Chapter 2

Orbital Dynamics of Particles

In this chapter, the fundamental interactions, required for constructing the Activated Molecular Cloud Cluster Model, are presented. All interactions will be referred again in Chapter 3 to describe the solar system formation.

2.1 The Keplerion Motion

The Keplerion motion or the elliptical motion of two masses under the influence of their mutual gravity is generally considered as the two-body problem. The system of two-body is shown in Figure 2.1.

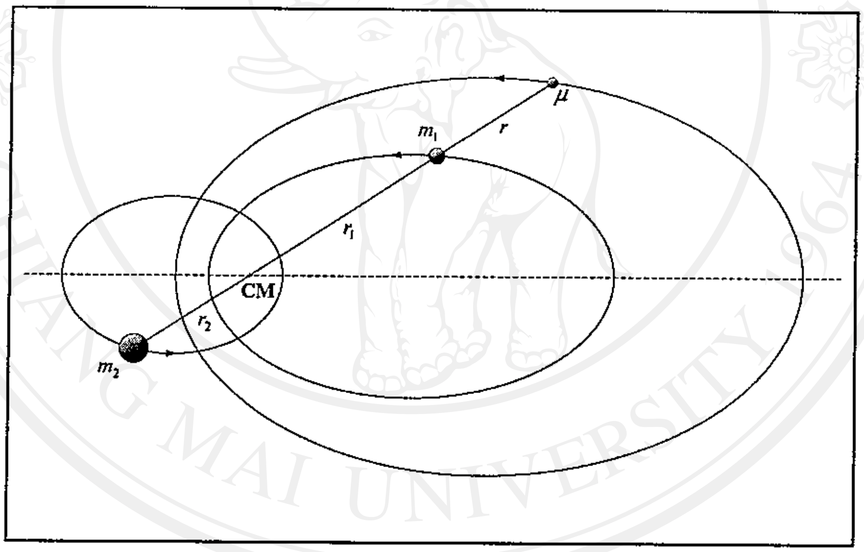

Figure 2.1 Keplerion motion or elliptical two-body system.

The center of mass of both particles is taken as the origin of system. The coordinate vector $\mathbf{r}_1$ and $\mathbf{r}_2$ can be defined by combining the equation

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0 \tag{2.1}$$

with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, then

$$\left.\begin{aligned} \mathbf{r}_1 &= \frac{m_2}{m_1+m_2}\mathbf{r} \\ \mathbf{r}_2 &= -\frac{m_1}{m_1+m_2}\mathbf{r} \end{aligned}\right\} \tag{2.2}$$

If there is no external force acts on the system, the total angular momentum of the system is conserved:

$$\mathbf{L}_{total} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i = \text{constant} \tag{2.3}$$

Its magnitude can be written alternatively as

$$\ell = |\mathbf{L}| = m_1 |\mathbf{r}_1|^2 \dot{\theta} + m_2 |\mathbf{r}_2|^2 \dot{\theta} \tag{2.4}$$

Substituting by Equation 2.2, gives

$$\ell = \mu r^2 \dot{\theta} \tag{2.5}$$

where μ is the reduced mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.6}$$

The general solution of the system can be solved from the conservation of energy relation:

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \\ &= \text{constant} \end{aligned} \tag{2.7}$$

or

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r) \tag{2.8}$$

Solving Equation 2.8 for $\dot{r}$, we have

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{\ell^2}{2\mu r^2}\right)} \tag{2.9}$$

This equation has two roots that give $\dot{r} = 0$; for elliptical orbit, these two roots are known as the turning point of motion.

We can write the equation of path in terms of r and θ by using the relation

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr \tag{2.10}$$

Then, substitute $\dot{\theta} = \frac{\ell}{\mu r^2}$ from Equation 2.5 and $\dot{r}$ from Equation 2.9 into Equation 2.10 with integration, we have

$$\theta(r) = \int \frac{\pm(\ell / r^2)dr}{\sqrt{2\mu\left(E - U(r) - \frac{\ell}{2\mu r^2}\right)}} + \text{constant} \qquad (2.11)$$

For the motion in gravitational field, the potential energy term is

$$U(r) = -\frac{k}{r} \qquad (2.12)$$

where $k = Gm_1m_2$. Then the Equation 2.11 becomes

$$\theta(r) = \int \frac{\pm(\ell / r^2)dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell}{2\mu r^2}\right)}} + \text{constant} \qquad (2.13)$$

