CHAPTER 3

MAIN RESULTS

In this chapter we consider the neutral system (1.1) and the Hopfiled neural network (1.2).
Throughout this paper, the following assumption is made on the systems (1.1) and (1.2).

Assumption 3.0.1 There exist nonnegative constants α_0, α_1 and α_2 such that

$$\|f(t,x(t),x[t-\tau],x[t-\sigma])\| \leq \alpha_0\|x(t)\| + \alpha_1\|x[t-\tau]\| + \alpha_2\|x[t-\sigma]\|,\ \forall t.$$

3.1 Neutral system

In this section, we will study the asymptotic stability of the equilibrium $x=0$ of neutral system

$$x^{'}(t) + Cx'[t-\tau] = Ax(t) + Bx[t-\sigma] + f(t,x(t),x[t-\tau],x[t-\sigma]) \qquad (1.1)$$

where $x(t) \in \Re^n$ is the state vector, τ and σ are positive constant time-delays, $A \in \Re^{n\times n}$, $B \in \Re^{n\times n}$ and $C \in \Re^{n\times n}$ are constant system matrices and $f(t,x(t),x[t-\tau],x[t-\sigma])$ is a nonlinear perturbation .

Theorem 3.1.1 *Let α_0, α_1, and α_2 be given in Assumption* 3.0.1*. The equilibrium $x=0$ of (1.1) is asymptotic stability if there exist positive definite matrices P, G_1 and G_2 and positive constants δ, γ such that for any positive constants α, β with $\alpha+\beta=1$, the following matrices*

$$\begin{aligned} F_1 &= A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K + M^TG_2^{-1}M + 3\delta\alpha_0^2I + \delta^{-1}P^2 + 3\gamma\alpha_0^2I, \\ F_2 &= 3\delta\alpha_1^2I + 3\gamma\alpha_1^2I + \gamma^{-1}C^TP^2C, and \\ F_3 &= 3\delta\alpha_2^2I + 3\gamma\alpha_2^2I + \frac{1}{\beta}H^TG_1^{-1}H \end{aligned} \qquad (3.2)$$

are negative definite, where $I = n\times n$ identity matrix, $K = C^TPA$, $M = B^TP$, $H = B^TPC$ and $f = f(t,x(t),x[t-\tau],x[t-\sigma])$.

Proof. We employ the Lyapunov functional

$$V(x(t)) = y^T(t)Py(t) + \int_{t-\tau}^{t} x^T(u)G_1x(u)du + \int_{t-\sigma}^{t} x^T(u)G_2x(u)du$$

where $y(t) = x(t) + Cx[t-\tau]$, by Eq. (2.9).
Then we have

$$\lambda_{\min}(P) \parallel y(t) \parallel^2 \leq V(x(t)). \tag{3.3}$$

Since P is a positive definite matrix, we conclude that $\lambda_{\min}(P) > 0$. Thus $V(x(t))$ is positive definite.
The time derivative of V along the solutions of (1.1) is given by

$$\begin{aligned}
V'(x(t)) &= y'^T(t)Py(t) + y^T(t)Py'(t) + x^T(t)G_1x(t) \\
&\quad -x^T[t-\tau]G_1x[t-\tau] + x^T(t)G_2x(t) \\
&\quad -x^T[t-\sigma]G_2x[t-\sigma] \\
&= (x^T(t)A^T + x^T[t-\sigma]B^T + f^T)P(x(t) + Cx[t-\tau]) \\
&\quad +(x^T(t) + x^T[t-\tau]C^T)P(Ax(t) + Bx[t-\sigma] + f) \\
&\quad +x^T(t)G_1x(t) - x^T[t-\tau]G_1x[t-\tau] \\
&\quad +x^T(t)G_2x(t) - x^T[t-\sigma]G_2x[t-\sigma] \\
&= x^T(t)A^TPx(t) + x^T(t)A^TPCx[t-\tau] \\
&\quad +x^T[t-\sigma]B^TPx(t) + x^T[t-\sigma]B^TPCx[t-\tau] \\
&\quad +f^TPx(t) + f^TPCx[t-\tau] + x^T(t)PAx(t) \\
&\quad +x^T(t)PBx[t-\sigma] + x^T(t)Pf + x^T[t-\tau]C^TPAx(t) \\
&\quad +x^T[t-\tau]C^TPBx[t-\sigma] + x^T[t-\tau]C^TPf \\
&\quad +x^T(t)G_1x(t) - x^T[t-\tau]G_1x[t-\tau] \\
&\quad +x^T(t)G_2x(t) - x^T[t-\sigma]G_2x[t-\sigma]
\end{aligned}$$

