CHAPTER 2

PRELIMINARIES

In this chapter, we give some notations and definitions that will be used in the later chapters.

2.1 Stability

2.1.1 Definitions

Consider the system described by

$$\dot{x} = f(x, t) \tag{2.1}$$

where $x \in \mathbb{R}^n$, $\dot{x} = \left[\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right]$ and f is a vector having components $f_i(x_1, ..., x_n, t)$, $i = 1, 2, ..., n$. We shall assume that the f_i are continuous and satisfy standard conditions, such as having continuous first partial derivatives so that the solution of (2.1) exists and is unique for given initial conditions. If f_i do not depend explicitly on t, (2.1) is called autonomous. (otherwise, nonautonomous). If $f(c, t) = 0$ for all t, where c is some constant vector, then it follow at once from (2.1) that if $x(t_0) = c$ then $x(t) = c$, for all $t \geq t_0$. Thus solutions starting at c remain there, and c is said to be an *equilibrium* or *critical point*. Clearly, by introducing new variables $\acute{x}_i = x_i - c_i$ we can arrange for the equilibrium point to be transferred to the origin; we shall assume that this has been done for any equilibrium point under consideration (there may well be several for a given system (2.1)) so that we then have $f(0, t) = 0$, $t \geq t_0$.

An equilibrium state $x = 0$ is said to be

1. **Stable** if for any positive scalar ε there exists a positive scalar δ such that $\|x(t_0)\|_e < \delta$ implies $\|x(t)\|_e < \varepsilon$, $t \geq t_0$, where $\|.\|_e$ is a standard Eucledian norm.

2. **Asymptotically stable** if it is stable and if in addition $x(t) \to 0$ as $t \to \infty$.

3. **Unstable** if it is not stable; that is, there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exist an $x(t_0)$ with $\|x(t_0)\|_e < \delta$ so that $\|x(t_1)\|_e \geq \varepsilon$ for some $t_1 > t_0$. If this holds for every $x(t_0)$ in $\|x(t_0)\|_e < \delta$ the equilibrium is completely unstable.

2.1.2 Algebraic Criteria for Linear Systems

Before studying nonlinear systems we return to the general continuous time linear system.

$$\dot{x} = Ax, \tag{2.2}$$

where A is a constant $n \times n$ matrix, and (2.2) may represent the closed or open loop system. Provided det $A \neq 0$, the only equilibrium point of (2.2) is the origin, so it is meaningful to refer to the stability of the system (2.2). The two basic results on which the development of linear system stability theory relies are now given.

Theorem 2.1.1 *The system (2.2) is asymptotically stable if and only if A is a stability matrix, i.e. all the characteristic roots λ_k of A have negative real parts; (2.2) is unstable if for some characteristic roots λ_k, $\Re e(\lambda_k) > 0$; and completely unstable if for all characteristic roots λ_k, $\Re e(\lambda_k) > 0$.*

See [3] for more details.

2.1.3 Lyapunov Theory

Consider autonomous system of nonlinear equations,

$$\dot{x} = f(x), \quad f(0) = 0. \tag{2.3}$$

We define a Lyapunov function $V(x)$ as follows:

1. $V(x)$ and all its partial derivatives $\dfrac{\partial V}{\partial x_i}$ are continuous.

2. $V(x)$ is positive definite, i.e. $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighbourhood $\| x \| \leq k$ of the origin.

3. The derivative of V with respect to (2.3), namely

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + ... + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 + ... + \frac{\partial V}{\partial x_n} f_n \end{aligned} \tag{2.4}$$

is negative semidefinite i.e. $\dot{V}(0) = 0$, and for all x in $\|x\| \leq k$, $V(\dot{x}) \leq 0$.

Notice that in (2.4) the f_i are the components of f in (2.3), so $\dot{V}$ can be determined directly from the system equations.

Theorem 2.1.2 *The origin of (2.3) is stable if there exists a Lyapunov function defined as above.*

Theorem 2.1.3 *The origin of (2.3) is asymptotically stable if there exists a Lyapunov function whose derivative (2.4) is negative definite.*

See [3] for more details.

2.1.4 Application of Lyapunov Theory to Linear Systems

The usefulness of linear theory can be extended by using the idea of linearization. Suppose the components of f in (2.1) are such that we can apply Taylor's theorem to obtain

$$f(x) = \acute{A}x + g(x), \tag{2.5}$$

using $f(0) = 0$. In (2.5) $\acute{A}$ denotes the $n \times n$ constant matrix having elements $(\partial f_i / \partial x_j)_{x=0}$, $g(0) = 0$ and the components of g have power series expansions in $x_1, x_2, ..., x_n$ beginning with terms of at least second degree. The system

$$\dot{x} = \acute{A}x \tag{2.6}$$

is called the *first approximation* to (2.1). We then have:

Theorem 2.1.4 *(Lyapunov's linearization theorem) If (2.6) is asymptotically stable, or unstable, then the origin for $\dot{x} = f(x)$, where $f(x)$ is given by (2.5), has the same stability property.*

See [3] for more details.

2.2 Routh-Hurwitz Theorem

Consider the characteristic equation of matrix A

$$det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n = 0 \tag{2.7}$$

determining the n eigenvalues λ of a real $n \times n$ square matrix A, where I is the identity matrix.