Integrated by using the new variable $u\left(\equiv \frac{1}{r}\right)$ gives

$$\cos\theta = \frac{\frac{\ell^2}{\mu k} \cdot \frac{1}{r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}} \qquad (2.14)$$

or

$$r = \frac{\alpha}{1 + \varepsilon\cos\theta} \qquad (2.15)$$

where

$$\left.\begin{aligned} \alpha &= \frac{\ell^2}{\mu k} \\ \varepsilon &= \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \end{aligned}\right\} \qquad (2.16)$$

Equation 2.15 is the equation of a conic section with the eccentricity ε and latus rectum 2α. The particle orbits with various values of eccentricity are illustrated in Figure 2.2.

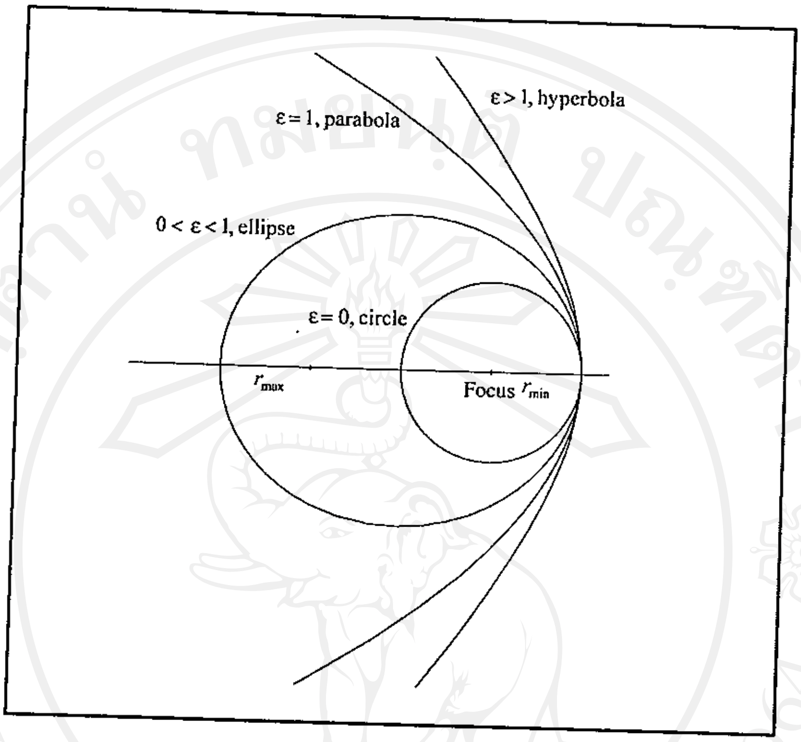

Figure 2.2 Graphs of a conic section with various eccentricities.

The orbital motion, which is of interest to us, is the elliptical motion where $0<\varepsilon<1$. This motion belongs to the Kepler's laws of planetary motion, so we can call it the Keplerion motion. The turning points of motion (r_{min} and r_{max}) can be derived from Equation 2.15 by taking $\theta=0$ and $\theta=\pi$ respectively. Thus

$$\left.\begin{aligned} r_{min} &= \frac{\alpha}{1+\varepsilon} \\ r_{max} &= \frac{\alpha}{1-\varepsilon} \end{aligned}\right\} \tag{2.17}$$

The radius of orbit extends back and forth between these two points. This behavior is called the *radial pulsation* with the pulsating period τ, given by

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)} \tag{2.18}$$

where a is the semi-major axis of orbit. Equation 2.18 is usually called the Kepler's third law of planetary motion.

The pulsation behavior is more obvious, when we look at a ring-construction system of particles, which all particles orbit about the central mass elliptically. Every particle has the same initial angular velocity ω_0 at the radius equal to r_{max} and their semi-major axes are laid in different angles. The sequence of particles motion with time is shown in Figure 2.3.