$$
\begin{aligned}
V'(x(t)) &= x^T(t)[A^TP + PA + G_1 + G_2]x(t) \\
&\quad + x^T(t)A^TPCx[t-\tau] + x^T[t-\tau]C^TPAx(t) \\
&\quad + x^T[t-\sigma]B^TPx(t) + x^T(t)PBx[t-\sigma] \\
&\quad + x^T[t-\sigma]B^TPCx[t-\tau] + x^T[t-\tau]C^TPBx[t-\sigma] \\
&\quad -(\alpha+\beta)x^T[t-\tau]G_1x[t-\tau] - x^T[t-\sigma]G_2x[t-\sigma] \\
&\quad + f^TPx(t) + x^T(t)Pf + f^TPCx[t-\tau] \\
&\quad + x^T[t-\tau]C^TPf \\
&= x^T(t)[A^TP + PA + G_1 + G_2]x(t) \\
&\quad + x^T(t)K^Tx[t-\tau] + x^T[t-\tau]Kx(t) \\
&\quad + x^T[t-\sigma]Mx(t) + x^T(t)M^Tx[t-\sigma] \\
&\quad + x^T[t-\sigma]Hx[t-\tau] + x^T[t-\tau]H^Tx[t-\sigma] \\
&\quad -\alpha x^T[t-\tau]G_1x[t-\tau] - \beta x^T[t-\tau]G_1x[t-\tau] \\
&\quad - x^T[t-\sigma]G_2x[t-\sigma] + f^TPx(t) + x^T(t)Pf \\
&\quad + f^TPCx[t-\tau] + x^T[t-\tau]C^TPf \\
&= x^T(t)[A^TP + PA + G_1 + G_2]x(t) \\
&\quad + x^T(t)K^Tx[t-\tau] + x^T[t-\tau]Kx(t) \\
&\quad -\frac{1}{\alpha}x^T(t)K^TG_1^{-1}Kx(t) - \alpha x^T[t-\tau]G_1x[t-\tau] \\
&\quad +\frac{1}{\alpha}x^T(t)K^TG_1^{-1}Kx(t) \\
&\quad + x^T[t-\sigma]Mx(t) + x^T(t)M^Tx[t-\sigma] \\
&\quad - x^T(t)M^TG_2^{-1}Mx(t) - x^T[t-\sigma]G_2x[t-\sigma] \\
&\quad + x^T(t)M^TG_2^{-1}Mx(t) \\
&\quad + x^T[t-\sigma]Hx[t-\tau] + x^T[t-\tau]H^Tx[t-\sigma] \\
&\quad -\frac{1}{\beta}x^T[t-\sigma]H^TG_1^{-1}Hx[t-\sigma] - \beta x^T[t-\tau]G_1x[t-\tau] \\
&\quad +\frac{1}{\beta}x^T[t-\sigma]H^TG_1^{-1}Hx[t-\sigma] \\
&\quad + f^TPx(t) + x^T(t)Pf \\
&\quad + f^TPCx[t-\tau] + x^T[t-\tau]C^TPf
\end{aligned}
$$

$$
\begin{aligned}
V'(x(t)) &= x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K + M^TG_2^{-1}M]x(t) \\
&\quad -(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t))^T \\
&\quad \times(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t)) \\
&\quad -(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Mx(t))^T \\
&\quad \times(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Mx(t)) \\
&\quad -(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Hx[t-\sigma])^T \\
&\quad \times(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Hx[t-\sigma]), \text{ by Eq. (2.10)} \\
&\quad +\frac{1}{\beta}x^T[t-\sigma]H^TG_1^{-1}Hx[t-\sigma] + f^TPx(t) \\
&\quad +x^T(t)Pf + f^TPCx[t-\tau] + x^T[t-\tau]C^TPf \qquad (3.4)
\end{aligned}
$$

Using Lemma 2.4.1, the term f^Tf on the right-hand side of (3.4) satisfies the following inequality:

$$
\begin{aligned}
f^Tf = \parallel f \parallel^2 &\leq (\alpha_0 \parallel x(t) \parallel + \alpha_1 \parallel x[t-\tau] \parallel + \alpha_2 \parallel x[t-\sigma] \parallel)^2 \\
&\leq 3\alpha_0^2 \parallel x(t) \parallel^2 + 3\alpha_1^2 \parallel x[t-\tau] \parallel^2 + 3\alpha_2^2 \parallel x[t-\sigma] \\
&= 3\alpha_0^2x^T(t)x(t) + 3\alpha_1^2x^T[t-\tau]x[t-\tau] \\
&\quad +3\alpha_2^2x^T[t-\sigma]x[t-\sigma]. \qquad (3.5)
\end{aligned}
$$

Again, by using Lemma 2.4.1 and (3.5), the other terms on the right-hand side of (3.4) satisfy

$$
\begin{aligned}
f^TPx(t) + x^T(t)Pf &\leq \delta f^Tf + \delta^{-1}x^T(t)PPx(t) \\
&\leq 3\delta\alpha_0^2x^T(t)x(t) + 3\delta\alpha_1^2x^T[t-\tau]x[t-\tau] \\
&\quad +3\delta\alpha_2^2x^T[t-\sigma]x[t-\sigma] + \delta^{-1}x^T(t)PPx(t).
\end{aligned}
$$

and

$$
\begin{aligned}
f^TPCx[t-\tau] + x^T[t-\tau]C^TPf &\leq \gamma f^Tf + \gamma^{-1}x^T[t-\tau]C^TPPCx[t-\tau] \\
&\leq 3\gamma\alpha_0^2x^T(t)x(t) + 3\gamma\alpha_1^2x^T[t-\tau]x[t-\tau] \\
&\quad +3\gamma\alpha_2^2x^T[t-\sigma]x[t-\sigma] \\
&\quad +\gamma^{-1}x^T[t-\tau]C^TPPCx[t-\tau].
\end{aligned}
$$

Where δ, γ are any positive scalars.

