{2n+1})\}).$$

By taking $n \to \infty$, we have

$$\begin{aligned} d(Sv, v) &\leq F(\max\{d(Sv, v), d(Sv, Sv), d(v, v)\}) \\ &\leq F(d(Sv, v)) \end{aligned}$$

so $Sv = v$. Since $SX \subset JX$, there exist $w \in X$ such that $Jw = Sv = v$ and

$$TJw = Tv = JTw \qquad (29)$$

since T and J are commute. We now have

$$d(Sx_{2n}, Tw) \leq F(\max\{d(Ix_{2n}, Jw), d(Ix_{2n}, Sx_{2n}), d(Jw, Tw)\})$$

take $n \to \infty$, we have

$$\begin{aligned} d(v, Tw) &\leq F(\max\{d(v, Jw), d(v, v), d(Jw, Tw)\}) \\ &\leq F(d(v, Tw)) \end{aligned}$$

so $Tw = v$ and from(29) we have $Tv = Jv$.

Since

$$d(S^2x_{2n}, Tv) \leq F(\max\{d(ISx_{2n}, Jv), d(ISx_{2n}, S^2x_{2n}), d(Jv, Tv)\}).$$

By taking $n \to \infty$, we have

$$\begin{aligned} d(Sv, Tv) &\leq F(\max\{d(Sv, Jv), d(Sv, Sv), d(Jv, Tv)\}) \\ &\leq F(d(Sv, Tv)) \end{aligned}$$

hence $Sv = Tv$ and $v = Tv = Jv$.

Since $TX \subset IX$, there exist $y \in X$ such that $Iy = Tv = v$ and

$$SIy = Sv = ISy. \qquad (30)$$

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Again by (7), we have

$$d(Sy,v) = d(Sy,Tv) \leq F(\max\{d(Iy,Jv), d(Iy,Sy), d(Jv,Tv)\})$$
$$\leq F(d(v,Sy))$$

so $Sy = v$ and from (30) we have $Iv = Sv = v$. Thus v is a common fixed point of S, T, I and J.

If the mapping T is continuous instead of S, then the proof that v is again a common fixed point of S, T, I and J is similar.

Finally, we will show that common fixed point of S, T, I and J is unique. Suppose $Sz = Tz = Iz = Jz = z$ and $Sv = Tv = Iv = Jv = v$, then

$$d(z,v) = d(Sz,Tv) \leq F(\max\{d(Iz,Jv), d(Iz,Sz), d(Jv,Tv)\})$$
$$\leq F(d(z,v))$$

so $z = v$. Therefore S, T, I and J have a unique common fixed point. □

Corollary 3.2.10 (cf. [4]) *Let (X,d) be a complete metric space and let $S, T, I, J : X \to X$ and S and I be commuting mappings and T and J be commuting mappings satisfying the inequality*

$$d(Sx,Ty) \leq c \cdot \max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty)\}, \forall x,y \in X,$$

where $0 \leq c < 1$. If $TX \subset IX$ and $SX \subset JX$ and if one of S, T, I and J is continuous, then S, T, I and J have a unique common fixed point.

Proof. Define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(t) = ct$ for all $t \in \mathbb{R}^+$. Then F is satisfied the condition in Theorem 3.2.9. Hence the corollary is obtained directly by Theorem 3.2.9. □

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Corollary 3.2.11 *Let (X,d) be a complete metric space and let $S, T, I : X \to X$ and S and I be commuting mappings and T and I be commuting mappings satisfying the inequality*

$$d(Sx,Ty) \leq F(\max\{d(Ix,Iy), d(Ix,Sx), d(Iy,Ty)\}), \forall x,y \in X,$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing continuous function such that* $F(t) < t$ *for each* $t > 0$. *If* $TX \subset IX$ *and* $SX \subset IX$ *and if one of* S, T *and* I *is continuous, then* S, T *and* I *have a unique common fixed point.*

Proof. Let $I = J$ in Theorem 3.2.9. Then all conditions of Theorem 3.2.9 are satisfied and so S, T and I have a unique common fixed point. □