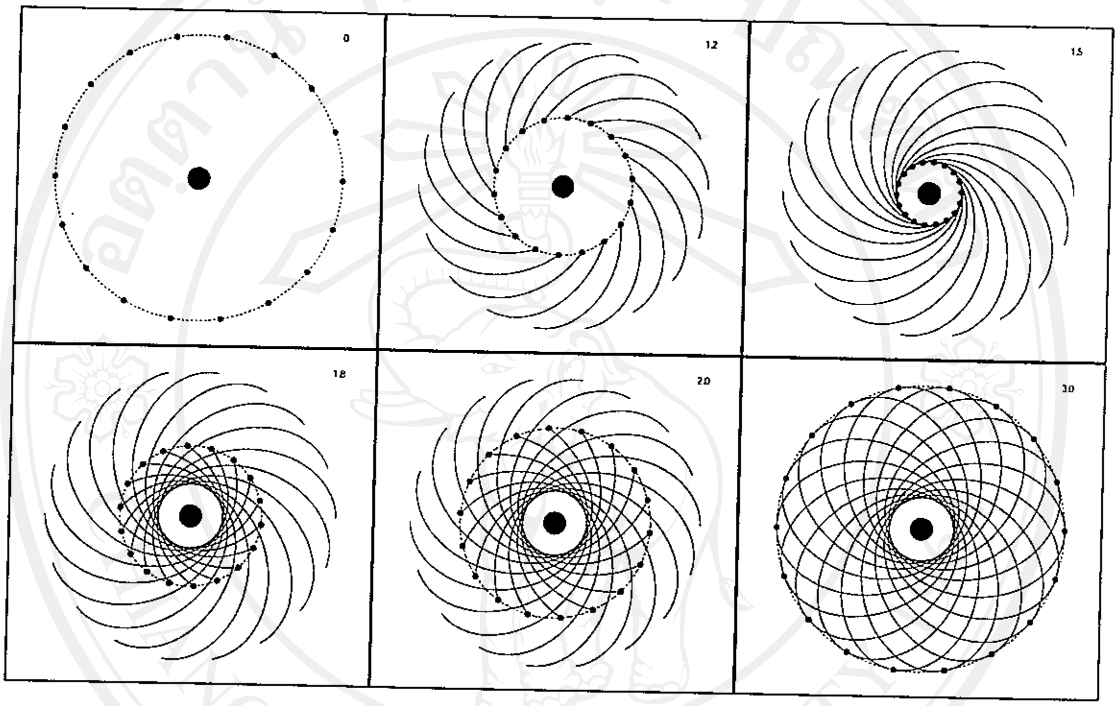

Figure 2.3 A ring of Keplerion particles moving elliptically back and forth around the central mass illustrates a radial pulsating system.

The radial pulsation concept will be used in Chapter 3 to describe the behavior of collapsing objects in a primordial solar system.

2.2 The Orbital Elastic Collision

In general, the elastic collision between two particles is usually the linear collision where linear momentums are exchanged. This section, we will concentrate on an elastic collision between the two Keplerion particles; each particle carries a linear momentum in both radial and azimuthal component, as shown in Figure 2.4. However, the two components are independent, so we can apply the method of linear collision to deal with both components.

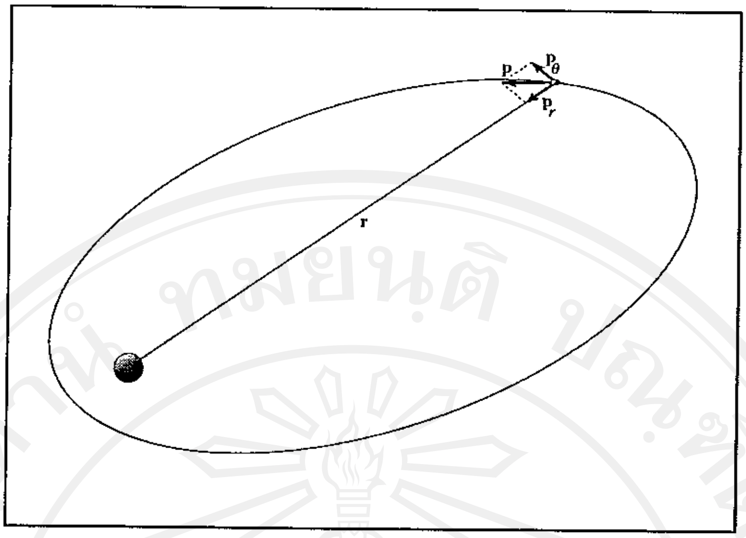

Figure 2.4 The two components of linear momentum.