Then, we obtain

$$\begin{aligned}
V^{'}(x(t)) \leq\ & x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K + M^TG_2^{-1}M]x(t) \\
& -(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t))^T \\
& \times(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t)) \\
& -(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Mx(t))^T \\
& \times(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Mx(t)) \\
& -(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Hx[t-\sigma])^T \\
& \times(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Hx[t-\sigma]) \\
& +3\delta\alpha_0^2x^T(t)x(t) + 3\delta\alpha_1^2x^T[t-\tau]x[t-\tau] \\
& +3\delta\alpha_2^2x^T[t-\sigma]x[t-\sigma] + \delta^{-1}x^T(t)PPx(t) \\
& +3\gamma\alpha_0^2x^T(t)x(t) + 3\gamma\alpha_1^2x^T[t-\tau]x[t-\tau] \\
& +3\gamma\alpha_2^2x^T[t-\sigma]x[t-\sigma] + \gamma^{-1}x^T[t-\tau]C^TPPCx[t-\tau] \\
& +\frac{1}{\beta}x^T[t-\sigma]H^TG_1^{-1}Hx[t-\sigma] \\
=\ & x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K \\
& +M^TG_2^{-1}M + 3\delta\alpha_0^2I + \delta^{-1}P^2 + 3\gamma\alpha_0^2I]x(t) \\
& +x^T[t-\tau][3\delta\alpha_1^2I + 3\gamma\alpha_1^2I + \gamma^{-1}C^TPPC]x[t-\tau] \\
& +x^T[t-\sigma][3\delta\alpha_2^2I + 3\gamma\alpha_2^2I + \frac{1}{\beta}H^TG_1^{-1}H]x[t-\sigma] \\
& -(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t))^T \\
& \times(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t)) \\
& -(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Mx(t))^T \\
& \times(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Mx(t)) \\
& -(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Hx[t-\sigma])^T \\
& \times(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Hx[t-\sigma])
\end{aligned}$$

Thus,

$$\begin{aligned} V'(x(t)) &\leq x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K \\ &\quad + M^TG_2^{-1}M + 3\delta\alpha_0^2 I + \delta^{-1}P^2 + 3\gamma\alpha_0^2 I]x(t) \\ &\quad + x^T[t-\tau][3\delta\alpha_1^2 I + 3\gamma\alpha_1^2 I + \gamma^{-1}C^TPPC]x[t-\tau] \\ &\quad + x^T[t-\sigma][3\delta\alpha_2^2 I + 3\gamma\alpha_2^2 I + \frac{1}{\beta}H^TG_1^{-1}H]x[t-\sigma] \\ &= x^T(t)F_1x(t) + x^T[t-\tau]F_2x[t-\tau] + x^T[t-\sigma]F_3x[t-\sigma] \\ &\leq \lambda_{\max}(F_1)\|x(t)\|^2 + \lambda_{\max}(F_2)\|x(t)\|^2 \qquad (3.6) \\ &\quad + \lambda_{\max}(F_3)\|x(t)\|^2, \text{by Eq. (2.9).} \end{aligned}$$

Since F_1, F_2 and F_3 are negative definite, we conclude that $\lambda_{\max}(F_1) < 0$, $\lambda_{\max}(F_2) < 0$ and $\lambda_{\max}(F_3) < 0$. Thus $V'(x(t))$ is negative definite. By (3.3) and (3.6), it now follows from Theorem 2.1.3 that the equilibrium $x = 0$ of system (1.1) is asymptotic stability. This completes the proof. □

3.2 Hopfiled neural network

In this section, we will study the asymptotic stability of the equilibrium $x = 0$ of the Hopfiled neural network

$$\begin{aligned} x^{'}(t) + Cx'[t-\tau] &= Ax(t) + BT(x[t-\sigma])x[t-\sigma] \\ &\quad + f(t, x(t), x[t-\tau], x[t-\sigma]) \end{aligned} \tag{1.2}$$

where $x(t) \in \Re^n$ is the state vector, τ and σ are positive constant time-delays, $A = diag\{a_1, ..., a_n\}$, $a_i > 0$, $i = 1, 2, ..., n$, $B \in \Re^{n\times n}$ and $C \in \Re^{n\times n}$ are constant system matrices, $T(x) = diag\{\sigma_1(x_1), ..., \sigma_n(x_n)\}$, $\sigma_i(x_i) = \frac{s_i(x_i)}{x_i}$ and $f(t, x(t), x[t-\tau], x[t-\sigma])$ is a nonlinear perturbation, where s_i is monotonically increasing for $i = 1, 2, ..., n$.

Theorem 3.2.1 *Let α_0, α_1, and α_2 be given in Assumption 3.0.1. The equilibrium $x = 0$ of (1.1) is asymptotic stability if there exist positive definite matrices P, G_1 and G_2 and positive constants δ, γ such that for any positive constants α, β with $\alpha + \beta = 1$, the following matrices*

$$\begin{aligned} F_4 &= A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K + W^TG_2^{-1}W + 3\delta\alpha_0^2I + \delta^{-1}P^2 + 3\gamma\alpha_0^2I, \\ F_5 &= 3\delta\alpha_1^2I + 3\gamma\alpha_1^2I + \gamma^{-1}C^TPPC, and \\ F_6 &= 3\delta\alpha_2^2I + 3\gamma\alpha_2^2I + \frac{1}{\beta}Z^TG_1^{-1}Z \end{aligned} \tag{3.7}$$

are negative definite, where $I = n \times n$ identity matrix, $K = C^TPA$, $W = PBT(x[t-\sigma])$, $Z = T(x[t-\sigma])B^TPC$ and $f = f(t, x(t), x[t-\tau], x[t-\sigma])$.