Corollary 3.2.12 *Let* S *and* T *be mappings of a complete metric space* (X, d) *into itself satisfying the inequality*

$$d(Sx, Ty) \leq F(\max\{d(x, y), d(x, Sx), d(y, Ty)\}), \forall x, y \in X,$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing continuous function such that* $F(t) < t$ *for each* $t > 0$. *Then* S *and* T *have a unique common fixed point.*

Proof. Let I and J be the identity mapping in Theorem 3.2.9. Then all conditions of Theorem 3.2.9 are satisfied and so S and T have a unique common fixed point. □

Corollary 3.2.13 *Let* I *and* J *be mappings of a complete metric space* (X, d) *onto itself satisfying the inequality*

$$d(x, y) \leq F(\max\{d(Ix, Jy), d(Ix, x), d(Jy, y)\}), \forall x, y \in X,$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing continuous function such that* $F(t) < t$ *for each* $t > 0$. *Then* I *and* J *have a unique common fixed point.*

Proof. Let S and T be the identity mapping in Theorem 3.2.9. Then all conditions of Theorem 3.2.9 are satisfied and so I and J have a unique common fixed point. □

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Corollary 3.2.14 *Let* (X, d) *be a complete metric space and let* $S, T, I, J : X \to X$ *and* S *and* I *be commuting mappings and* T *and* J *be commuting mappings satisfying the inequality*

$$d(Sx, Ty) \leq F(\max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \frac{1}{2}d(Ix, Ty), \frac{1}{2}d(Jy, Sx)\}),$$

for all $x, y \in X$, *where* $F : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing continuous function such that* $F(t) < t$ *for each* $t > 0$. *If* $TX \subset IX$ *and* $SX \subset JX$ *and if one of* S, T, I

and J is continuous, then S, T, I and J have a unique common fixed point.

Proof. Let $x, y \in X$, we have

$$\begin{aligned} d(Ix, Ty) &\le d(Ix, Sx) + d(Sx, Ty) \\ &\le 2\max\{d(Ix, Sx), d(Sx, Ty), d(Ix, Jy), d(Jy, Ty)\} \end{aligned}$$

so

$$\frac{1}{2}d(Ix, Ty) \le \max\{d(Ix, Sx), d(Sx, Ty), d(Ix, Jy), d(Jy, Ty)\}$$

and similarly

$$\frac{1}{2}d(Jy, Sx) \le \max\{d(Ix, Sx), d(Sx, Ty), d(Ix, Jy), d(Jy, Ty)\}.$$

Thus

$$\begin{aligned} d(Sx, Ty) &\le F(\max\{d(Ix, Sx), d(Sx, Ty), d(Ix, Jy), d(Jy, Ty)\}) \\ &\le F(\max\{d(Ix, Sx), d(Ix, Jy), d(Jy, Ty)\}) \end{aligned}$$

Hence all the conditions of Theorem 3.2.9 are satisfied. Therefore the corollary is obtained by Theorem 3.2.9. □

Corollary 3.2.15 (cf. [4]) *Let (X, d) be a complete metric space and let $S, T, I, J : X \to X$ and S and I be commuting mappings and T and J be commuting mappings satisfying the inequality*

$$d(Sx, Ty) \le c \cdot \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty), \frac{1}{2}d(Ix, Ty), \frac{1}{2}d(Jy, Sx)\},$$

for all $x, y \in X$ where $0 \le c < 1$. If $TX \subset IX$ and $SX \subset JX$ and if one of S, T, I and J is continuous, then S, T, I and J have a unique common fixed point.

Proof Define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(t) = ct$ for all $t \in \mathbb{R}^+$. Then F is satisfied the condition in Corollary 3.2.14. Hence the corollary is obtained directly by Corollary 3.2.14. □

Corollary 3.2.16 *Let S and T be mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Sx, Ty) \le F(\max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}d(x, Ty), \frac{1}{2}d(y, Sx)\}),$$

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for all $x, y \in X$ where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous function such that $F(t) < t$ for each $t > 0$. Then S and T have a unique common fixed point.