It will be more convenient if we use the center-of-mass (CM) coordinate system to describe the collision, as illustrated in Figure 2.5. Although the velocity of each particle is not constant throughout its orbit, the velocity of collision is certainly to be able to use the velocity at a collision point.

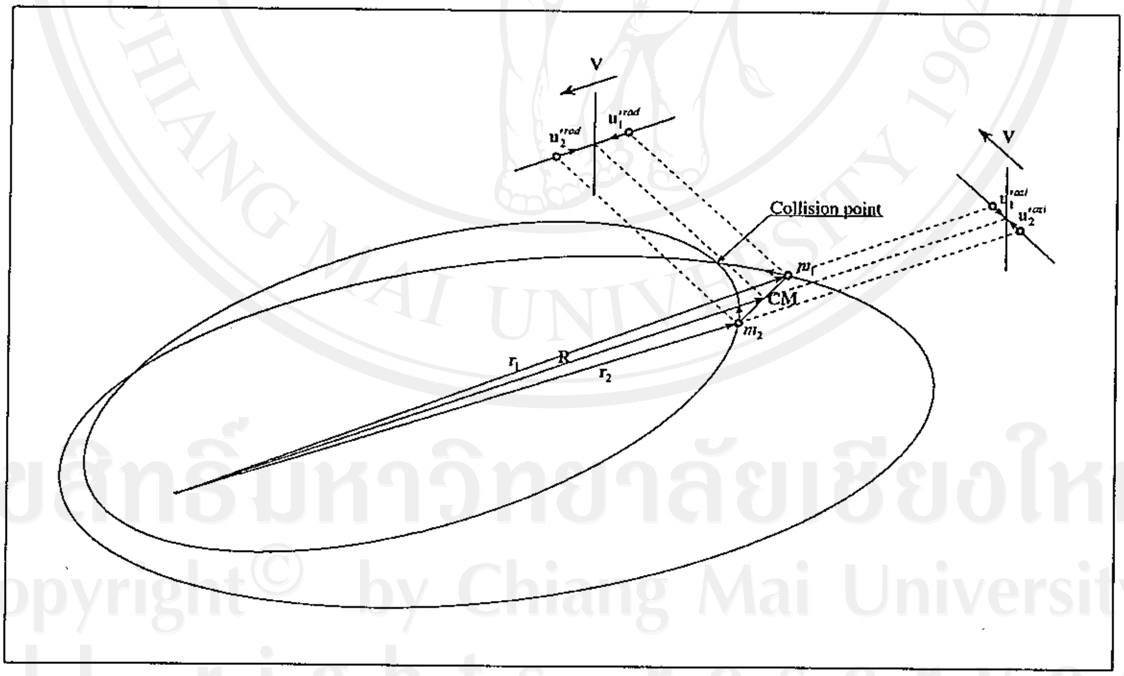

Figure 2.5 The projection paths illustrate the motion of particles in CM system.

For CM system, the CM coordinate $\mathbf{R}$ is defined by the relation

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1 + m_2)\mathbf{R} \qquad (2.19)$$

Differentiating with respect to time gives us the velocity relation

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = (m_1 + m_2)\mathbf{V} \tag{2.20}$$

or

$$\mathbf{V} = \frac{m_1}{m_1 + m_2}\mathbf{u}_1 + \frac{m_2}{m_1 + m_2}\mathbf{u}_2 \tag{2.21}$$

where $\mathbf{V} = \dot{\mathbf{R}}$ is the velocity of CM and $\mathbf{u}_i = \dot{\mathbf{r}}_i$ is the original velocity that represents both radial and azimuthal component.

From Figure 2.6, the velocity of particles in CM system are given by

$$\left.\begin{aligned} \mathbf{u}_1' &= \mathbf{u}_1 - \mathbf{V} \\ \mathbf{u}_2' &= \mathbf{u}_2 - \mathbf{V} \end{aligned}\right\} \tag{2.22}$$

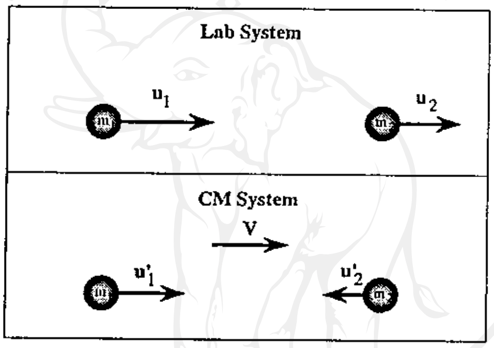

Figure 2.6 Motion of two particles in the Lab and CM frames.