Proof. We employ the Lyapunov functional

$$V(x(t)) = y^T(t)Py(t) + \int_{t-\tau}^{t} x^T(u)G_1x(u)du + \int_{t-\sigma}^{t} x^T(u)G_2x(u)du$$

where $y(t) = x(t) + Cx[t-\tau]$, by Eq. (2.9).

Then we have

$$\lambda_{\min}(P) \parallel y(t) \parallel^2 \leq V(x(t)). \tag{3.8}$$

Since P is a positive definite matrix, we conclude that $\lambda_{\min}(P) > 0$. Thus $V(x(t))$ is positive definite.

The time derivative of V along the solutions of (1.2) is given by

$$\begin{aligned}
V'(x(t)) &= y'^T(t)Py(t) + y^T(t)Py'(t) + x^T(t)G_1x(t) \\
&\quad -x^T[t-\tau]G_1x[t-\tau] + x^T(t)G_2x(t) \\
&\quad -x^T[t-\sigma]G_2x[t-\sigma] \\
&= (x^T(t)A^T + x^T[t-\sigma]T(x[t-\sigma])B^T + f^T)P \\
&\quad \times(x(t) + Cx[t-\tau]) \\
&\quad +(x^T(t) + x^T[t-\tau]C^T)P \\
&\quad \times(Ax(t) + BT(x[t-\sigma])x[t-\sigma] + f) \\
&\quad +x^T(t)G_1x(t) - x^T[t-\tau]G_1x[t-\tau] \\
&\quad +x^T(t)G_2x(t) - x^T[t-\sigma]G_2x[t-\sigma] \\
&= x^T(t)A^TPx(t) + x^T(t)PAx(t) \\
&\quad +x^T(t)A^TPCx[t-\tau] + x^T[t-\tau]C^TPAx(t) \\
&\quad +x^T[t-\sigma]T(x[t-\sigma])B^TPx(t) \\
&\quad +x^T(t)PBT(x[t-\sigma])x[t-\sigma] \\
&\quad +x^T[t-\sigma]T(x[t-\sigma])B^TPCx[t-\tau] \\
&\quad +x^T[t-\tau]C^TPBT(x[t-\sigma])x[t-\sigma] \\
&\quad +f^TPx(t) + f^TPCx[t-\tau] \\
&\quad +x^T(t)Pf + x^T[t-\tau]C^TPf \\
&\quad +x^T(t)G_1x(t) - x^T[t-\tau]G_1x[t-\tau] \\
&\quad +x^T(t)G_2x(t) - x^T[t-\sigma]G_2x[t-\sigma]
\end{aligned}$$

$$
\begin{aligned}
V'(x(t)) &= x^T(t)[A^TP+PA+G_1+G_2]x(t) \\
&\quad +x^T(t)A^TPCx[t-\tau]+x^T[t-\tau]C^TPAx(t) \\
&\quad +x^T[t-\sigma]T(x[t-\sigma])B^TPx(t) \\
&\quad +x^T(t)PBT(x[t-\sigma])x[t-\sigma] \\
&\quad +x^T[t-\sigma]B^TPCT(x[t-\sigma])x[t-\tau] \\
&\quad +x^T[t-\tau]C^TPBT(x[t-\sigma])x[t-\sigma] \\
&\quad -(\alpha+\beta)x^T[t-\tau]G_1x[t-\tau]-x^T[t-\sigma]G_2x[t-\sigma] \\
&\quad +f^TPx(t)+x^T(t)Pf+f^TPCx[t-\tau]+x^T[t-\tau]C^TPf \\
&= x^T(t)[A^TP+PA+G_1+G_2]x(t) \\
&\quad +x^T(t)K^Tx[t-\tau]+x^T[t-\tau]Kx(t) \\
&\quad +x^T[t-\sigma]Wx(t)+x^T(t)W^Tx[t-\sigma] \\
&\quad +x^T[t-\sigma]Zx[t-\tau]+x^T[t-\tau]Z^Tx[t-\sigma] \\
&\quad -\alpha x^T[t-\tau]G_1x[t-\tau]-\beta x^T[t-\tau]G_1x[t-\tau] \\
&\quad -x^T[t-\sigma]G_2x[t-\sigma]+f^TPx(t)+x^T(t)Pf \\
&\quad +f^TPCx[t-\tau]+x^T[t-\tau]C^TPf \\
&= x^T(t)[A^TP+PA+G_1+G_2]x(t) \\
&\quad +x^T(t)K^Tx[t-\tau]+x^T[t-\tau]Kx(t) \\
&\quad -\frac{1}{\alpha}x^T(t)K^TG_1^{-1}Kx(t)-\alpha x^T[t-\tau]G_1x[t-\tau] \\
&\quad +\frac{1}{\alpha}x^T(t)K^TG_1^{-1}Kx(t) \\
&\quad +x^T[t-\sigma]Wx(t)+x^T(t)W^Tx[t-\sigma] \\
&\quad -x^T(t)W^TG_2^{-1}Wx(t)-x^T[t-\sigma]G_2x[t-\sigma] \\
&\quad +x^T(t)W^TG_2^{-1}Wx(t) \\
&\quad +x^T[t-\sigma]Zx[t-\tau]+x^T[t-\tau]Z^Tx[t-\sigma] \\
&\quad -\frac{1}{\beta}x^T[t-\sigma]Z^TG_1^{-1}Zx[t-\sigma]-\beta x^T[t-\tau]G_1x[t-\tau] \\
&\quad +\frac{1}{\beta}x^T[t-\sigma]Z^TG_1^{-1}Zx[t-\sigma] \\
&\quad +f^TPx(t)+x^T(t)Pf+f^TPCx[t-\tau]+x^T[t-\tau]C^TPf
\end{aligned}
$$