Proof. Let I and J be the identity mapping in Corollary 3.2.14. Then all conditions of Corollary 3.2.14 are satisfied and so S and T have a unique common fixed point. □

Corollary 3.2.17 *Let I and J be mappings of a complete metric space (X, d) onto itself satisfying the inequality*

$$d(x, y) \le F(\max\{d(Ix, Jy), d(Ix, x), d(Jy, y), \frac{1}{2}d(Ix, y), \frac{1}{2}d(Jy, x)\}),$$

for all $x, y \in X$ where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous function such that $F(t) < t$ for each $t > 0$. Then I and J have a unique common fixed point.

Proof. Let S and T be the identity mapping in Corollary 3.2.14. Then all conditions of Corollary 3.2.14 are satisfied and so I and J have a unique common fixed point. □

3.3 Fixed Point Theory of Composition of Mappings

Theorem 3.3.1 *Let (X, d) be complete metric spaces and let (Y, d') be metric spaces. If $T : X \to Y$ and $S : Y \to X$ satisfying the inequalities*

$$d'(Tx, TSy) \le F(\max\{d(x, Sy), d'(y, Tx), d'(y, TSy)\}) \tag{31}$$

$$d(Sy, STx) \le F(\max\{d'(y, Tx), d(x, Sy), d(x, STx)\}) \tag{32}$$

for all $x \in X$ and for all $y \in Y$, where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous function such that $F(t) < t$ for each $t > 0$, and if there is $x \in X$ such that the sequence (y_n), define by $y_n = T(ST)^{n-1}x$ converges, then ST has a unique fixed point in X and TS has a unique fixed point in Y.

Proof. Define the sequence (x_n) in X by $x_n = (ST)^n x$ for $n = 1, 2, ...$

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If $d(x_n, x_{n+1}) \neq d'(y_n, y_{n+1})$, by (31) we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d((ST)^n x, (ST)^{n+1} x) \\
&\leq F(\max\{d'(T(ST)^{n-1}x, T(ST)^n x), d((ST)^n x, (ST)^n x), \\
&\quad d((ST)^n x, (ST)^{n+1} x)\}) \\
&\leq F(\max\{d'(y_n, y_{n+1}), d(x_n, x_n), d(x_n, x_{n+1})\}) \\
&\leq F(d'(y_n, y_{n+1})) \\
&< d'(y_n, y_{n+1}).
\end{aligned}$$

Thus

$$d(x_n, x_{n+1}) \leq d'(y_n, y_{n+1}). \tag{33}$$

By (31), we have

$$\begin{aligned}
d'(y_n, y_{n+1}) &= d'(T(ST)^{n-1}x, T(ST)^n x) \\
&\leq F(\max\{d((ST)^{n-1}x, (ST)^n x), d'(T(ST)^{n-1}x, T(ST)^{n-1}x), \\
&\quad d'(T(ST)^{n-1}x, T(ST)^n x)\}) \\
&\leq F(\max\{d(x_{n-1}, x_n), d'(y_n, y_n), d'(y_n, y_{n+1})\}) \\
&\leq F(\max\{d(x_{n-1}, x_n), d'(y_n, y_{n+1})\}).
\end{aligned}$$

If $d'(y_n, y_{n+1}) = d(x_{n-1}, x_n)$, then $d'(y_n, y_{n+1}) \leq F(d(x_{n-1}, x_n))$.

If $d'(y_n, y_{n+1}) \neq d(x_{n-1}, x_n)$, we have $d'(y_n, y_{n+1}) \leq F(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n)$.

Hence

$$d'(y_n, y_{n+1}) \leq d(x_{n-1}, x_n) \tag{34}$$

and

$$d'(y_n, y_{n+1}) \leq F(d(x_{n-1}, x_n)). \tag{35}$$

From (33) and (34), we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \tag{36}$$

and from (33) and (35), we have

$$d(x_n, x_{n+1}) \leq F(d(x_{n-1}, x_n)). \tag{37}$$

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It follows from (33),(34) and (36) that

$$d(x_n, x_{n+1}) \leq d'(y_n, y_{n+1}) \leq d(x_{n-1}, x_n) \leq d'(y_{n-1}, y_n) \leq \ldots \leq d'(y_1, y_2) \leq d(x_0, x_1)$$

so the sequence $(d(x_n, x_{n+1}))_{n=1}^{\infty}$ and the sequence $(d'(y_n, y_{n+1}))_{n=1}^{\infty}$ are nonincreasing sequence of positive real numbers, hence they have limits.