In CM system, the total momentum of the two particles is zero:

$$m_1\mathbf{u}_1' + m_2\mathbf{u}_2' = m_1\mathbf{v}_1' + m_2\mathbf{v}_2' = 0 \tag{2.23}$$

where $\mathbf{v}'$ is the velocity of particle after collision in the CM frame. From this equation, we have

$$\left.\begin{aligned} \mathbf{u}_2' &= -\frac{m_1}{m_2}\mathbf{u}_1' \\ \mathbf{v}_2' &= -\frac{m_1}{m_2}\mathbf{v}_1' \end{aligned}\right\} \tag{2.24}$$

In an elastic collision, the energy is also conserved in both Laboratory and CM system. We can write

$$T_{before} + U_{before} = T_{after} + U_{after}$$
$$\frac{1}{2}m_1|\mathbf{u}_1'|^2 + \frac{1}{2}m_2|\mathbf{u}_2'|^2 = \frac{1}{2}m_1|\mathbf{v}_1'|^2 + \frac{1}{2}m_2|\mathbf{v}_2'|^2 \quad (2.25)$$

where $U_{befor} = U_{after}$ at collision point. Substituted by Equation 2.24 yields

$$\left.\begin{aligned} |\mathbf{v}_1'| &= |\mathbf{u}_1'| \\ |\mathbf{v}_2'| &= |\mathbf{u}_2'| \end{aligned}\right\} \quad (2.26)$$

The directions of recoil particles must be opposite, so

$$\left.\begin{aligned} \mathbf{v}_1' &= -\mathbf{u}_1' = \mathbf{V} - \mathbf{u}_1 \\ \mathbf{v}_2' &= -\mathbf{u}_2' = \mathbf{V} - \mathbf{u}_2 \end{aligned}\right\} \quad (2.27)$$

Using Equation 2.22 to transform the velocities in CM system to Laboratory system, we have

$$\left.\begin{aligned} \mathbf{v}_1 &= \mathbf{v}_1' + \mathbf{V} = 2\mathbf{V} - \mathbf{u}_1 \\ \mathbf{v}_2 &= \mathbf{v}_2' + \mathbf{V} = 2\mathbf{V} - \mathbf{u}_2 \end{aligned}\right\} \quad (2.28)$$

Substituted $\mathbf{V}$ from Equation 2.21 into 2.28 and rearranged gives us

$$\mathbf{v}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\mathbf{u}_1 + \left(\frac{2m_2}{m_1 + m_2}\right)\mathbf{u}_2 \quad (2.29a)$$

and

$$\mathbf{v}_2 = \left(\frac{2m_1}{m_1 + m_2}\right)\mathbf{u}_1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\mathbf{u}_2 \quad (2.29b)$$

the recoil velocities of the two particles in Laboratory system.

Now we use the Equation 2.29a and 2.29b to determine the recoil velocities of particles in each component to find the new quantity of the angular momentum and the total energy. Note that, first things we must know about the system are the original values of the angular momentum and the total energy of particles.

Radial Velocities

The original velocity of particles in radial component at the collision point can be determined from Equation 2.9 by substituting $U = -\frac{k}{r}$, we have

$$\left(u_1^{ori}\right)_{rad} = \dot{r}_1 = \pm\sqrt{\frac{2}{\mu}\left(E_1^{ori} + \frac{k}{r} - \frac{\ell_1^{ori^2}}{2\mu r^2}\right)} \qquad (2.30a)$$

and

$$\left(u_2^{ori}\right)_{rad} = \dot{r}_2 = \pm\sqrt{\frac{2}{\mu}\left(E_2^{ori} + \frac{k}{r} - \frac{\ell_2^{ori^2}}{2\mu r^2}\right)} \qquad (2.30a)$$

Then the recoil velocity of particles become

$$\left(v_1^{rec}\right)_{rad} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\left(u_1^{ori}\right)_{rad} + \left(\frac{2m_2}{m_1 + m_2}\right)\left(u_2^{ori}\right)_{rad} \qquad (2.31a)$$

and

$$\left(v_2^{rec}\right)_{rad} = \left(\frac{2m_1}{m_1 + m_2}\right)\left(u_1^{ori}\right)_{rad} - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\left(u_2^{ori}\right)_{rad} \qquad (2.31b)$$