$$
\begin{aligned}
V'(x(t)) &= x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K + W^TG_2^{-1}W]x(t) \\
&\quad -(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t))^T \\
&\quad \times(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t)) \\
&\quad -(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Wx(t))^T \\
&\quad \times(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Wx(t)) \\
&\quad -(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Zx[t-\sigma])^T \\
&\quad \times(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Zx[t-\sigma]), \text{by Eq. (2.10)} \\
&\quad +\frac{1}{\beta}x^T[t-\sigma]Z^TG_1^{-1}Zx[t-\sigma] + f^TPx(t) \\
&\quad +x^T(t)Pf + f^TPCx[t-\tau] + x^T[t-\tau]C^TPf. \qquad (3.9)
\end{aligned}
$$

Using Lemma 2.4.1, the term f^Tf on the right-hand side of (3.9) satisfies the following inequality:

$$
\begin{aligned}
f^Tf = \| f \|^2 &\leq (\alpha_0 \| x(t) \| + \alpha_1 \| x[t-\tau] \| + \alpha_2 \| x[t-\sigma] \|)^2 \\
&\leq 3\alpha_0^2 \| x(t) \|^2 + 3\alpha_1^2 \| x[t-\tau] \|^2 + 3\alpha_2^2 \| x[t-\sigma] \\
&= 3\alpha_0^2x^T(t)x(t) + 3\alpha_1^2x^T[t-\tau]x[t-\tau] \\
&\quad +3\alpha_2^2x^T[t-\sigma]x[t-\sigma]. \qquad (3.10)
\end{aligned}
$$

Again, by using Lemma 2.4.1 and (3.10), the other terms on the right-hand side of (3.9) satisfy

$$
\begin{aligned}
f^TPx(t) + x^T(t)Pf &\leq \delta f^Tf + \delta^{-1}x^T(t)PPx(t) \\
&\leq 3\delta\alpha_0^2x^T(t)x(t) + 3\delta\alpha_1^2x^T[t-\tau]x[t-\tau] \\
&\quad +3\delta\alpha_2^2x^T[t-\sigma]x[t-\sigma] + \delta^{-1}x^T(t)PPx(t).
\end{aligned}
$$

and

$$
\begin{aligned}
f^TPCx[t-\tau] + x^T[t-\tau]C^TPf &\leq \gamma f^Tf + \gamma^{-1}x^T[t-\tau]C^TPPCx[t-\tau] \\
&\leq 3\gamma\alpha_0^2x^T(t)x(t) + 3\gamma\alpha_1^2x^T[t-\tau]x[t-\tau] \\
&\quad +3\gamma\alpha_2^2x^T[t-\sigma]x[t-\sigma] \\
&\quad +\gamma^{-1}x^T[t-\tau]C^TPPCx[t-\tau].
\end{aligned}
$$

Where δ, γ are any positive.

Then, we obtain

$$\begin{aligned}
V^{'}(x(t)) \leq\ & x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K + W^TG_2^{-1}W]x(t) \\
& -(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t))^T \\
& \times(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t)) \\
& -(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Wx(t))^T \\
& \times(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Wx(t)) \\
& -(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Zx[t-\sigma])^T \\
& \times(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Zx[t-\sigma]) \\
& +3\delta\alpha_0^2x^T(t)x(t) + 3\delta\alpha_1^2x^T[t-\tau]x[t-\tau] \\
& +3\delta\alpha_2^2x^T[t-\sigma]x[t-\sigma] + \delta^{-1}x^T(t)PPx(t) \\
& +3\gamma\alpha_0^2x^T(t)x(t) + 3\gamma\alpha_1^2x^T[t-\tau]x[t-\tau] \\
& +3\gamma\alpha_2^2x^T[t-\sigma]x[t-\sigma] + \gamma^{-1}x^T[t-\tau]C^TPPCx[t-\tau] \\
& +\frac{1}{\beta}x^T[t-\sigma]Z^TG_1^{-1}Zx[t-\sigma] \\
=\ & x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K \\
& +W^TG_2^{-1}W + 3\delta\alpha_0^2I + \delta^{-1}P^2 + 3\gamma\alpha_0^2I]x(t) \\
& +x^T[t-\tau][3\delta\alpha_1^2I + 3\gamma\alpha_1^2I + \gamma^{-1}C^TPPC]x[t-\tau] \\
& +x^T[t-\sigma][3\delta\alpha_2^2I + 3\gamma\alpha_2^2I + \frac{1}{\beta}Z^TG_1^{-1}Z]x[t-\sigma] \\
& -(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t))^T \\
& \times(\sqrt{\alpha}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\alpha}}G_1^{\frac{-1}{2}}Kx(t)) \\
& -(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Wx(t))^T \\
& \times(G_2^{\frac{1}{2}}x[t-\sigma] - G_2^{\frac{-1}{2}}Wx(t)) \\
& -(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Zx[t-\sigma])^T \\
& \times(\sqrt{\beta}G_1^{\frac{1}{2}}x[t-\tau] - \frac{1}{\sqrt{\beta}}G_1^{\frac{-1}{2}}Zx[t-\sigma])
\end{aligned}$$

