CHAPTER 3

MAIN RESULTS

In this chapter we introduce a simple, smooth and adaptive controller for resolving the control and synchronization problems of the perturbed Chua's circuit system.

3.1 Chua's Circuit System

In this section, We will study the stability of the equilibrium points of Chua's circuit system described by the following dynamics system :

$$\begin{aligned} \dot{x} &= p\big(y - \frac{1}{7}(2x^3 - x)\big) \\ \dot{y} &= x - y + z \\ \dot{z} &= -qy \end{aligned} \tag{3.1}$$

where x, y and z are the state variables, p and q are positive real parameters. The system (3.1) has three equilibrium points

$$E_+ = (\sqrt{0.5}, 0, -\sqrt{0.5}),\ E_0 = (0,0,0),\ E_- = (-\sqrt{0.5}, 0, \sqrt{0.5}).$$

Theorem 3.1.1 *If p and q are satisfies either inequality $q < \frac{5p}{7}$ or $q < \frac{10p^2}{49} + \frac{5p}{7}$, then the three equilibrium points E_+, E_0, and E_- of the system (3.1) are unstable.*

Proof. The Jacobian matrix of the system (3.1) about the equilibrium point $E = (\overline{x}, \overline{y}, \overline{z})$ is

$$J_0 = \begin{bmatrix} \frac{p(-6\overline{x}^2+1)}{7} & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix}.$$

The characteristic equation of J_0 is

$$\lambda^3 + \Big(1 + \frac{p(6\overline{x}^2 - 1)}{7}\Big)\lambda^2 + \Big(q + p\big(\frac{6\overline{x}^2 - 8}{7}\big)\Big)\lambda + \frac{pq(6\overline{x}^2 - 1)}{7} = 0.$$

Let

$$\varphi_1 = 1 + \frac{p(6\overline{x}^2 - 1)}{7}, \quad \varphi_2 = q + p\Big(\frac{(6\overline{x}^2 - 8)}{7}\Big) \text{ and } \varphi_3 = \frac{pq(6\overline{x}^2 - 1)}{7}.$$

Firstly, if $\overline{x} = 0$, then $\varphi_3 < 0$. According to Routh-Hurwitz criteria the equilibrium point (0,0,0) is unstable.

Secondly, if $\overline{x} = \pm\sqrt{0.5}$, we have

$$\varphi_1 = 1 + \frac{2p}{7}, \quad \varphi_2 = q - \frac{5p}{7} \text{ and } \varphi_3 = \frac{2pq}{7}.$$

We can see that if $q < \frac{5p}{7}$ then $\varphi_2 < 0$ and if $q < \frac{10p^2}{49} + \frac{5p}{7}$ then $\varphi_1\varphi_2 < \varphi_3$. According to Routh-Hurwitz criteria the equilibrium point $E_+ = (\sqrt{0.5}, 0, -\sqrt{0.5})$ and $E_- = (-\sqrt{0.5}, 0, \sqrt{0.5})$ are unstable. □

3.1.1 Numerical Simulations

We give numerical experiments to demonstrate the effectiveness of the proposed control scheme. Fourth-order Runge-Kutta method is used to solve the differential equations with time step 0.01. The parameters p and q are chosen as $p = 10$ and $q = \frac{100}{7}$. The initial states are taken as $x = 0.65$, $y = 0$ and $z = 0$. Fig. 3.1 shows the chaotic behavior of the states x, y and z of the system (3.1) with time in xy-plane. Fig. 3.2 shows the chaotic behavior of the states x, y and z of the system (3.1) with time in xz-plane. Fig. 3.3 shows the chaotic behavior of the states x, y and z of the system (3.1) with time in yz-plane.

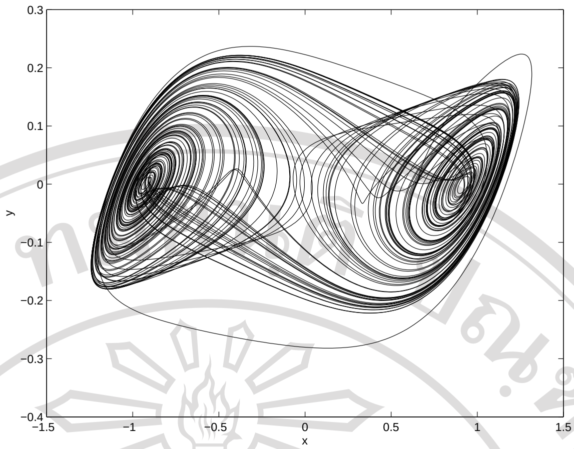

Figure 3.1: The chaotic attractor of Chua's circuit system (3.1) in the xy-plane.

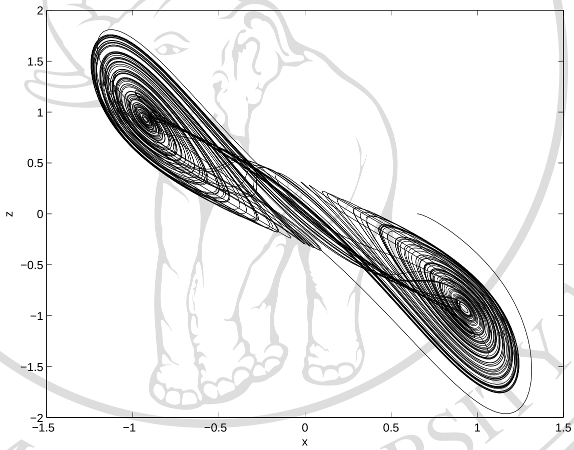

Figure 3.2: The chaotic attractor of Chua's circuit system (3.1) in the xz-plane.

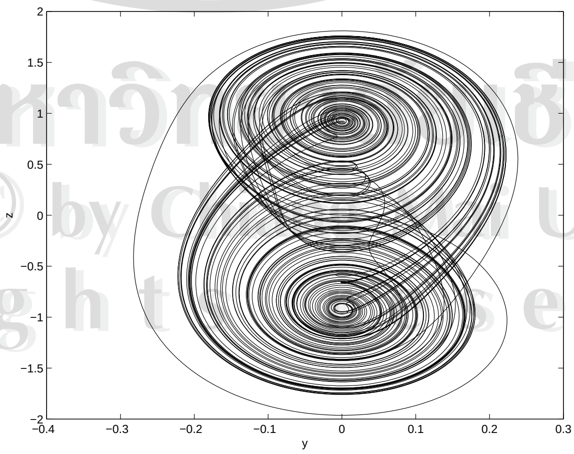

Figure 3.3: The chaotic attractor of Chua's circuit system (3.1) in the yz-plane.

3.2 The Perturbed Chua's Circuit System

We will study the perturbed Chua's circuit system described by the following dynamics system :

$$\begin{aligned} \dot{x} &= p\big(y - \frac{1}{7}(2x^3 - x)\big) \\ \dot{y} &= x - y + z \\ \dot{z} &= -qy + rx^2 \end{aligned} \tag{3.2}$$

where x, y and z are the state variables, p, q and r are positive real parameters. The system (3.2) has three equilibrium points

$$E_0 = (0,0,0),\ E_1 = (\alpha_1, \beta_1, \gamma_1),\ E_2 = (\alpha_2, \beta_2, \gamma_2)$$

where

$$\alpha_1 = \frac{7r + \sqrt{49r^2 + 8q^2}}{4q}, \quad \beta_1 = \frac{r\alpha_1^2}{q}, \quad \gamma_1 = \beta_1 - \alpha_1$$
$$\alpha_2 = \frac{7r - \sqrt{49r^2 + 8q^2}}{4q}, \quad \beta_2 = \frac{r\alpha_2^2}{q}, \quad \gamma_2 = \beta_2 - \alpha_2.$$

Theorem 3.2.1 *For p, q and r are positive real parameters, the equilibrium point $E_0 = (0,0,0)$ is unstable.*

Proof. The Jacobian matrix of the system (3.2) at the equilibrium point $E_0 = (0,0,0)$ is given by

$$J_0 = \begin{bmatrix} \frac{p}{7} & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{bmatrix}.$$

The characteristic equation of J_0 is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$a_1 = 1 - \frac{p}{7}$$

$$a_2 = q - \frac{8p}{7}$$
$$a_3 = -\frac{pq}{7}.$$

We see that $a_3 < 0$ does not satisfy the Routh-Hurwitz criteria, and so the equilibrium point $E_0 = (0, 0, 0)$ is unstable. □

Theorem 3.2.2 *For $p = 10$ and $q = \frac{100}{7}$ the equilibrium point $E_1 = (\alpha_1, \beta_1, \gamma_1)$ is*

(i) asymptotically stable if $r > r_1$;

(ii) unstable if $0 < r < r_1$;

where r_1 is the unique positive root of equation (3.5), ($r_1 \approx 4.51841$).