Azimuthal Velocities

We can determine the original velocity of particles in azimuthal component from the angular momentum relation (Equation 2.5)

$$\ell = \mu r^2\dot{\theta} = \text{constant} \qquad (2.5)$$

which constant until the collision begins. The azimuthal velocities are, therefore,

$$\left(u_1^{ori}\right)_{azi} = \omega_1^{ori} r = \frac{\ell_1^{ori}}{m_1 r} \qquad (2.32a)$$

and

$$\left(u_2^{ori}\right)_{azi} = \omega_2^{ori} r = \frac{\ell_2^{ori}}{m_2 r} \qquad (2.32b)$$

The recoil velocity of particles become

$$\left(v_1^{rec}\right)_{azi} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\left(u_1^{ori}\right)_{azi} + \left(\frac{2m_2}{m_1 + m_2}\right)\left(u_2^{ori}\right)_{azi} \qquad (2.33a)$$

and

$$\left(v_2^{rec}\right)_{azi} = \left(\frac{2m_1}{m_1 + m_2}\right)\left(u_1^{ori}\right)_{azi} - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\left(u_2^{ori}\right)_{azi} \qquad (2.33b)$$

Angular Momentum and Total Energy of Particles

The angular momentum of both particles are given by multiplying Equation 2.33a and 2.33b with $m_1 r$ and $m_2 r$ respectively, and then, rearranged by using Equation 2.32a and 2.32b, we have

$$\ell_1^{rec} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\ell_1^{ori} + \left(\frac{2m_1}{m_1 + m_2}\right)\ell_2^{ori} \tag{2.34a}$$

and

$$\ell_2^{rec} = \left(\frac{2m_2}{m_1 + m_2}\right)\ell_1^{ori} - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\ell_2^{ori} \tag{2.34b}$$

The total energy after collision can be defined in terms of momentum and potential energy as

$$E_i^{rec} = \frac{1}{2}\frac{p_i^{rec^2}}{\mu} + \frac{1}{2}\frac{\ell_i^{rec^2}}{\mu r^2} + U_i(r) \tag{2.35}$$

where p_i^{rec} is the radial momentum of recoil particle determined by the same manner as angular momentum. Thus

$$p_1^{rec} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)p_1^{ori} + \left(\frac{2m_1}{m_1 + m_2}\right)p_2^{ori} \tag{2.36a}$$

and

$$p_2^{rec} = \left(\frac{2m_2}{m_1 + m_2}\right)p_1^{ori} - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)p_2^{ori} \tag{2.36b}$$

After collision, the total energy of each particle may changes but the sum of them is always constant.

$$E_1^{ori} + E_2^{ori} = E_1^{rec} + E_2^{rec} = \text{constant} \tag{2.37}$$

The changing of angular momentum and total energy gives particle a new orbit that will be mentioned in Chapter 3.

2.3 Motion in the Resisting Medium

The motion of every object must be retarded in the resisting medium. The retarding force is generally proportional to velocity and cross-sectional area of the object. Although the exact equation of the retarding forces is more complicated, the power-law approximation is still useful in many cases (Jerry, 1995); it can be written as

$$\mathbf{F}_{retard} = -kv^n \frac{\mathbf{v}}{v} \tag{2.38}$$

where k is a positive constant and $\frac{\mathbf{v}}{v}$ is a unit vector in the direction of velocity $\mathbf{v}$.

Experimentally, for a relatively small object moving in air, $n \cong 1$ for velocities less than about 24 m/s and $n \cong 2$ for higher velocities up to the velocity of sound (~330 m/s). For higher above these ranges, the retarding force is linearly proportional to velocity or $n \cong 1$ again. The example of retarding force, which $n = 2$ can be expressed as

$$\mathbf{F} = -\frac{1}{2} c\rho A v^2 \frac{\mathbf{v}}{v} \tag{2.39}$$

where c is the dimensionless drag coefficient, ρ is the air density and A is the cross-sectional area of the object. This equation is known as the Prandtl expression for the air resistance (Jerry, 1995).