Thus,

$$
\begin{aligned}
V^{'}(x(t)) \leq\ & x^T(t)[A^TP + PA + G_1 + G_2 + \frac{1}{\alpha}K^TG_1^{-1}K \\
& + W^TG_2^{-1}W + 3\delta\alpha_0^2 I + \delta^{-1}P^2 + 3\gamma\alpha_0^2 I]x(t) \\
& + x^T[t-\tau][3\delta\alpha_1^2 I + 3\gamma\alpha_1^2 I + \gamma^{-1}C^TPPC]x[t-\tau] \\
& + x^T[t-\sigma][3\delta\alpha_2^2 I + 3\gamma\alpha_2^2 I + \frac{1}{\beta}Z^TG_1^{-1}Z]x[t-\sigma] \\
=\ & x^T(t)F_4x(t) + x^T[t-\tau]F_5x[t-\tau] + x^T[t-\sigma]F_6x[t-\sigma] \\
\leq\ & \lambda_{\max}(F_4)\|x(t)\|^2 + \lambda_{\max}(F_5)\|x(t)\|^2 \\
& + \lambda_{\max}(F_6)\|x(t)\|^2, \text{by Eq. (2.9).} \qquad (3.11)
\end{aligned}
$$

Since F_4, F_5 and F_6 are negative definite, we conclude that $\lambda_{\max}(F_4) < 0$, $\lambda_{\max}(F_5) < 0$ and $\lambda_{\max}(F_6) < 0$. Thus $V^{'}(x(t))$ is negative

definite. By (3.8) and (3.11), it now follows from Theorem 2.1.3 that the equilibrium $x = 0$ of system (1.2) is asymptotic stability. This completes the proof. □

3.3 Numerical Examples

Example 3.3.1 Consider the neutral system :

$$x'(t) + Cx'[t-\tau] = Ax(t) + Bx[t-\sigma] + f(t, x(t), x[t-\tau], x[t-\sigma]) \qquad (3.12)$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0.5 \\ -0.2 & -0.2 \end{bmatrix},\ C = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix},$$

$f(t, x(t), x[t-\tau], x[t-\sigma]) = \begin{bmatrix} x_1(t)sinx_1(t) + x_1[t-\tau]sinx_1(t) \\ x_2[t-\sigma]cosx_2(t) \end{bmatrix}$, $\tau = 0.3$ and $\sigma = 0.3$, and the initial condition of the system is as follows:

$$x(t) = [0.5e^t, -0.5e^{-t}]^T, \text{ for any } -0.3 \le t \le 0.$$

Let

$$\begin{aligned} \|f(t, x(t), x[t-\tau], x[t-\sigma])\|^2 &= (x_1(t)sinx_1(t) + x_1[t-\tau]sinx_1(t))^2 \\ &\quad + x_2^2[t-\sigma]cos^2x_2(t) \\ &\le 2x_1^2(t)sin^2x_1(t) + 2x_1^2[t-\tau]sin^2x_1(t) \\ &\quad + x_2^2[t-\sigma]cos^2x_2(t) \\ &\le 2\|x(t)\|^2 + 2\|x[t-\tau]\|^2 + \|x[t-\sigma]\|^2, \end{aligned}$$

we obtain

$$\|f(t, x(t), x[t-\tau], x[t-\sigma])\| \le \sqrt{2}\|x(t)\| + \sqrt{2}\|x[t-\tau]\| + \|x[t-\sigma]\|.$$

Let

$$P = \begin{bmatrix} 0.0462 & -0.0356 \\ -0.0356 & 0.1187 \end{bmatrix},\ G_1 = \begin{bmatrix} 0.7666 & -1.0195 \\ -1.0195 & 1.3969 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0.0564 & 0.0750 \\ 0.0750 & 0.1028 \end{bmatrix},\ \alpha_0 = \sqrt{2},\ \alpha_1 = \sqrt{2},\ \alpha_2 = 1,$$

and

$$F_1 = \begin{bmatrix} -3.8915 - 0.689\frac{1}{\alpha} + 6\delta + 0.0034\frac{1}{\delta} + 6\gamma & 0.3698 + 0.0125\frac{1}{\alpha} - 0.0058\frac{1}{\delta} \\ 0.3698 + 0.0125\frac{1}{\alpha} - 0.0058\frac{1}{\delta} & -4.5846 - 1.9630\frac{1}{\alpha} + 6\delta + 0.0154\frac{1}{\delta} + 6\gamma \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -4.5618 + 6\delta + 6\gamma + 0.0039\frac{1}{\gamma} & 0.5232 - 0.0296\frac{1}{\gamma} \\ 0.5232 - 0.0296\frac{1}{\gamma} & -4.5761 + 6\delta + 6\gamma + 0.0258\frac{1}{\gamma} \end{bmatrix},$$

$$F_3 = \begin{bmatrix} -3.3596 - 1.2560\frac{1}{\beta} + 6\delta + 6\gamma & 0.0273 + 0.4621\frac{1}{\beta} \\ 0.0273 + 0.4621\frac{1}{\beta} & -4.1756 - 0.5970\frac{1}{\beta} + 6\delta + 6\gamma \end{bmatrix}.$$