Proof. The Jacobian matrix of the system (3.2) at the equilibrium point $E_1 = (\alpha_1, \beta_1, \gamma_1)$ is given by

$$J_1 = \begin{bmatrix} \frac{p(-6\alpha_1^2+1)}{7} & p & 0 \\ 1 & -1 & 1 \\ 2r\alpha_1 & -q & 0 \end{bmatrix}.$$

The characteristic equation of J_1 is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= 1 + \frac{p(6\alpha_1^2 - 1)}{7} \\ a_2 &= q + \frac{p(6\alpha_1^2 - 8)}{7} \\ a_3 &= \frac{pq(6\alpha_1^2 - 1)}{7} + 2pr\alpha_1. \end{aligned} \tag{3.3}$$

Substituting α_1 in (3.3), we get the equation

$$\begin{aligned} a_1 &= 1 + \frac{p}{7}\Big[\frac{588r^2}{q^2} + \frac{108r}{16q^2}\sqrt{49r^2 + 8q^2} + 2\Big] \\ a_2 &= q + \frac{p}{7}\Big[\frac{588r^2}{q^2} + \frac{108r}{16q^2}\sqrt{49r^2 + 8q^2} - 5\Big] \\ a_3 &= \frac{pq}{7}\Big[\frac{588r^2}{q^2} + \frac{108r}{16q^2}\sqrt{49r^2 + 8q^2} + 2\Big] \\ &\quad + \frac{pr}{2q}\Big[7r + \sqrt{49r^2 + 8q^2}\Big]. \end{aligned} \tag{3.4}$$

Substituting $p = 10$ and $q = \frac{100}{7}$ in (3.4), we obtain $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$ for all $r \geq 0$. Let

$$\begin{aligned} f_1(r) = a_1a_2 - a_3 \;\; = \;\; & q + \frac{p}{7}\Big[\frac{588r^2}{q^2} + \frac{108r}{16q^2}\sqrt{49r^2+8q^2} - 5\Big] \\ & + \frac{36p^2}{3136q^2}\Big[19208r^4 + 1372r^3\sqrt{49r^2+8q^2} + 3136q^2r^2 \\ & + 224q^2r\sqrt{49r^2+8q^2}\Big] - \frac{54p^2}{3136q^2}\Big[98r^2 + 8q^2 \qquad (3.5) \\ & + 14r\sqrt{49r^2+8q^2}\Big] + \frac{8p^2}{49} - \frac{pr}{2q}\Big[7r + \sqrt{49r^2+8q^2}\Big]. \end{aligned}$$

For $p = 10$ and $q = \frac{100}{7}$, we have $f_1(0) < 0$ and $f_1(\infty) > 0$, so that $f_1(r)$ has at least one positive root. Since $f_1'(r) > 0$ for all $r \geq 0$, $f_1(r)$ has only one positive root. Let r_1 be the positive root of (3.5), for $0 < r < r_1$ we have $f_1(r) < 0$ and for $r > r_1$ we have $f_1(r) > 0$.

When $r > r_1$, $f_1(r) > 0$ satisfies the Routh-Hurwitz criteria. Thus, the equilibrium point $E_1 = (\alpha_1, \beta_1, \gamma_1)$ is asymptotically stable if $r > r_1$.

However, when $0 < r < r_1$, we have that $f_1(r) < 0$ which does not satisfy the Routh-Hurwitz criteria therefore the equilibrium point $E_1 = (\alpha_1, \beta_1, \gamma_1)$ is unstable. □

Theorem 3.2.3 *For $p = 10$ and $q = \frac{100}{7}$ the equilibrium point $E_2 = (\alpha_2, \beta_2, \gamma_2)$ is*

(i) asymptotically stable if $r > r_2$.

(ii) unstable if $0 < r < r_2$.

where r_2 is the unique positive root of equation (3.8), ($r_2 \approx 0.73672$).

Proof. The Jacobian matrix of the system (3.2) at the equilibrium point $E_2 = (\alpha_2, \beta_2, \gamma_2)$ is given by

$$J_2 = \begin{bmatrix} \frac{p(-6\alpha_2^2+1)}{7} & p & 0 \\ 1 & -1 & 1 \\ 2r\alpha_2 & -q & 0 \end{bmatrix}.$$

The characteristic equation of J_2 is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= 1+\frac{p(6\alpha_2^2-1)}{7} \\ a_2 &= q+\frac{p(6\alpha_2^2-8)}{7} \\ a_3 &= \frac{pq(6\alpha_2^2-1)}{7}+2pr\alpha_2. \end{aligned} \tag{3.6}$$

Substituting α_2 in (3.6), we get the equation

$$\begin{aligned} a_1 &= 1+\frac{p}{7}\Big[\frac{588r^2}{q^2}-\frac{108r}{16q^2}\sqrt{49r^2+8q^2}+2\Big] \\ a_2 &= q+\frac{p}{7}\Big[\frac{588r^2}{q^2}-\frac{108r}{16q^2}\sqrt{49r^2+8q^2}-5\Big] \\ a_3 &= \frac{pq}{7}\Big[\frac{588r^2}{q^2}-\frac{108r}{16q^2}\sqrt{49r^2+8q^2}+2\Big] \\ &\quad +\frac{pr}{2q}\Big[7r+\sqrt{49r^2+8q^2}\Big]. \end{aligned} \tag{3.7}$$

Substituting $p = 10$ and $q = \frac{100}{7}$ in (3.7), we obtain $a_1 > 0$, $a_2 > 0$ and $a_3 > 0$ for all $r \geq 0$. Let

$$\begin{aligned} f_2(r) = a_1a_2 - a_3 &= q+\frac{p}{7}\Big[\frac{588r^2}{q^2}-\frac{108r}{16q^2}\sqrt{49r^2+8q^2}-5\Big] \\ &\quad +\frac{36p^2}{3136q^2}\Big[19208r^4-1372r^3\sqrt{49r^2+8q^2}+3136q^2r^2 \\ &\quad -224q^2r\sqrt{49r^2+8q^2}\Big]-\frac{54p^2}{3136q^2}\Big[98r^2+8q^2 \\ &\quad -14r\sqrt{49r^2+8q^2}\Big]+\frac{8p^2}{49}-\frac{pr}{2q}\Big[7r-\sqrt{49r^2+8q^2}\Big]. \end{aligned} \tag{3.8}$$

For $p = 10$ and $q = \frac{100}{7}$, we have $f_2(0) < 0$ and $f_2(\infty) > 0$, so that $f_2(r)$ has at least one the positive root. Since $f_2'(r) > 0$ for all $r \geq 0$, $f_2(r)$ has exactly one positive root. Let r_2 be the positive root of (3.8). For $0 < r < r_2$ we have $f_2(r) < 0$ and when $r > r_2$ we have $f_2(r) > 0$.

When $r > r_2$, satisfies the Routh-Hurwitz criteria when $r > r_2$. Thus, the equilibrium point $E_2 = (\alpha_2, \beta_2, \gamma_2)$ is asymptotically stable when $r > r_2$.

However, when $0 < r < r_2$, $f_2(r) < 0$ which does not satisfy the Routh-Hurwitz criteria therefore the equilibrium point $E_2 = (\alpha_2, \beta_2, \gamma_2)$ is unstable. □

3.2.1 Numerical Simulations

We give numerical experiments to demonstrate the effectiveness of the proposed control scheme. Fourth-order Runge-Kutta method is used to solve the differential equations with time step 0.01. The parameters p, q and r are chosen as $p = 10$, $q = \frac{100}{7}$, and $r = 0.07$. The initial states are taken as $x = 0.65$, $y = 0$ and $z = 0$. Fig. 3.4 shows the chaotic behavior of the states x, y and z of the system (3.2) with time in xy-plane. Fig.3.5 shows chaotic the behavior of the states x, y and z of the system (3.2) with time in xz-plane. Fig. 3.6 shows the chaotic behavior of the states x, y and z of the system (3.2) with time in yz-plane.

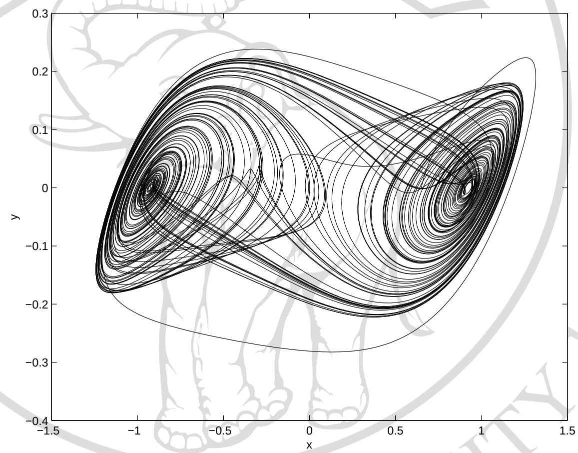

Figure 3.4: The chaotic attractor of the perturbed Chua's circuit system (3.2) in the xy-plane.

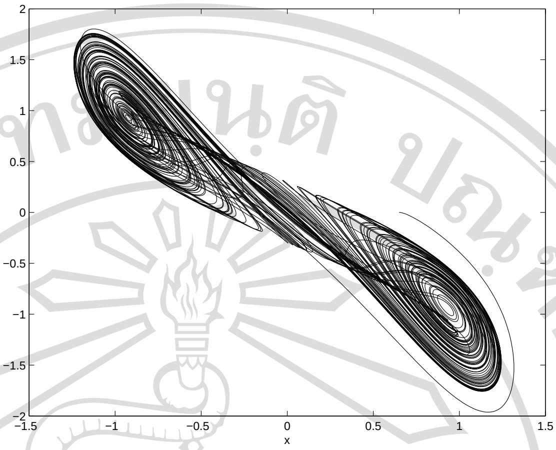

Figure 3.5: The chaotic attractor of the perturbed Chua's circuit system (3.2) in the xz-plane.