In general, the motion of the object is attenuated in the component where the retarding force exists. However, in the non-stationary medium, the motion of the object may be whether attenuated or enhanced during its journey. In this case, the retarding force can be expressed in term of relative velocity between the object and the medium, thus for the Keplerion motion

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^3}\mathbf{r} - k|\mathbf{v}-\mathbf{u}|^n \frac{(\mathbf{v}-\mathbf{u})}{|\mathbf{v}-\mathbf{u}|} \tag{2.40}$$

where $\mathbf{u}$ is the velocity field of the medium. We can solve this equation by numerical calculation; the results of simulations are shown in Section 4.2.

2.4 The Tidal Induction

The tidal induction is the interaction between two clusters of particles, which are moving with Keplerion orbit relative to each other as illustrated in Figure 2.7.

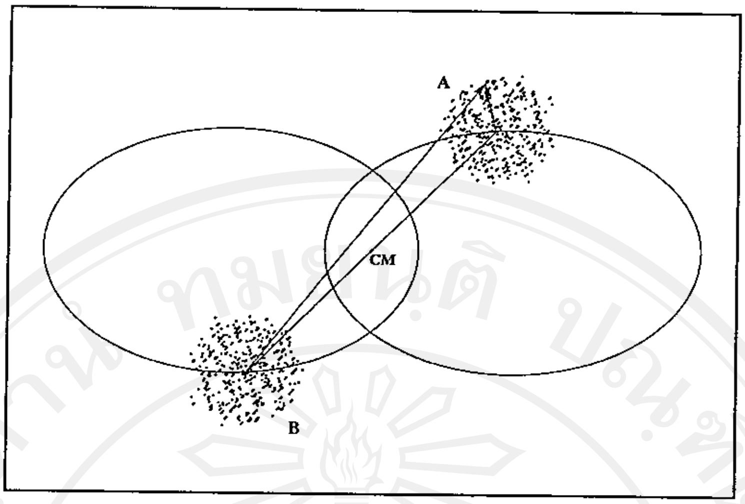

Figure 2.7 Two particles clusters, moving with Keplerion orbits and interact each other with tidal force.

The equation of motion of particle i^{th} in cluster **A** is given by

$$\frac{d^2\mathbf{r}_i}{dt^2} = -\sum_{\substack{j=1 \\ j\neq i}}^{N_A} \frac{Gm_j}{r_{ij}^3}\mathbf{r}_{ij} - \sum_{k=1}^{N_B} \frac{Gm_k}{r_{ik}^3}\mathbf{r}_{ik} \qquad (2.41)$$

where N_A and N_B are the total number of particles in cluster **A** and **B** respectively. For the rotational frame of reference, the equation of motion can be written as

$$\frac{d^2\mathbf{r}_i'}{dt^2} = \frac{d^2\mathbf{r}_i}{dt^2} - 2\boldsymbol{\omega}\times\dot{\mathbf{r}}_i' - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i') - \dot{\boldsymbol{\omega}}\times\mathbf{r}_i' \qquad (2.42)$$

where $\boldsymbol{\omega}$ is the angular velocity of the system varying with time. The both equations can be solved by numerical calculation; the simulations are already shown in Section 4.3.

From Equation 2.42 above, we see that the motion of particle is influenced by the gravitational and the rotational forces. The interaction can be classified by its strength into two kinds: high-density-regions induction and mass transfer. The strength itself depends on many variables such as the mass of clusters (M_A and M_B), the total distance R, the angular velocity ω, etc; every variable except mass are time dependent. We can determine which interaction will be taken place by using the Lagrange points of the system. The Lagrange points represent the points that all relevant forces are equilibrium as shown in Figure 2.8.

