CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some notations and definitions and theorems that will be used in our research.

2.1 Some Basic Concept of Continuous functions

Definition 2.1.1 Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$, and let $c \in A$. We say that f is *continuous at* c if, given any neighborhood $N_\epsilon(f(c))$ of $f(c)$ there exists a neighborhood $N_\delta(c)$ of c such that if x is any point of $A \cap N_\delta(c)$, then $f(x)$ belongs to $N_\epsilon(f(c))$.

Definition 2.1.2 Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$, If $B \subset A$, we say that f is *continuous onB* if f is continuous at every point of B .

Theorem 2.1.3 *Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$, and let $c \in A$. Then the following conditions are equivalent.*
(i) f is continuous at c; that is, given any neighborhood $N_\epsilon(f(c))$ of $f(c)$ there exists a neighborhood $N_\delta(c)$ of c such that if x is any point of $A \cap N_\delta(c)$, then $f(x)$ belongs to $N_\epsilon(f(c))$.
(ii) Given any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$
(iii) If (x_n) is any sequence of real numbers such that $x_n \in A$ for all $n \in \mathbf{N}$ and (x_n) converges to c, then the sequence $(f(x_n))$ converges to $f(c)$.

See [4] for more details.

Definition 2.1.4 Let $A \subseteq \mathbf{R}$ and let $f : A \to \mathbf{R}$ is said to be *bounded on* A if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$.

Theorem 2.1.5 (Boundedness Theorem) *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbf{R}$ be continuous on I. Then f is bounded on I*

See [4] for more details.

Definition 2.1.6 Let $A \subseteq \mathbf{R}$, let $f : A \to \mathbf{R}$. We say that f is *uniformly continuous* on A if for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $x, u \in A$ are any numbers satisfying $|x - u| < \delta(\epsilon)$, then $|f(x) - f(u)| < \epsilon$.

Theorem 2.1.7 (Uniform Continuty Theorem) *Let I be a closed bounded interval and let $f : I \to \mathbf{R}$ be continuous on I. Then f is uniformly continuous on I*

See [4] for more details.

Definition 2.1.8 Let $A \subseteq \mathbf{R}$ and let $f : A \to \mathbf{R}$. If there exists a constant $M > 0$ such that

$$|f(x) - f(u)| \leq M|x - u|$$

for all $x, u \in A$, then f is said to be a *Lipschitz function* (or to satisfy a *Lipschitz condition*) on A.

Theorem 2.1.9 (Rolle's Theorem) *Suppose that f is continuous on a closed interval $I := [a, b]$, that the deivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.*

See [4] for more details.

2.2 Some Concepts from Functional Analysis

2.2.1 Metric Space

Definition 2.2.1 A *metric space* is a pair (X, d),where X is a set and d a metric on X, that is $d : X \times X \longrightarrow \mathbb{R}$ is a function such that the following three conditions are satisfied by all x, y and z in X;

(i) $d(x, x) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

Definition 2.2.2 Let (X, d) be a metric space. A sequence (x_n) in X is said to *converges to* $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$. Note that, whenever the limit exists, it is unique.

Definition 2.2.3 A sequence (x_n) in a metric space $X = (X, d)$ is said to be *Cauchy sequence* if every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that
$d(x_m, x_n) < \epsilon$ for every $m, n > N$.
The space X is said to be *complete* if every Cauchy sequence in X converges.

Theorem 2.2.4 *Every convergent sequence in a metric space is a Cauchy sequence.*

See [10] for more details.

2.2.2 Normed Space

Definition 2.2.5 Let X be a vector space. A norm on X is a real-valued function $\| \cdot \|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:
(i) $\| x \| \geq 0$ and $\| x \| = 0$ if and only if $x = 0$.
(ii) $\| \alpha x \| = | \alpha | \| x \|$
(iii) $\| x + y \| \leq \| x \| + \| y \|$
The ordered pair $(X, \| \cdot \|)$ is called a *normed space*. When there is no danger of confusion, it is customary to use the same symbol, such as X,to denote *the normed space*.

Definition 2.2.6 A complete normed space is said to be *Banach space*.

Example 1 Space $C[a, b]$. This space is a Banach space with norm given by

$$\|f\| = \max_{x \in J} |f(x)|$$

where $J = [a, b]$.

2.2.3 Space of Continuous functions

Let Ω be open bounded subset of $\mathbf{R}^n$

Definition 2.2.7 $C^0(\overline{\Omega})$=$\{f \in C^0(\Omega)$:f is bounded and uniformly continuous in $\Omega\}$ with the norm

$$\|f\|_{C^0(\overline{\Omega})} = \sup_{x\in\Omega} |f(x)|$$

If $k \in \mathbf{N}$, $C^k(\Omega)$ is the set of functions $f \in C^0(\Omega)$ whose derivatives of order $\leq k$ exist and are continuous. For $f \in C^k(\Omega)$ the notation

$$D^\alpha f = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \quad , \quad D_j = \frac{\partial}{\partial x_j}$$

for $|\alpha| \leq k$ where $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

Definition 2.2.8 $C^k(\overline{\Omega})$=$\{f \in C^k(\Omega)$:$D^\alpha f \in C^0(\overline{\Omega})$, $|\alpha| \leq k\}$ with the norm

$$\|f\|_{C^k(\overline{\Omega})} = \max_{0\leq|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha f(x)|$$

Note that $C^k(\overline{\Omega})$ is a Banach space.