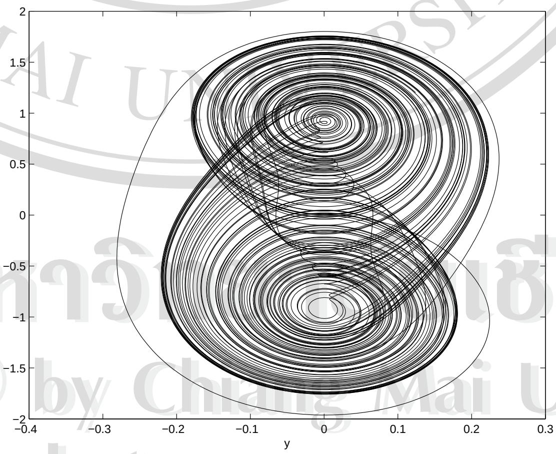

Figure 3.6: The chaotic attractor of the perturbed Chua's circuit system (3.2) in the yz-plane.

3.3 Controlling Chaos of the Perturbed Chua's Circuit System

In this section, the chaos of system (3.2) is controlled to one of the three equilibrium points of the system. Feedback control and adaptive control are applied to achieve this goal.

3.3.1 Feedback Control Method

Let us consider the controlled system of the system (3.2) which has the form

$$\begin{aligned} \dot{x} &= p\big(y - \frac{1}{2}(2x^3 - x)\big) + u_1 \\ \dot{y} &= x - y + z + u_2 \\ \dot{z} &= -qy + rx^2 + u_3 \end{aligned} \tag{3.9}$$

where u_1, u_2 and u_3 are external control inputs which will drag the chaotic trajectory (x, y, z) of the perturbed Chua's circuit system to $E = (\overline{x}, \overline{y}, \overline{z})$ which is one of the three steady states E_0, E_1 and E_2. Let the control laws take the following form

$$u_1 = -k_1(x - \overline{x}), u_2 = -k_2(y - \overline{y}), u_3 = -k_3(z - \overline{z})$$

where k_1, k_2 and k_3 are a positive feedback gain.

Stabilizing the equilibrium point $E = (\overline{x}, \overline{y}, \overline{z})$

In order to suppress chaos to $E = (\overline{x}, \overline{y}, \overline{z})$, we introduce the external control laws $u_1 = -k_1(x - \overline{x}), u_2 = -k_2(y - \overline{y}), u_3 = -k_3(z - \overline{z})$ with x, y and z as the feedback variables into system (3.9). Hence the controlled system (3.9) has the following form:

$$\begin{aligned} \dot{x} &= p\big(y - \frac{1}{2}(2x^3 - x)\big) - k_1(x - \overline{x}) \\ \dot{y} &= x - y + z - k_2(y - \overline{y}) \\ \dot{z} &= -qy + rx^2 - k_3(z - \overline{z}). \end{aligned} \tag{3.10}$$

The controlled system (3.10) has the equilibrium point $E = (\overline{x}, \overline{y}, \overline{z})$. The system (3.10) can be stabilized to the steady state $E = (\overline{x}, \overline{y}, \overline{z})$ if $k_i \geq k_i^*$, $i = 1, 2, 3$ is satisfied and the system parameters are constant and known.

Let us consider $\xi_1 = x - \overline{x}$, $\xi_2 = y - \overline{y}$, $\xi_3 = z - \overline{z}$, $\alpha = p\Big(\frac{1}{7}\big(-6\overline{x}^2 + 1\big)\Big)$, and $\beta = 2r\overline{x}$.

Theorem 3.3.1 *The equilibrium point* $E = (\overline{x}, \overline{y}, \overline{z})$ *of the system (3.10) is asymptotically stable provided that* $k_1 > k_1^* = \alpha + p + \frac{\beta^2 p}{2q}$, $k_2 > k_2^* = 0$ *and* $k_3 > k_3^* = \frac{1}{2}$.

Proof. The Jacobian matrix of the system (3.10) about the equilibrium point $E = (\overline{x}, \overline{y}, \overline{z})$ is

$$J = \begin{bmatrix} \frac{p(-6\overline{x}^2+1)}{7} - k_1 & p & 0 \\ 1 & -1 - k_2 & 1 \\ 2r\overline{x} & -q & -k_3 \end{bmatrix}. \tag{3.11}$$

The linearized system of (3.11) is given by

$$\begin{aligned} \dot{\xi_1} &= (\alpha - k_1)\xi_1 + p\xi_2 \\ \dot{\xi_2} &= \xi_1 - (1 + k_2)\xi_2 + \xi_3 \\ \dot{\xi_3} &= \beta\xi_1 - q\xi_2 - k_3\xi_3. \end{aligned} \tag{3.12}$$

We study the stability of the equilibrium point $(0, 0, 0)$ of the system (3.12) Consider the Lyapunov function $V(\xi_1, \xi_2, \xi_3)$ in the form

$$V(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\Big[\frac{p}{q}\xi_1^2 + q\xi_2^2 + \xi_3^2\Big]. \tag{3.13}$$

The time derivative of V in the neighbourhood of $(0, 0, 0)$ is

$$\begin{aligned} \dot{V} &= \frac{q}{p}\xi_1\dot{\xi_1} + q\xi_2\dot{\xi_2} + \xi_3\dot{\xi_3} \\ &= \frac{q}{p}\xi_1\big[(\alpha - k_1)\xi_1 + p\xi_2\big] + q\xi_2\big[\xi_1 - (1 + k_2)\xi_2 + \xi_3\big] \\ &\quad +\xi_3\big[\beta\xi_1 - q\xi_2 - k_3\xi_3\big] \\ &= \frac{q}{p}(\alpha - k_1)\xi_1^2 + q\xi_1\xi_2 + q\xi_1\xi_2 - q\xi_2^2 + q\xi_2\xi_3 - qk_2\xi_2^2 \\ &\quad +\beta\xi_1\xi_3 - q\xi_2\xi_3 - k_3\xi_3^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{q}{p}(\alpha - k_1)\xi_1^2 - q(\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2) + q\xi_1^2 - qk_2\xi_2^2 \\
&\quad -\frac{1}{2}(\xi_3^2 - 2\beta\xi_1\xi_2 + \beta^2\xi_1^2) + \frac{\beta^2}{2}\xi_1^2 + \frac{\xi_3^2}{2} - k_3\xi_3^2 \\
&= -\frac{q}{p}\Big(k_1 - \alpha - p - \frac{\beta^2 p}{2q}\Big)\xi_1^2 - q(\xi_1 - \xi_2)^2 - qk_2\xi_2^2 \\
&\quad -\frac{1}{2}\big(\xi_3 - \beta\xi_1\big)^2 - \big(k_3 - \frac{1}{2}\big)\xi_3^2.
\end{aligned}
$$

It is clear that $\dot{V} < 0$ if $k_1 > k_1^* = \alpha + p + \frac{\beta^2 p}{2q}$, $k_2 > k_2^* = 0$ and $k_3 > k_3^* = \frac{1}{2}$. According to Lyapunov stability theory the equilibrium point $(0, 0, 0)$ is asymptotically stable. □

Numerical Simulations

We give numerical experiments to demonstrate the effectiveness of the proposed control scheme. Fourth-order Runge-Kutta method is used to solve the differential equations with time step 0.01. The parameters p, q and r are chosen as $p = 10$, $q = \frac{100}{7}$, and $r = 0.07$ to ensure the existence of chaos in the absence of control. The initial states are taken as $x = 0.65$, $y = 0$ and $z = 0$. The equilibrium point $E_0 = (0, 0, 0)$ of the system (3.2) is stabilized for $k_1 = 11.4$, $k_2 = 0.1$ and $k_3 = 0.55$. Fig. 3.7 shows the chaos is suppressed to the equilibrium point E_0 with time. The control is active at $t = 10$. The equilibrium point $E_1 = (0.71573, 0.00251, -0.71322)$ of the system (3.2) is stabilized for $k_1 = 8.75$, $k_2 = 0.1$ and $k_3 = 0.55$. Fig. 3.8 shows the chaotic trajectory can be stabilized to the equilibrium point E_1 with time. The control is active at $t = 10$. The equilibrium point $E_2 = (-0.69858, 0.00239, 0.70098)$ of the system (3.2) is stabilized for $k_1 = 8.75$, $k_2 = 0.1$ and $k_3 = 0.55$. Fig. 3.9 shows the chaotic trajectory can be stabilized to the equilibrium point E_2 with time. The control is active at $t = 10$.

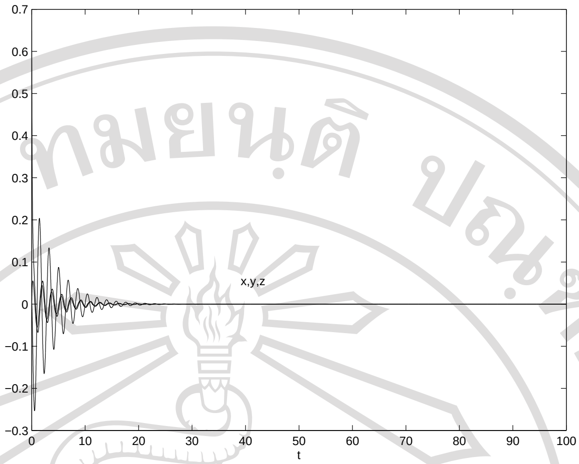

Figure 3.7: The stabilization of the equilibrium point E_0 of the system (3.2). The control law $u_1 = -11.4x, u_2 = -0.1y, u_3 = -0.55z$ is activated at $t = 10$.