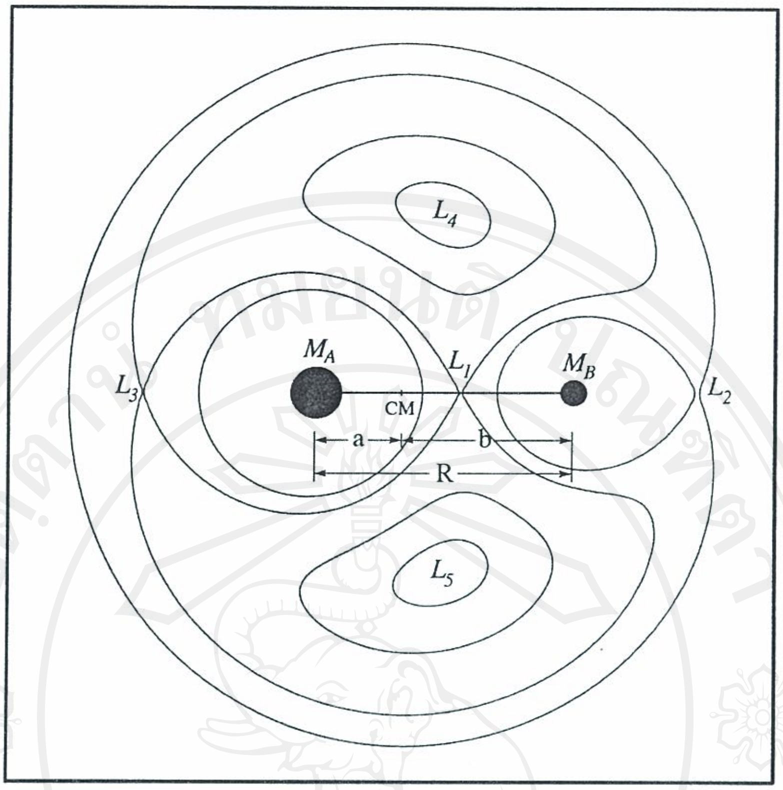

Figure 2.8 Contour of equipotential and Lagrange Points $L_1 - L_5$.

The interactions can be distinguished by considering the size of each cluster when they reach the pericenter of orbit ($R = r_{min}$, Equation 2.17). At this position, if the radius of cluster approximately approaches to the point L_1, the particles at the rim of cluster can move through this point; the mass is transferred. If it does not, the high-density-regions induction can be occurred instead.

All Lagrange points $(L_1 - L_5)$ can be determined by approximated that the gravitational and the centrifugal term in Equation 2.42 are dominate; because at pericenter the angular velocity ω is maximum while the angular acceleration $\dot{\omega}$ is zero (the azimuthal force vanishes) and the Coriolis term is not included. Therefore, the net acceleration in x and y axis are given by

$$g_x = -\frac{GM_A(x+a)}{((x+a)^2+y^2)^{\frac{3}{2}}} + \frac{GM_B(x-b)}{((x-b)^2+y^2)^{\frac{3}{2}}} + \omega_{max}^2 x \qquad (2.43a)$$

and

$$g_y = -\frac{GM_A y}{((x+a)^2+y^2)^{\frac{3}{2}}} - \frac{GM_B y}{((x-b)^2+y^2)^{\frac{3}{2}}} + \omega_{max}^2 y \qquad (2.43b)$$

where

$$
\left.\begin{aligned}
a &= \frac{R}{\left(1+\dfrac{M_A}{M_B}\right)} \\
b &= \frac{R}{\left(1+\dfrac{M_B}{M_A}\right)}
\end{aligned}\right\} \tag{2.44}
$$

are the distance from CM of the system. The angular velocity from Equation 2.5 is

$$
\omega = \frac{\ell}{\mu R^2} \tag{2.45}
$$

Substituting ω and $y = 0$ into Equation 2.43a we have

$$
g_x = -\frac{GM_A}{(x+a)^2} + \frac{GM_B}{(x-b)^2} + \omega_{max}^2 x \tag{2.46}
$$

At Lagrange points $g_x = g_y = 0$ then the solutions in x-axis for $L_1 - L_5$ are

$$
\{x_1, x_2, x_3, x_4, x_5\} \tag{2.47}
$$

From Figure 2.8 x_4 and x_5 are the same, substituted x in Equation 2.43b by x_4 or x_5 with $g_y = 0$ gives

$$
\{y_4, y_5\} \tag{2.48}
$$

Combining the solutions from Equation 2.47 and 2.48 gives the positions of Lagrange points

$$
\left.\begin{aligned}
L_1 &= (x_1, 0) \\
L_2 &= (x_2, 0) \\
L_3 &= (x_3, 0) \\
L_4 &= (x_4, y_4) \\
L_5 &= (x_5, y_5)
\end{aligned}\right\} \tag{2.49}
$$

However, the Lagrange Point we interest is L_1; it indicates that what interaction can be taken place. For the high-density-regions induction, the particles in both clusters can be induced in the regions between the Lagrange points (L_1, L_2 and L_3) and the center of clusters (see Section 4.3).