We show that the sequence $(d(x_n, x_{n+1}))_{n=1}^{\infty}$ and the sequence $(d'(y_n, y_{n+1}))_{n=1}^{\infty}$ have the zero limit. Suppose $\lim_{n\to\infty} d(x_n, x_{n+1}) = L > 0$. By taking $n \to \infty$ in (37), we have by continuity of F that $L \leq F(L) < L$, a contradiction, so $L = 0$, hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. By (34), it implies that $\lim_{n\to\infty} d'(y_n, y_{n+1}) = 0$. Let $a_n = d(x_n, x_{n+1})$ and $b_n = d'(y_n, y_{n+1})$, Then $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$ and $a_n \leq b_n \leq a_{n-1} \leq b_{n-1}$.

Next, we will show that (x_n) is a Cauchy sequence in X.

To show this, suppose not. Then there exists $\epsilon > 0$ and strictly increasing sequences of positive integer (m_k) and (n_k) with $m_k > n_k \geq k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \tag{38}$$

and m_k is the smallest positive integers greater than n_k for which (38) hold.

Since $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ and

$$\begin{aligned} \epsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \epsilon \\ &\leq d(x_k, x_{k-1}) + \epsilon \end{aligned}$$

we obtain that $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$.

And we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq a_{m_k} + d(x_{m_k+1}, x_{n_k+1}) + a_{n_k} \\ &\leq 2a_{n_k} + d(x_{m_k+1}, x_{n_k+1}) \\ &\leq 2a_{n_k} + d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k+1}) \\ &\leq 3a_{n_k} + d(x_{m_k}, x_{n_k+1}) \end{aligned} \tag{39}$$

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and by (32),

$$\begin{aligned}
d(x_{m_k}, x_{n_k+1}) &= d((ST)^{m_k}x, (ST)^{n_k+1}x) \\
&\le F(\max\{d'(T(ST)^{m_k-1}x, T(ST)^{n_k}x), d((ST)^{n_k}x, (ST)^{m_k}x), \\
&\quad d((ST)^{n_k}x, (ST)^{n_k+1}x)\}) \\
&\le F(\max\{d'(y_{m_k}, y_{n_k+1}), d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1})\}) \\
&\le F(\max\{d'(y_{m_k}, y_{n_k+1}), d(x_{n_k}, x_{m_k}), a_{n_k}\}) \qquad (40)
\end{aligned}$$

from (39) and (40), we have

$$d(x_{m_k}, x_{n_k}) \le 3a_{n_k} + F(\max\{d'(y_{m_k}, y_{n_k+1}), d(x_{n_k}, x_{m_k}), a_{n_k}\})$$

By taking $k \to \infty$, we have $\epsilon \le F(\epsilon) < \epsilon$, a contradiction. Thus (x_n) is a Cauchy sequence in X.

Since X is a complete and Y is convergent, (x_n) has a limit in X, say z and (y_n) has a limit in Y, say w. By (31), we have

$$\begin{aligned}
d'(Tz, y_n) &= d'(Tz, T(ST)^{n-1}x) \\
&\le F(\max\{d(z, (ST)^{n-1}x), d'(T(ST)^{n-2}x, Tz), d'(T(ST)^{n-2}x, T(ST)^{n-1}x)\}) \\
&\le F(\max\{d(z, x_{n-1}), d'(y_{n-1}, Tz), d'(y_{n-1}, y_n)\}).
\end{aligned}$$