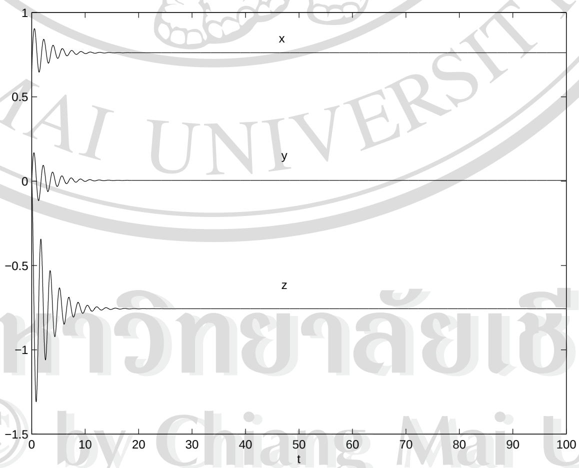

Figure 3.8: The stabilization of the equilibrium point E_1 of the system (3.2). The control law $u_1 = -8.75(x - 0.71573), u_2 = -0.1(y - 0.00251), u_3 = -0.55(z + 0.71322)$ is activated at $t = 10$.

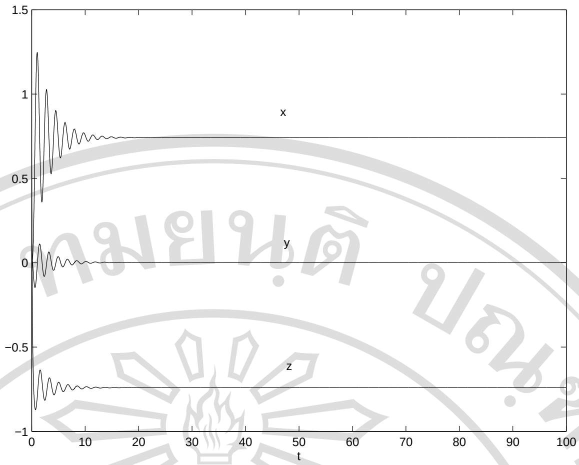

Figure 3.9: The stabilization of the equilibrium point E_2 of the system (3.2). The control law $u_1 = -8.75(x + 0.69858), u_2 = -0.1(y - 0.00239), u_3 = -0.55(z + 0.70098)$ is activated at $t = 10$.

3.3.2 Adaptive Control with two Controllers

In this case the control law is

$$u_1 = -g(x - \overline{x}), u_2 = 0, u_3 = -\big(k(z - \overline{z}) + r(x - \overline{x})^2 + 2r\overline{x}(x - \overline{x})\big). \qquad (3.14)$$

where g and k are updated according to the following adaptive algorithm :

$$\begin{aligned} \dot{g} &= \mu(x - \overline{x})^2 \\ \dot{k} &= \rho(z - \overline{z})^2 \end{aligned} \qquad (3.15)$$

where μ, ρ are adaption gains. Then the controlled system (3.9) has the following form:

$$\begin{aligned} \dot{x} &= p\Big(y - \frac{1}{7}(2x^3 - x)\Big) - g(x - \overline{x}) \\ \dot{y} &= x - y + z \\ \dot{z} &= -qy + rx^2 - \big(k(z - \overline{z}) + r(x - \overline{x})^2 + 2r\overline{x}(x - \overline{x})\big) \\ \dot{g} &= \mu(x - \overline{x})^2 \\ \dot{k} &= \rho(z - \overline{z})^2. \end{aligned} \qquad (3.16)$$

Let us consider $\alpha = p\Big(\frac{1}{7}\big(-6\overline{x}^2 + 1\big)\Big)$.

Theorem 3.3.2 *Assume that g^* and k^* are real satisfy the inequality $g^* \geq p + \alpha$ and $k^* \geq 0$. The equilibrium point $E = (\overline{x}, \overline{y}, \overline{z})$ of the system (3.16) is asymptotically stable.*

Proof. Let us consider the Lyapunov function

$$\begin{aligned} V &= \frac{1}{2}\Big[\frac{q}{p}(x-\overline{x})^2 + q(y-\overline{y})^2 + (z-\overline{z})^2 + \frac{q}{\mu p}(g-g^*)^2 \\ &\quad + \frac{1}{\rho}(k-k^*)^2\Big]. \end{aligned} \tag{3.17}$$

The time derivative of V in the neighbourhood of the equilibrium point $E = (\overline{x}, \overline{y}, \overline{z})$ of the system (3.16) is

$$\begin{aligned} \dot{V} &= \frac{q}{p}(x-\overline{x})\dot{x} + q(y-\overline{y})\dot{y} + (z-\overline{z})\dot{z} + \frac{q}{\mu p}(g-g^*)\dot{g} \\ &\quad + \frac{1}{\rho}(k-k^*)\dot{k}. \end{aligned} \tag{3.18}$$

Substituting (3.16) in (3.18), we obtain

$$\begin{aligned} \dot{V} &= \frac{q}{p}(x-\overline{x})\Big[p\big(y - \frac{1}{7}(2x^3 - x)\big) - g(x-\overline{x})\Big] + q(y-\overline{y})[x - y + z] \\ &\quad + (z-\overline{z})\big[-qy + rx^2 - k(z-\overline{z}) - r(x-\overline{x})^2 - 2r\overline{x}(x-\overline{x})\big] \\ &\quad + \frac{q}{p}(g-g^*)(x-\overline{x})^2 + (k-k^*)(z-\overline{z})^2. \end{aligned}$$

Let $\eta_1 = x - \overline{x}, \eta_2 = y - \overline{y}$ and $\eta_3 = z - \overline{z}$. We have

$$\begin{aligned} \dot{V} &= q\eta_1\Big[\eta_2 + \overline{y} - \frac{1}{7}\big(2(\eta_1+\overline{x})^3 - \eta_1 - \overline{x}\big)\Big] - \frac{qg}{p}\eta_1^2 + q\eta_2[\eta_1 + \overline{x} - \eta_2 - \overline{y} \\ &\quad + \eta_3 + \overline{z}] + \eta_3[-q(\eta_2+\overline{y}) + r(\eta_1+\overline{x})^2 - r\eta_1^2 - 2r\overline{x}\eta_1 - k\eta_3] \\ &\quad + \frac{q}{p}(g-g^*)\eta_1^2 + (k-k^*)\eta_3^2 \\ &= q\eta_1\Big[\eta_2 + \overline{y} - \frac{1}{7}\big(2(\eta_1^3 + 3\eta_1^2\overline{x} + 3\eta_1\overline{x}^2 + \overline{x}^3) - \eta_1 - \overline{x}\big)\Big] - \frac{qg}{p}\eta_1^2 + q\eta_1\eta_2 \\ &\quad + q\overline{x}\eta_2 - q\eta_2^2 - q\overline{y}\eta_2 + q\eta_2\eta_3 + q\overline{z}\eta_2 - q\eta_2\eta_3 - q\overline{y}\eta_3 + r\eta_1^2\eta_3 + 2r\overline{x}\eta_1\eta_3 \\ &\quad + r\overline{x}^2\eta_3 - r\eta_1^2\eta_3 - 2r\overline{x}\eta_1\eta_3 - k\eta_3^2 + \frac{q}{p}(g-g^*)\eta_1^2 + (k-k^*)\eta_3^2 \\ &= q\eta_1\Big[\eta_2 - \frac{1}{7}(2\eta_1^3 - \eta_1) - \frac{1}{7}(6\overline{x}^2\eta_1 + 6\overline{x}\eta_1^2)\Big] + q\eta_1\big(\overline{y} - \frac{1}{7}(2\overline{x}^3 - \overline{x})\big) \\ &\quad - \frac{qg}{p}\eta_1^2 + q\eta_2(\overline{x} - \overline{y} + \overline{z}) + q\eta_1\eta_2 - q\eta_2^2 + \eta_3(-q\overline{y} + r\overline{x}^2) - k\eta_3^2 + \frac{qg}{p}\eta_1^2 \\ &\quad - \frac{qg^*}{p}\eta_1^2 + k\eta_3^2 - k^*\eta_3^2. \end{aligned}$$