By using Matlab's, we may show that F_1, F_2, and F_3 are negative definite, and by Sec. 2.4.3, we obtain $0.0511+0.1130\beta+0.2078\alpha < \delta < 0.1450+0.0430\beta+0.2110\alpha$ and $0.2705+0.2436\beta+0.3370\alpha < \gamma < 0.4023+0.5130\beta+0.4478\alpha$ for any positive constants α, β with $\alpha+\beta = 1$.

It follows from Theorem 3.1.1 that the equilibrium $x = 0$ of neutral system (3.12) is asymptotic stability.

We illustrate 3.3.1 when $\alpha = 0.5$, $\beta = 0.5$, $0.2115 < \delta < 0.272$, and $0.5608 < \gamma < 0.8827$ by using the fourth-order Runge-Kutta method with step size 0.001 in Figs. 3.1 and 3.2. In these figures, one can see that the neutral system is asymptotically stable.

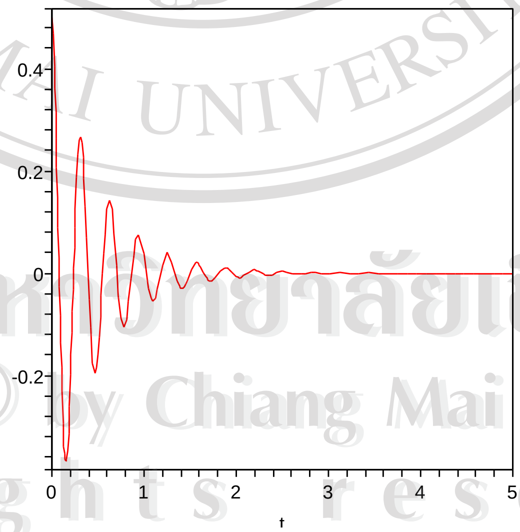

Figure 3.1: the time response of the state x_1 of the system (3.12)

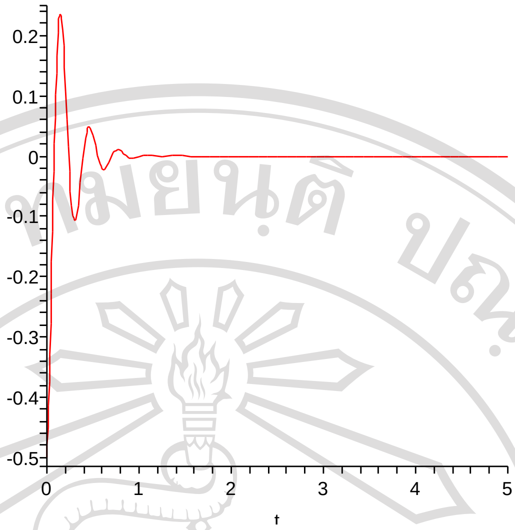

Figure 3.2: the time response of the state x_2 of the system (3.12)

Example 3.3.2 The Hopfiled neural network :

$$\begin{aligned} x^{'}(t) + Cx'[t-\tau] &= Ax(t) + BT(x[t-\sigma])x[t-\sigma] \\ &\quad + f(t, x(t), x[t-\tau], x[t-\sigma]) \end{aligned} \tag{3.13}$$

Let

$s_i(x(t)) = 0.5(|x_i(t)+1| - |x_i(t)-1|)$, $A = \begin{bmatrix} -2.6 & 0 \\ 0 & -1.1 \end{bmatrix}$, $B = \begin{bmatrix} 1.1 & 1 \\ -0.2 & 0.1 \end{bmatrix}$,

$C = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}$,

$T(x[t-\sigma]) = \begin{bmatrix} 0.5(|x_1[t-\sigma]+1| - |x_1[t-\sigma]-1|) & 0 \\ 0 & 0.5(|x_2[t-\sigma]+1| - |x_2[t-\sigma]-1|) \end{bmatrix}$

, $f(t, x(t), x[t-\tau], x[t-\sigma]) = \begin{bmatrix} x_1(t)sinx_1(t) + x_1[t-\tau]sinx_1(t) \\ x_2[t-\sigma]cosx_2(t) \end{bmatrix}$, $\tau = 0.3$ and $\sigma = 0.3$, and the initial condition of the system is as follows:

$$x(t) = [0.5, 0.5]^T, \text{ for any } -0.3 \le t \le 0.$$

By following inequality

$$\begin{aligned}\|f(t,x(t),x[t-\tau],x[t-\sigma])\|^2 &= (x_1(t)sinx_1(t)+x_1[t-\tau]sinx_1(t))^2\\ &\quad +x_2^2[t-\sigma]cos^2x_2(t)\\ &\le 2x_1^2(t)sin^2x_1(t)+2x_1^2[t-\tau]sin^2x_1(t)\\ &\quad +x_2^2[t-\sigma]cos^2x_2(t)\\ &\le 2\|x(t)\|^2+2\|x[t-\tau]\|^2+\|x[t-\sigma]\|^2,\end{aligned}$$

we obtain

$$\|f(t,x(t),x[t-\tau],x[t-\sigma])\| \le \sqrt{2}\|x(t)\|+\sqrt{2}\|x[t-\tau]\|+\|x[t-\sigma]\|.$$