By taking $n \to \infty$, we have

$$\begin{aligned}
d'(Tz, w) &\le F(\max\{d(z, z), d'(w, Tz), d'(w, w)\}) \\
&\le F(d'(Tz, w)),
\end{aligned}$$

hence $Tz = w$. And by (32), we have

$$\begin{aligned}
d(Sw, x_n) &= d(Sw, (ST)^n x) \\
&\le F(\max\{d'(w, T(ST)^{n-1}x), d((ST)^{n-1}x, Sw), d((ST)^{n-1}x, (ST)^n x)\}) \\
&\le F(\max\{d'(w, y_n), d(x_n, Sw), d(x_{n-1}, x_n)\}).
\end{aligned}$$

By taking $n \to \infty$, we have

$$\begin{aligned}
d(Sw, z) &\le F(\max\{d'(w, w), d(z, Sw), d(z, z)\}) \\
&\le F(d(Sw, z)),
\end{aligned}$$

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so $Sw = z$ and so $STz = Sw = z$ and $TSw = Tz = w$. Thus z is a fixed point of ST and w is a fixed point of TS.

Suppose there is $z' \in X$ such that $STz' = z'$ and $z \neq z'$.

By (31), we have

$$\begin{aligned} d(z', z) &= d(STz', STz) \\ &\leq F(\max\{d'(Tz', Tz), d(z, STz'), d(z, STz)\}) \\ &\leq F(\max\{d'(Tz', Tz), d(z, z')\}) \\ &\leq F(d'(Tz', Tz)) \\ &< d'(Tz', Tz) \end{aligned}$$

and by (32), we have

$$\begin{aligned} d'(Tz', Tz) &= d'(Tz', TSTz) \\ &\leq F(\max\{d(z', STz), d'(Tz, Tz'), d'(Tz, TSTz)\}) \\ &\leq F(\max\{d(z', z), d'(Tz, Tz')\}) \\ &\leq F(d(z', z)) \end{aligned}$$

so $d(z', z) < F(d(z', z)) < d(z', z)$, a contradiction, hence $z = z'$ and ST has a unique fixed point.

Similarly, TS has a unique fixed point. □

Corollary 3.3.2 (cf. [5]) *Let (X, d) and (Y, d') be a complete metric space. If $T : X \to Y$ and $S : Y \to X$ satisfying the inequalities*

$$d'(Tx, TSy) \leq c \cdot \max\{d(x, Sy), d'(y, Tx), d'(y, TSy)\} \quad (41)$$

$$d(Sy, STx) \leq c \cdot \max\{d'(y, Tx), d(x, Sy), d(x, STx)\} \quad (42)$$

for all $x \in X$ and for all $y \in Y$, where $0 \leq c < 1$, then ST has a unique fixed point in X and TS has a unique fixed point in Y.

Proof. Define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(t) = ct$ for all $t \in \mathbb{R}^+$. Then F is satisfied the condition in Theorem 3.3.1. and for $x \in X$ the sequence (y_n) defined as in

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Theorem 3.3.1 is Cauchy sequence as seen in [5], so (y_n) converges. Hence the corollary is obtained directly by Theorem 3.3.1. □

Corollary 3.3.3 *Let (X, d) be a complete metric space. If $S, T : X \to X$ satisfying the inequalities*

$$d(Tx, TSy) \leq F(\max\{d(x, Sy), d(y, Tx), d(y, TSy)\}) \tag{43}$$

$$d(Sy, STx) \leq F(\max\{d(y, Tx), d(x, Sy), d(x, STx)\}) \tag{44}$$

for all $x, y \in X$, where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous function such that $F(t) < t$ for each $t > 0$ and if there is $x \in X$ such that the sequence (y_n), define by $y_n = T(ST)^{n-1}x$ converges, then ST has a unique fixed point and TS has a unique fixed point. Further if fixed point of ST is fixed point of TS, then S and T has a unique fixed point.

Proof. By Theorem 3.3.1, we obtain that each of ST and TS has a unique fixed point. Now, suppose that ST and TS have the same unique fixed point, say x. Then $STx = x$ and $TSx = x$. So $(TS)(Tx) = Tx$ and $(ST)(Sx) = Sx$, thus Tx and Sx are fixed point of TS and ST, respectively. By the uniqueness of their fixed point, it follows that $Tx = Sx = x$. □

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