Since $(\overline{x},\overline{y},\overline{z})$ is an equilibrium point of the uncontrolled system (3.2), $\dot{V}$ becomes

$$\begin{aligned}
\dot{V} &= q\eta_1\eta_2 - \frac{2}{7}q\eta_1^4 + \frac{q}{7}\eta_1^2 - \frac{6}{7}q\overline{x}^2\eta_1^2 - \frac{6}{7}q\overline{x}\eta_1^3 + q\eta_1\eta_2 - q\eta_2^2 - \frac{qg^*}{p}\eta_1^2 - k^*\eta_3^2 \\
&= 2q\eta_1\eta_2 - \frac{2}{7}q\eta_1^4 - q\eta_1^2\Big(\frac{6}{7}\overline{x}^2 - \frac{1}{7} + \frac{g^*}{p}\Big) - \frac{6}{7}q\overline{x}\eta_1^3 - q\eta_2^2 - k^*\eta_3^2 \\
&= -q\eta_2^2 + 2q\eta_1\eta_2 - q\eta_1^2\Big(\frac{g^*}{p} + \frac{(6\overline{x}^2-1)}{7}\Big) - \frac{2}{7}q\eta_1^4 - \frac{6}{7}q\overline{x}\eta_1^3 - k^*\eta_3^2 \\
&= -q(\eta_2^2 - 2q\eta_1\eta_2 + \eta_1^2) + q\eta_1^2 - q\eta_1^2\Big(\frac{g^*}{p} + \frac{(6\overline{x}^2-1)}{7}\Big) - \frac{2}{7}q\eta_1^4 - \frac{6}{7}q\overline{x}\eta_1^3 - k^*\eta_3^2 \\
&= -q(\eta_2-\eta_1)^2 - q\eta_1^2\Big(-1 + \frac{g^*}{p} + \frac{(6\overline{x}^2-1)}{7}\Big) - \frac{2}{7}q\eta_1^4 - \frac{6}{7}q\overline{x}\eta_1^3 - k^*\eta_3^2 \\
&= -q(\eta_2-\eta_1)^2 - q\eta_1^2\Big(\frac{g^*}{p} - \big(1 + \frac{(-6\overline{x}^2+1)}{7}\big)\Big) - \frac{2}{7}q\eta_1^4 - \frac{6}{7}q\overline{x}\eta_1^3 - k^*\eta_3^2 \\
&= -q(\eta_2-\eta_1)^2 - \frac{q}{p}\eta_1^2\Big(g^* - p - p\big(\frac{1}{7}(-6\overline{x}^2+1)\big)\Big) - \frac{2}{7}q\eta_1^4 - \frac{6}{7}q\overline{x}\eta_1^3 - k^*\eta_3^2 \\
&= -q(\eta_2-\eta_1)^2 - \frac{q}{p}(g^* - p - \alpha)\eta_1^2 - \frac{2}{7}q\eta_1^4 - \frac{6}{7}q\overline{x}\eta_1^3 - k^*\eta_3^2.
\end{aligned}$$

It is clear that for positive parameters p,q,r,μ and ρ, if we choose $g^* \geq p+\alpha$, $k^* \geq 0$ and $|\eta_1|$ is sufficiently small then $\dot{V}$ is negative semidefinite and Lyapunov function V in (3.17) is positive definite implies that the equilibrium points of the system (3.16) are stable, i.e. $\eta_1,\eta_2,\eta_3 \in L_\infty$. We integrate both side of $\dot{V}$ with respect to time which yields

$$\begin{aligned}
\int_0^\infty \frac{dV(\tau)}{d\tau}d\tau &= -\int_0^\infty q\big(\eta_2(\tau)-\eta_1(\tau)\big)^2 d\tau - \int_0^\infty \frac{q}{p}(g^*-p-\alpha)\eta_1^2(\tau)d\tau \\
&\quad - \int_0^\infty \frac{2}{7}q\eta_1^4(\tau)d\tau - \int_0^\infty \frac{6}{7}q\overline{x}\eta_1^3(\tau)d\tau - \int_0^\infty k^*\eta_3^2(\tau)d\tau.
\end{aligned}$$

Thus,

$$\begin{aligned}
V(\infty) - V(0) &= -\int_0^\infty q\big(\eta_2(\tau)-\eta_1(\tau)\big)^2 d\tau - \int_0^\infty \frac{q}{p}(g^*-p-\alpha)\eta_1^2(\tau)d\tau \\
&\quad - \int_0^\infty \frac{2}{7}q\eta_1^4(\tau)d\tau - \int_0^\infty \frac{6}{7}q\overline{x}\eta_1^3(\tau)d\tau - \int_0^\infty k^*\eta_3^2(\tau)d\tau \\
V(0) - V(\infty) &= \int_0^\infty q\big(\eta_2(\tau)-\eta_1(\tau)\big)^2 d\tau + \int_0^\infty \frac{q}{p}(g^*-p-\alpha)\eta_1^2(\tau)d\tau \\
&\quad + \int_0^\infty \frac{2}{7}q\eta_1^4(\tau)d\tau + \int_0^\infty \frac{6}{7}q\overline{x}\eta_1^3(\tau)d\tau + \int_0^\infty k^*\eta_3^2(\tau)d\tau.
\end{aligned}$$

Since $\dot{V}$ is negative or zero, V is either decreasing or constant which gives $V(0) \geq V(\infty) \geq 0$. Then we obtain

$$\int_0^\infty q\big(\eta_2(\tau)-\eta_1(\tau)\big)^2 d\tau + \int_0^\infty \frac{q}{p}(g^* - p - \alpha)\eta_1^2(\tau)d\tau + \int_0^\infty \frac{2}{7}q\eta_1^4(\tau)d\tau + \int_0^\infty \frac{6}{7}q\overline{x}\eta_1^3(\tau)d\tau + \int_0^\infty k^*\eta_3^2(\tau)d\tau \leq V(0) < \infty.$$

It follows that

$$\sqrt{\int_0^\infty q\big(\eta_2(\tau)-\eta_1(\tau)\big)^2 d\tau} < \infty,$$

$$\sqrt{\int_0^\infty \frac{q}{p}(g^* - p - \alpha)\eta_1^2(\tau)d\tau} < \infty,$$

$$\sqrt{\int_0^\infty k^*\eta_3^2(\tau)d\tau} < \infty$$

which indicates, according to Definition 2.1.1, that $\eta_1, \eta_2, \eta_3 \in L_2$. We can use (3.16) to show that $\dot{\eta_1}, \dot{\eta_2}, \dot{\eta_3} \in L_\infty$. By Proposition 2.1.2 we obtain $\eta_1, \eta_2, \eta_3 \longrightarrow 0$ as $t \longrightarrow \infty$, i.e. $x \longrightarrow \overline{x}$, $y \longrightarrow \overline{y}$, $z \longrightarrow \overline{z}$. □

Numerical Simulations

Numerical experiments are carried out to investigate controlled systems by using fourth-order Runge-Kutta method with time step 0.01. The parameters p, q, r, μ and ρ are chosen as $p = 10$, $q = \frac{100}{7}$, $r = 0.07$, $\mu = 1$ and $\rho = 1$ to ensure the existence of chaos in the absence of control. The initial states are taken as $x = 0.65$, $y = 0$ and $z = 0$. The initial value of parameters g and k are set to be 0 in this simulation. Fig. 3.10-3.12 show time response for the states x, y and z of the controlled system (3.9) after applying adaptive control. The changing parameters g and k are depicted in Fig. 3.13-3.14.

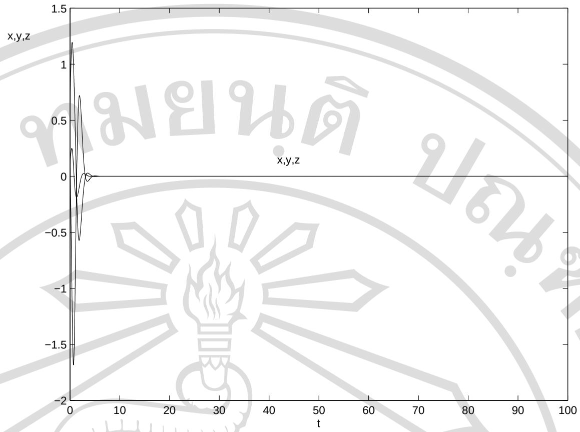

Figure 3.10: The time response of the states x, y and z of the controlled system (3.16), where $\overline{x} = 0, \overline{y} = 0$ and $\overline{z} = 0$.

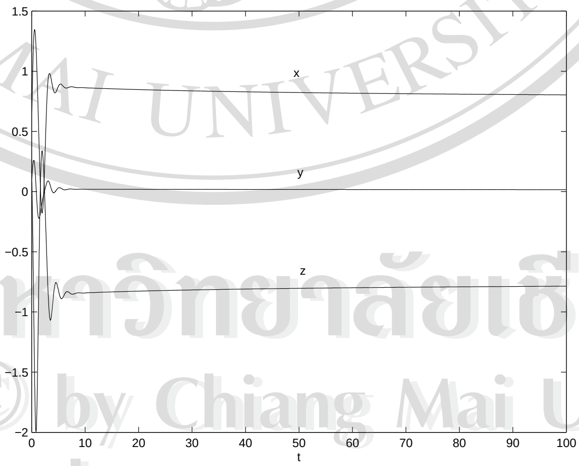

Figure 3.11: The time response of the states x, y and z of the controlled system (3.16), where $\overline{x} = 0.71573, \overline{y} = 0.00251$ and $\overline{z} = -0.71322$.

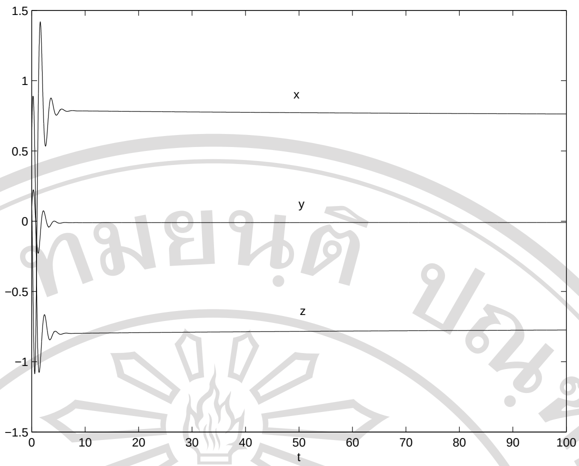

Figure 3.12: The time response of the states x, y and z of the controlled system (3.16), where $\overline{x} = -0.69858, \overline{y} = 0.00239$ and $\overline{z} = 0.70098$.