Theorem 2.2.1 *The $n \times n$ Hurwitz matrix associated with $a(\lambda)$ in (2.7) is*

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & \cdot & \cdot & \cdot & a_{2n-1} \\ 1 & a_2 & a_4 & \cdot & \cdot & \cdot & a_{2n-2} \\ 0 & a_1 & a_3 & \cdot & \cdot & \cdot & a_{2n-3} \\ 0 & 1 & a_2 & \cdot & \cdot & \cdot & a_{2n-4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & a_n \end{bmatrix}$$

where $a_r = 0$, $r > n$. Let H_i denote the i(th) leading principle minor of H. Then all the roots of $a(\lambda)$ have negative real parts ($a(\lambda)$ is a Hurwitz polynomial) if and only if $H_i > 0$, $i = 1, 2, ..., n-1$.

If $n = 3$ then

$$|\lambda I - A| = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

In this case all of the eigenvalues λ have negative real parts if

$$H_1 > 0, H_2 > 0,$$

or

(1) $a_1 > 0$,

and (2) $\begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0$ or $a_1a_2 - a_3 > 0$.

Since we have assumed that the a_i are real it is easy to derive a simple *necessary* condition for asymptotic stability:

Theorem 2.2.2 *If the a_i in (2.7) are real and $a(\lambda)$ corresponds to an asymptotically stable system, then*

$$a_i > 0, i = 1, 2, ..., n.$$

2.3 Fourth-Order Runge-Kutta Method

In order to solve an initial-value problem

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0$$

where $x = [x_1, x_2, \ldots, x_n]^T$ and $f = [f_1, f_2, \ldots, f_n]^T$.

The best known Runge-Kutta method of the first stage and fourth order is given by

$$X_{i+1} = X_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= hf(t_i, X_i) \\ k_2 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_1}{2}) \\ k_3 &= hf(t_i + \frac{h}{2}, X_i + \frac{k_2}{2}) \\ k_4 &= hf(t_i + h, X_i + k_3) \end{aligned}$$

where X_i is an approximation of $x(t_i)$ when $X_i = [X_{i1}, X_{i2}, \ldots, X_{in}]^T$, $t_i = t_0 + ih$, h is step size and $k_i = [k_{i1}, k_{i2}, \ldots, k_{in}]^T, \forall i = 1, \ldots, 4$.

2.4 Matrix Types

The following section follow from [4].

2.4.1 Symmetric Matrix

A real $n \times n$ matrix A is called *symmetric* if

$$A^T = A.$$

2.4.2 Positive Definite Matrix

Consider a real $n \times n$ matrix A, A is called *positive definite* if

$$x^T Ax > 0$$

for all nonzero vectors $x \in \mathbb{R}^n$, where x^T denotes the transpose of x.
A *positive definite* matrix is a Symmetric Matrix in which all of whose eigenvalues are positive. Or symmetric matrix A is called *positive definite* if and only if $D_i > 0$, $i = 1, 2, ..., n$, where D_i denotes leading principal minors.

2.4.3 Negative Definite Matrix

Consider a real $n \times n$ matrix A, A is called *negative definite* if

$$x^T Ax < 0$$

for all nonzero vectors $x \in \mathbb{R}^n$, where x^T denotes the transpose of x.
A *Negative definite* matrix is a Symmetric Matrix in which all of whose eigenvalues are negative. Or symmetric matrix A is called *negative definite* if and only if $(-1)^i D_i > 0$, $i = 1, 2, ..., n$, where D_i denotes leading principal minors.
If A satisfies none of the above then it is indefinite.

2.4.4 The Rayleigh Quotient

The set of values assumed by the quadratic from $x^T Ax$ on sphere $x^T x = 1$ is precisely the same set taken by the quadratic from $y^T \Lambda y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_N y_N^2$ on $y^T y$=1, $\Lambda = T^T AT$, $y = Tx$, with T orthogonal. Let us henceforth suppose that

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$$

We readily obtain the representations

$$\lambda_1 = \max \frac{\mathrm{y^T \Lambda y}}{\mathrm{y^T y}} = \max \frac{\mathrm{x^T Ax}}{\mathrm{x^T x}}$$
$$\lambda_N = \max \frac{\mathrm{y^T \Lambda y}}{\mathrm{y^T y}} = \max \frac{\mathrm{x^T Ax}}{\mathrm{x^T x}} \quad (2.8)$$

The quotient

$$q(x) = \frac{x^T A x}{x^T x}$$

is often called the *Rayleigh quotient*.
From the relation in (2.8), we see that for all x we have

$$\lambda_1 \geq \frac{x^T A x}{x^T x} \geq \lambda_N. \tag{2.9}$$

2.4.5 Square Roots Matrix

Since a positive definite matrix represents a natural generalization of a positive number, it is interesting to inquire whether or not a positive definite matrix possesses a positive definite square root.
Proceeding as in Sec. 2.4.4, we can define $A^{\frac{1}{2}}$ by means of the relation

$$A^{\frac{1}{2}} = T \begin{pmatrix} \lambda_1^{\frac{1}{2}} & & & 0 \\ & \lambda_2^{\frac{1}{2}} & & \\ & & \ddots & \\ 0 & & & \lambda_N^{\frac{1}{2}} \end{pmatrix} T^T. \tag{2.10}$$

Lemma 2.4.1 [5] *For any real vector D and E with appropriate dimension and any positive scalar δ, we have*

$$DE + E^T D^T \leq \delta D D^T + \delta^{-1} E^T E.$$