Let

$$P=\begin{bmatrix}0.9906 & 0.8820\\ 0.8820 & 5.7641\end{bmatrix},\ G_1=\begin{bmatrix}0.0725 & -0.0389\\ -0.0389 & 0.0567\end{bmatrix},$$

$$G_2=\begin{bmatrix}0.0512 & -0.0657\\ -0.0657 & 0.0834\end{bmatrix},\ \alpha_0=\sqrt{2},\ \alpha_1=\sqrt{2},\ \alpha_2=1,$$

and

$$F_4=\begin{bmatrix}-6.7894-2.584\frac{1}{\alpha}+6\delta+1.5558\frac{1}{\delta}+6\gamma & 7.0653+0.9589\frac{1}{\alpha}+5.9576\frac{1}{\delta}\\ 7.9653+0.9589\frac{1}{\alpha}+5.9576\frac{1}{\delta} & -0.5742-0.089\frac{1}{\alpha}+6\delta+34.0027\frac{1}{\delta}+6\gamma\end{bmatrix},$$

$$F_5=\begin{bmatrix}-0.7246+6\delta+6\gamma+0.0457\frac{1}{\gamma} & 0.7681+0.1578\frac{1}{\gamma}\\ 0.7681+0.1578\frac{1}{\gamma} & -52.6802+6\delta+6\gamma+0.1258\frac{1}{\gamma}\end{bmatrix},$$

$$F_6=\begin{bmatrix}-2.8839-3.2560\frac{1}{\beta}+6\delta+6\gamma & 0.0766+0.0021\frac{1}{\beta}\\ 0.0766+0.0021\frac{1}{\beta} & -5.5050-0.3760\frac{1}{\beta}+6\delta+6\gamma\end{bmatrix}.$$

By using Matlab's, we may show that F_4, F_5, and F_6 are negative definite, and by Sec. 2.4.3, we obtain $0.1011+0.4430\alpha+0.5401\beta<\delta<0.4136+0.2598\alpha+0.2607\beta$ and $0.0215+0.1693\alpha+0.3798\beta<\gamma<0.1128+0.3831\alpha+0.4430\beta$ for any positive constants α, β with $\alpha+\beta=1$.

It follows from Theorem 3.2.1 that the equilibrium $x=0$ of the Hopfiled neural network (3.13) is asymptotic stability.

We illustrate 3.3.2 when $\alpha=0.4$, $\beta=0.6$, $0.6042<\delta<0.6739$, and $0.3171<\gamma<0.5318$ by using the fourth-order Runge-Kutta method with step size 0.001 in Figs. 3.3 and 3.4. In these figures, one can see that the Hopfiled neural network is asymptotically stable.

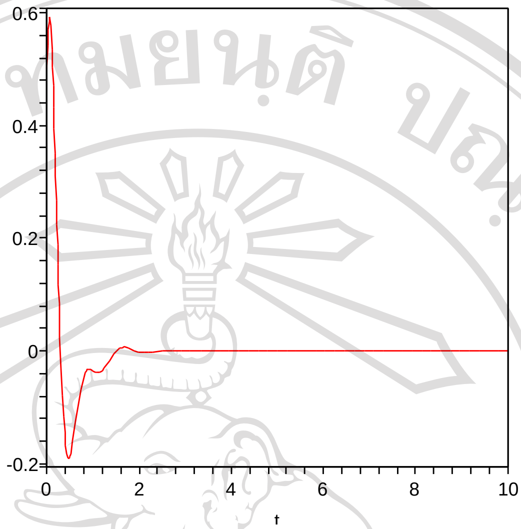

Figure 3.3: the time response of the state x_1 of the system (3.13)

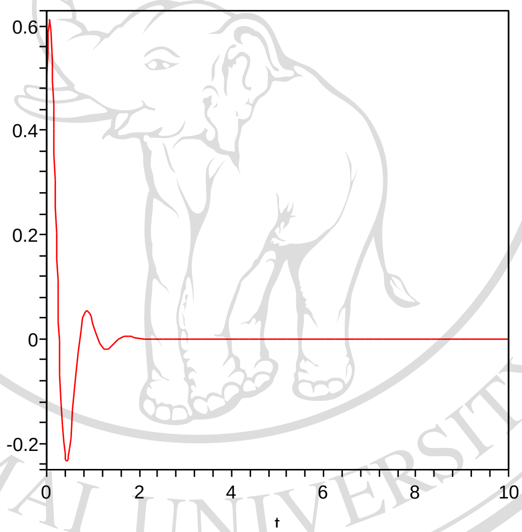

Figure 3.4: the time response of the state x_2 of the system (3.13)