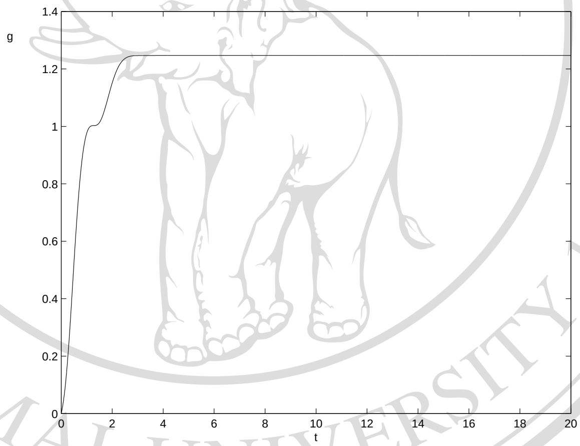

Figure 3.13: Changing of parameter g of the adaptive control.

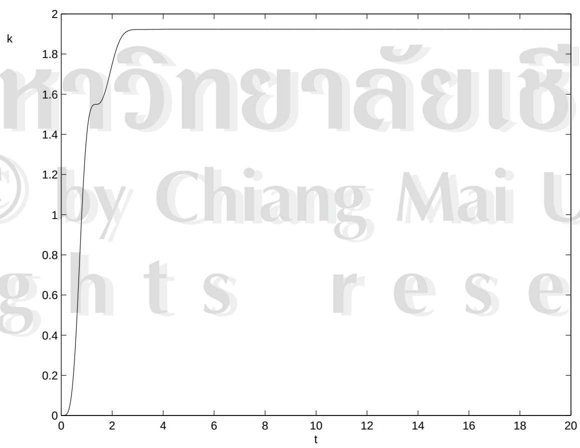

Figure 3.14: Changing of parameter k of the adaptive control.

3.4 Synchronization of the Perturbed Chua's Circuit System

To begin with, the definition of chaos synchronization used in this thesis is given below.

For two nonlinear chaotic systems:

$$\dot{x} = f(t,x) \tag{3.19}$$

$$\dot{y} = g(t,y) + u(t,x,y) \tag{3.20}$$

where $x, y \in \mathbb{R}^n$, $f, g \in C^r[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $u \in C^r[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $r \geq 1$, $\mathbb{R}^+$ is the set of non-negative real numbers. Assume that (3.19) is the drive system, and (3.20) is the response system, $u(t,x,y)$ is the control vector. Response system and drive system are said to be *synchronic* if $\forall x(t_0), y(t_0) \in \mathbb{R}^n$,

$$\lim_{t\to\infty} \parallel x(t) - y(t) \parallel = 0.$$

3.4.1 Synchronization of the Perturbed Chua's Circuit System Using Active Control

In this section, we assume that we have two perturbed Chua's circuit system and that the drive system (with the subscript 1) drives the response system (with subscript 2). The systems are

$$\begin{aligned} \dot{x_1} &= p\big(y_1 - \frac{1}{7}(2x_1^3 - x_1)\big) \\ \dot{y_1} &= x_1 - y_1 + z_1 \\ \dot{z_1} &= -qy_1 + rx_1^2 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \dot{x_2} &= p\big(y_2 - \frac{1}{7}(2x_2^3 - x_2)\big) + u_1(t) \\ \dot{y_2} &= x_2 - y_2 + z_2 + u_2(t) \\ \dot{z_2} &= -qy_2 + rx_2^2 + u_3(t). \end{aligned} \tag{3.22}$$

We have introduced three control functions $u_1(t), u_2(t), u_3(t)$ in (3.22). Our goal is to determine the control functions $u_1(t), u_2(t)$ and $u_3(t)$. In order to estimate the control functions, we subtract (3.21) from (3.22). We define the error system as the difference between system (3.21) and the controlled system (3.22). Let us define the state errors between the response system (3.22) that is to be controlled and the controlling system (3.21) as

$$\begin{aligned} e_x &= x_2 - x_1 \\ e_y &= y_2 - y_1 \\ e_z &= z_2 - z_1. \end{aligned} \tag{3.23}$$

Subtracting (3.21) from (3.22) and using the notations in (3.23) yields

$$\begin{aligned} \dot{e_x} &= p\Big(e_y - \frac{1}{7}\big((2x_2^3 - 2x_1^3) - e_x\big)\Big) + u_1(t) \\ \dot{e_y} &= e_x - e_y + e_z + u_2(t) \\ \dot{e_z} &= -qe_y + re_x(x_2 + x_1) + u_3(t). \end{aligned} \tag{3.24}$$

We define active control functions $u_1(t), u_2(t)$ and $u_3(t)$ as follows

$$\begin{aligned} u_1(t) &= V_1(t) + \frac{2p}{7}\Big(x_2^3 - x_1^3\Big) \\ u_2(t) &= V_2(t) - e_x - e_z \\ u_3(t) &= V_3(t) + qe_y - e_z - re_x(x_2 + x_1). \end{aligned} \tag{3.25}$$

Hence, the error system (3.24) becomes

$$\begin{aligned} \dot{e_x} &= \frac{p}{7}e_x + pe_y + V_1(t) \\ \dot{e_y} &= -e_y + V_2(t) \\ \dot{e_z} &= -e_z + V_3(t). \end{aligned} \tag{3.26}$$

The error system (3.26) is a linear system with a control input $V_1(t), V_2(t)$ and $V_3(t)$ as function of the error states e_x, e_y and e_z. There are many possible choices for the controls $V_1(t), V_2(t)$ and $V_3(t)$. We choose

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = A \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix}$$

where A is a 3×3 constant matrix. Let the matrix A is chosen in the following form

$$A = \begin{bmatrix} \frac{-p-7}{7} & -p & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

With this particular choice of A, (3.26) has the eigenvalues $-1, -2$ and -2. This choice will lead to the error states e_x, e_y and e_z converge to zero as time t tends to infinity and this implies the synchronization of the Perturbed Chua's Circuit System. □

Numerical Simulations

Fourth-order Runge-Kutta integration method is used to solve two system of differential equations (3.21) and (3.22) with time step size 0.01. We select the parameter of (3.21) as follow: $p = 10$, $q = \frac{100}{7}$ and $r = 0.07$ to ensure the chaotic behavior of perturbed Chua's circuit system. The initial values of the drive system are $x_1(0) = 0.65$, $y_1(0) = 0$ and $z_1(0) = 0$ and the initial values of the response system are $x_2(0) = 0.2$, $y_2(0) = 0.1$ and $z_2(0) = 0.1$. Then the initial values of the error system are $e_x(0) = -0.45$, $e_y(0) = 0.1$ and $e_z(0) = 0.1$.

The results of the simulation of the two identical perturbed Chua's circuit systems without active control are shown in Fig. 3.15 (displays x_1 and x_2), Fig. 3.16 (displays y_1 and y_2), Fig. 3.17 (displays z_1 and z_2). Fig. 3.18-3.20 show the synchronization is occurred after applying active control at $t = 10$. Fig.3.21 shows the state errors (e_x, e_y, e_z) of perturbed Chua's circuit system of equations with the active control activated.

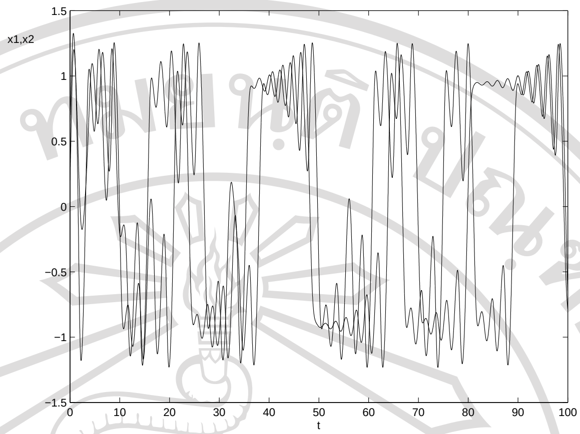

Figure 3.15: The states x_1, x_2 of the coupled perturbed Chua's circuit system of equations with the active control deactivated.

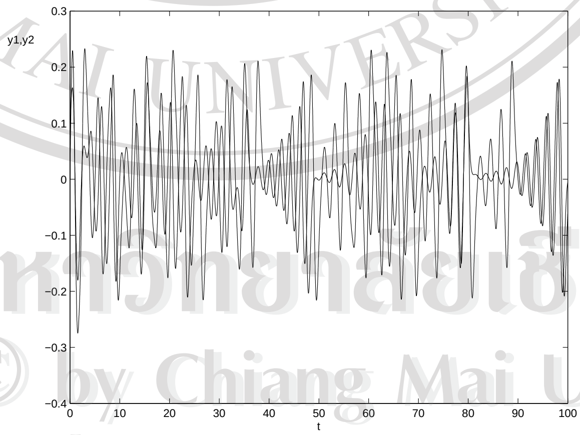

Figure 3.16: The states y_1, y_2 of the coupled perturbed Chua's circuit system of equations with the active control deactivated.

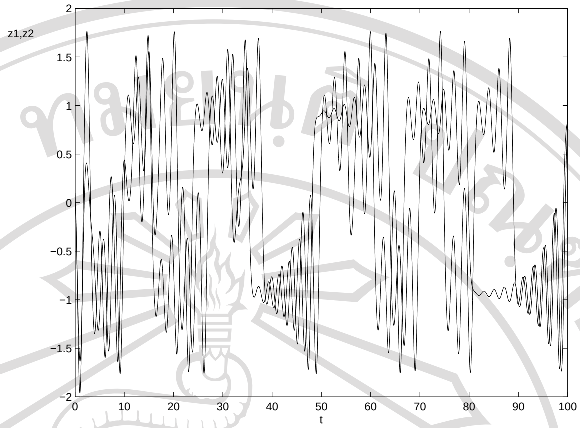

Figure 3.17: The states z_1, z_2 of the coupled perturbed Chua's circuit system of equations with the active control deactivated.

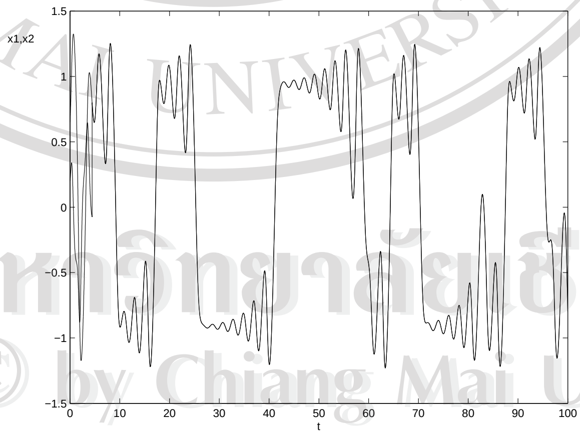

Figure 3.18: The states x_1, x_2 of the coupled perturbed Chua's circuit system of equations with the active control activated.

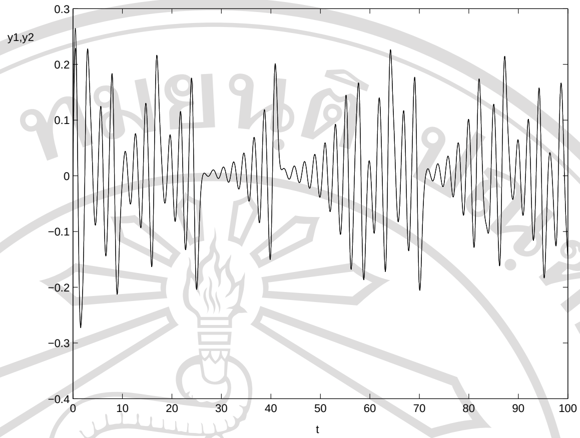

Figure 3.19: The states y_1, y_2 of the coupled perturbed Chua's circuit system of equations with the active control activated.

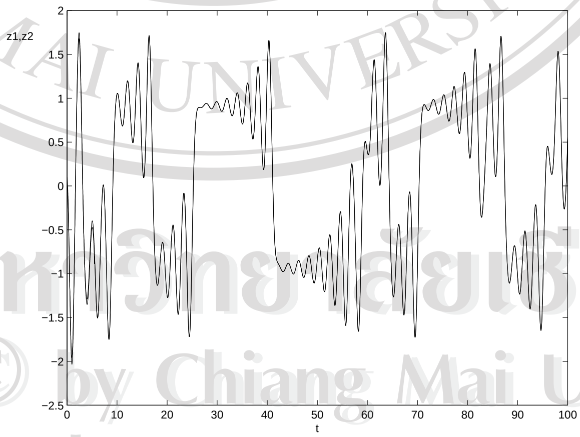

Figure 3.20: The states z_1, z_2 of the coupled perturbed Chua's circuit system of equations with the active control activated.

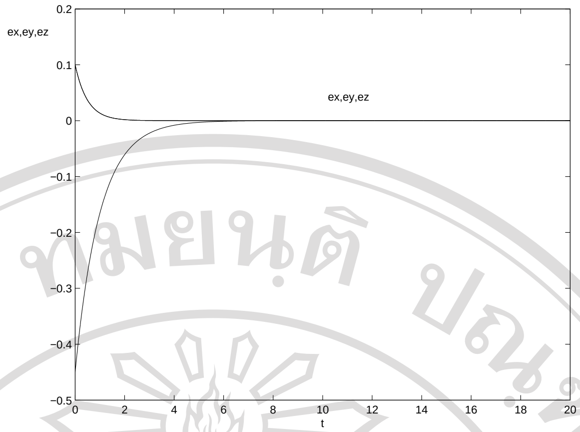

Figure 3.21: The states error (e_x, e_y, e_z) of perturbed Chua's circuit system of equations with the active control activated.

3.4.2 Adaptive Synchronization of the Perturbed Chua's circuit System

This section considers adaptive synchronization of perturbed Chua's circuit system. This approach can synchronize the chaotic systems when the parameters of the drive system are fully unknown and different with those of the response system. The synchronization problem of perturbed Chua's circuit systems with fully unknown parameters will be studied in which the adaptive controller will be introduced.

We assume that we have two perturbed Chua's circuit systems and that the drive system (with the subscript 1) is to control the response (with subscript 2). The drive and response systems are defined as follows: The drive system is

$$
\begin{aligned}
\dot{x_1} &= p\big(y_1 - \frac{1}{7}(2x_1^3 - x_1)\big) \\
\dot{y_1} &= x_1 - y_1 + z_1 \\
\dot{z_1} &= -qy_1 + rx_1^2
\end{aligned}
\tag{3.27}
$$

where the parameters p, q and r are unknown or uncertain, and the response system is

$$\begin{aligned} \dot{x_2} &= p_1\big(y_2 - \frac{1}{7}(2x_2^3 - x_2)\big) - u_1 \\ \dot{y_2} &= x_2 - y_2 + z_2 - u_2 \\ \dot{z_2} &= -q_1 y_2 + r_1 x_2^2 - u_3 \end{aligned} \tag{3.28}$$

where p_1, q_1 and r_1 are parameters of the response system which need to be estimated, and $u = [u_1, u_2, u_3]^T$ is the controller. We choose

$$\begin{aligned} u_1 &= k_1 e_x - \frac{2p}{7}\Big(x_2^3 - x_1^3\Big) \\ u_2 &= k_2 e_y \\ u_3 &= k_3 e_z + r(x_2^2 - x_1^2) \end{aligned} \tag{3.29}$$

where e_x, e_y and e_z are the error states which are defined as follows:

$$\begin{aligned} e_x &= x_2 - x_1 \\ e_y &= y_2 - y_1 \\ e_z &= z_2 - z_1 \end{aligned} \tag{3.30}$$

The parameters p_1, q_1, and r_1 satisfy the system

$$\begin{aligned} \dot{p_1} &= f_{p_1} = -\gamma\Big(\frac{2}{7}x_2^3 e_x - \frac{1}{7}x_2 e_x - y_2 e_x\Big) \\ \dot{q_1} &= f_{q_1} = \theta y_2 e_z \\ \dot{r_1} &= f_{r_1} = -\delta x_2^2 e_z \end{aligned} \tag{3.31}$$

where $k_1, k_2, k_3 \geq 0$ and γ, θ, δ are positive real constants.

Theorem 3.4.1 *If k_1, k_2 and k_3 are chosen to satisfy the following matrix inequality,*

$$P = \begin{bmatrix} k_1 - \frac{p}{7} & -\frac{1}{2}\Big(p+1\Big) & 0 \\ -\frac{1}{2}\Big(p+1\Big) & k_2 + 1 & -\frac{1}{2}\Big(1-q\Big) \\ 0 & -\frac{1}{2}\Big(1-q\Big) & k_3 \end{bmatrix} > 0 \tag{3.32}$$

or the inequalities,

$$
\begin{array}{rrcl}
(i) & A & = & k_1 - \frac{p}{7} > 0 \\
(ii) & B & = & A(k_2+1) - \frac{1}{4}\Big(p+1\Big)^2 > 0 \\
(iii) & C & = & A(k_2+1)k_3 - A\frac{1}{4}\Big(p+1\Big)^2 - \frac{k_3}{4}\Big(p+1\Big)^2 > 0
\end{array}
\tag{3.33}
$$

then the two perturbed Chua's circuit systems (3.27) and (3.28) can be synchronized under the adaptive control of (3.29) and (3.31).

Proof. It is easy to see from (3.27) and (3.28) that the error system can be obtained as follows:

$$
\begin{array}{rcl}
\dot{e_x} & = & p_1\big(y_2 - \frac{1}{7}(2x_2^3 - x_2)\big) - p\big(y_1 - \frac{1}{7}(2x_1^3 - x_1)\big) - u_1 \\
\dot{e_y} & = & x_2 - y_2 + z_2 - x_1 + y_1 - z_1 - u_2 \\
\dot{e_z} & = & -q_1y_2 + r_1x_2^2 + qy_1 - rx_1^2 - u_3.
\end{array}
\tag{3.34}
$$

Let $e_p = p_1 - p, e_q = q_1 - q$ and $e_r = r_1 - r$. Choose the Lyapunov function as follows:

$$
V(t) \;=\; \frac{1}{2}\Big(e_x^2 + e_y^2 + e_z^2 + \frac{1}{\gamma}e_p^2 + \frac{1}{\theta}e_q^2 + \frac{1}{\delta}e_r^2\Big). \tag{3.35}
$$

then the differentiation of V along trajectories of (3.35) is

$$
\begin{array}{rcl}
\dot{V} & = & e_x\dot{e_x} + e_y\dot{e_y} + e_z\dot{e_z} + \frac{1}{\gamma}e_p\dot{e_p} + \frac{1}{\theta}e_q\dot{e_q} + \frac{1}{\delta}e_r\dot{e_r} \\
& = & e_x[p_1\big(y_2 - \frac{1}{7}(2x_2^3 - x_2)\big) - p\big(y_1 - \frac{1}{7}(2x_1^3 - x_1)\big) - u_1] \\
& & +e_y[x_2 - y_2 + z_2 - x_1 + y_1 - z_1 - u_2] + e_z[-q_1y_2 + r_1x_2^2 \\
& & +qy_1 - rx_1^2 - u_3] + \frac{1}{\gamma}e_pf_p + \frac{1}{\theta}e_qf_q + \frac{1}{\delta}e_rf_r \\
& = & e_x[p_1y_2 - py_1 + py_2 - py_2] - e_x\Big[\frac{2p_1x_2^3}{7} - \frac{2px_1^3}{7} + \frac{2px_2^3}{7} - \frac{2px_2^3}{7}\Big] \\
& & +e_x\Big[\frac{p_1x_2}{7} - \frac{px_1}{7} + \frac{px_2}{7} - \frac{px_2}{7}\Big] + e_y[(x_2 - x_1) - (y_2 - y_1) + (z_2 - z_1)] \\
& & +e_z[-q_1y_2 + qy_1 + qy_2 - qy_2] + e_z[r_1x_2^2 - rx_1^2 + rx_2^2 - rx_2^2] \\
& & -e_xu_1 - e_yu_2 - e_zu_3 + \frac{1}{\gamma}e_pf_p + \frac{1}{\theta}e_qf_q + \frac{1}{\delta}e_rf_r
\end{array}
$$

$$
\begin{aligned}
\dot{V} &= e_x[e_p y_2 + p e_y] - e_x\Big[\frac{2e_p x_2^3}{7} + \frac{2p}{7}(x_2^3 - x_1^3)\Big] + e_x\Big[\frac{e_p x_2}{7} + \frac{p e_x}{7}\Big] \\
&\quad e_y[e_x - e_y + e_z] - e_z[e_q y_2 + q e_y] + e_z[e_r x_2^2 + r(x_2^2 - x_1^2)] \\
&\quad -e_x u_1 - e_y u_2 - e_z u_3 + \frac{1}{\gamma} e_p f_p + \frac{1}{\theta} e_q f_q + \frac{1}{\delta} e_r f_r \\
&= y_2 e_p e_x + p e_x e_y - \frac{2x_2^3 e_p e_x}{7} - \frac{2p}{7}(x_2^3 - x_1^3)e_x + \frac{x_2 e_p e_x}{7} + \frac{p e_x^2}{7} \\
&\quad e_x e_y - e_y^2 + e_y e_z - y_2 e_q e_z - q e_y e_z + x_2^2 e_r e_z + r(x_2^2 - x_1^2)e_z \\
&\quad -k_1 e_x^2 + \frac{2p}{7}(x_2^3 - x_1^3)e_x - k_2 e_y^2 - k_3 e_z^2 - r(x_2^2 - x_1^2)e_z \\
&\quad + \frac{1}{\gamma} e_p f_p + \frac{1}{\theta} e_q f_q + \frac{1}{\delta} e_r f_r \\
&= -\Big(k_1 - \frac{p}{7}\Big)e_x^2 - (k_2+1)e_y^2 - k_3 e_z^2 + (p+1)e_x e_y + (1-q)e_y e_z \\
&\quad + e_p\Big[\frac{1}{\gamma} f_p + y_2 e_x + \frac{x_2 e_x}{7} - \frac{2x_2^3 e_x}{7}\Big] + e_q\Big[\frac{1}{\theta} f_q - y_2 e_z\Big] + e_r\Big[\frac{1}{\delta} f_r + x_2^2 e_z\Big] \\
&\le -\Big(k_1 - \frac{p}{7}\Big)e_x^2 - (k_2+1)e_y^2 - k_3 e_z^2 + (p+1)|e_x e_y| + (1-q)|e_y e_z| \\
&= -e^T P e
\end{aligned}
$$

where $e = [|e_x| \ |e_y| \ |e_z|]^T$, P is as in (3.32). Thus, the differentiation of $V(t)$ is negative semidefinite, which implies that the origin of error system (3.34) is stable, i.e. $e_x(t), e_y(t), e_z(t) \in L_\infty$. By the same argument in the proof of Theorem 3.3.2, we obtain the errors system (3.34) tend to zero as t tends to infinity. Therefore, the response system (3.28) is synchronizing with the drive system (3.27) under the controller (3.29) and a parameter estimation update law (3.31), provided that the condition (3.33) are satisfied. □

Numerical Simulations

The numerical simulations are carried out using the fourth-order Runge-Kutta method. The initial states are $x_1(0) = 0.65$, $y_1(0) = 0$ and $z_1(0) = 0$ for the drive system and $x_2(0) = 0.2$, $y_2(0) = 0.1$ and $z_2(0) = 0.1$ for the response system. The parameters of the drive system are $p = 10$, $q = \frac{100}{7}$ and $r = 0.07$. The control parameters are chosen as follows $k_1 = 3$, $k_2 = 32$, $k_3 = 13$ which satisfy (3.33) and $\gamma = \theta = \delta = 1$. The initial values of the parameters p_1, q_1 and r_1 are all chosen to be 0 the response system synchronizes with the drive system is shown in Fig. (3.22). The changing parameters of p_1, q_1 and r_1 are shown in Fig. (3.23)-(3.25).

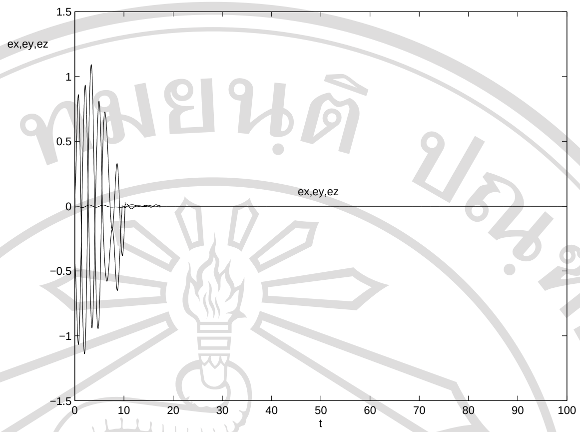

Figure 3.22: Synchronization error (e_x,e_y, e_z) states for system (3.27) and (3.28) with time t.

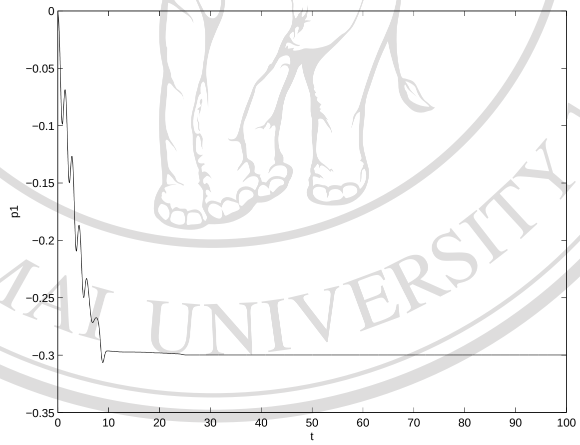

Figure 3.23: Changing parameter p_1 of system (3.28) with time t.

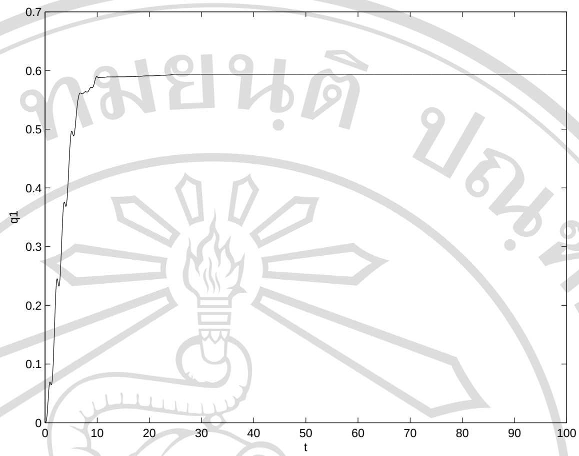

Figure 3.24: Changing parameter q_1 of system (3.28) with time t.

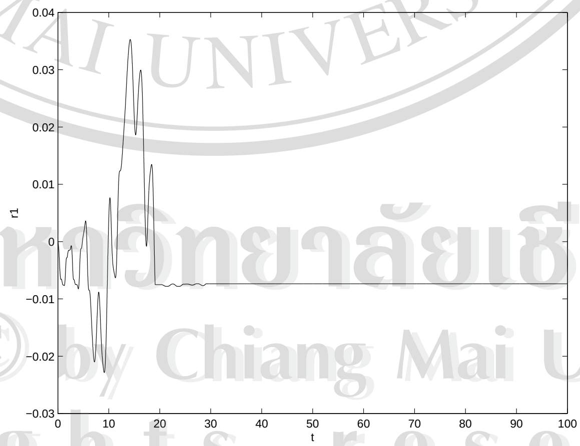

Figure 3.25: Changing parameter r_1 of system (3.28) with time t.