2.2.4 Sequences of functions

Consider a sequence of functions

$$f_n : X \to Y \quad (n = 1, 2, \ldots),$$

where (Y, σ) is matric space. For the moment the space X need not carry a metric.

Definition 2.2.9 (Pointwise Convergence) *Suppose that there is a function $f : X \to Y$ such that, for each x in X,*

$$\sigma(f_n(x), f(x)) \to 0$$

as $n \to \infty$. We then say that (f_n) converges to f on X, or, more specially,(f_n) converges pointwise to f on X

Definition 2.2.10 (Uniform Convergence) *Suppose that $f_n : X \to Y$ is a sequence of functions on set X to a metric space (Y, σ). The sequence (f_n) is said to converges uniformly to $f : X \to Y$ if*

$$\sup_{x \in X} \sigma(f_n(x), f(x)) \to 0 \quad as \quad n \to \infty. \tag{2.1}$$

The $\epsilon - N$ criterion for the condition (2.1) is often useful. Convergence for each separate x in X means that, given $\epsilon > 0$, there exists $N = N(\epsilon, x)$ such that $\sigma(f_n(x), f(x)) < \epsilon$ for all $n \geq N$. The convergence is uniform if and only if it is possible to choose $N = N(\epsilon)$, independent of x, such that

$$\sigma(f_n(x), f(x)) < \epsilon \quad , \forall n \geq N, \forall x \in X.$$

We note that in the space $B(X, Y)$ of bounded functions $f : X \to (Y, \sigma)$ was defined by

$$\rho(f, g) = \sup_{x \in X} \sigma(f(x), g(x)).$$

Thus, for functions in $B(X, Y)$, uniform convergence is identical with convergence in the metric space $(B(X, Y), \rho)$. If, in particular, Y is $\mathbf{R}^1$, then

$$\rho(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

and, for bounded real valued functions, uniform convergence is the same as convergence in the metric space $(B(X), \rho)$.

Theorem 2.2.11 (General Principle of Uniform Convergence)
A necessary and sufficient condition for the sequence (f_n) of real or complex valued functions on set X to be uniformly convergent is that, given $\epsilon > 0$, there exists N_0 such that

$$\sup_{x \in X} |f_m(x) - f_n(x)| < \epsilon \quad , m, n \geq N_0$$

See [5] for more details.

2.3 Differential Equations

Definition 2.3.1 An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a *differential equation (DE).*

Example 2 For examples of differential equations we list the following:

$$\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0, \tag{2.2}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t, \tag{2.3}$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v, \tag{2.4}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{2.5}$$

Definition 2.3.2 A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an *ordinary differential equation (ODE).*

Example 3 Equations (2.2) and (2.3) are ordinary differential equations. In equation (2.2) the variable x is the single independent variable, and y is a dependent variable. In equation (2.3) the independent variable is t, whereas x is dependent.

Definition 2.3.3 A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a *partial differential equation (PDE).*

Example 4 Equations (2.4) and (2.5) are partial differential equations. In equation (2.4) the variables s and t is are independent variables, and v is a dependent variable. In equation (2.5) there are three independent variables: x, y, and z in this equation u is dependent.

Definition 2.3.4 The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Example 5 The ordinary differential equation (2.2) is of the second order, since the highest derivative involved is a second derivative. Equation (2.3) is an ordinary differential equation of the fourth order. The partial differential equations (2.4) and (2.5) are of the first and second orders, respectively.

Definition 2.3.5 A linear differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x),$$

where a_0 is not identically zero.

Example 6 The following ordinary differential equations are both linear. In each case y is the dependent variable.

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0,$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x.$$

Definition 2.3.6 A nonlinear differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in, the form

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + ... + a_{n-1}\frac{dy}{dx} + a_n y = b(x),$$

where $a_0, a_1, ..., a_n$ are functions of $x, y, y^{(1)}, ..., y^{(n-1)}$.

Example 7 The following ordinary differential equations are all nonlinear:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0,$$

$$\frac{d^2y}{dx^2} + 5(\frac{dy}{dx})^3 + 6y = 0.$$

2.4 Initial and Boundary Value Problems

2.4.1 Initial Value Problems

We are often interested in solving a differential equation subject to prescribed side conditions. That is the conditions that are imposed on the unknown solution $y = y(x)$ or its derivatives. On some interval I containing x_0, the problem

$$\frac{d^n y}{dx^n} = f(x, y, y', ..., y^{(n-1)}),$$

$$y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, ..., y_{n-1}$ are arbitrarily specified real constants, is called an *initial value problem (IVP)*. For example,

$$y' = \sin x, \quad y(\pi) = 1,$$

and

$$y'' + 5y' = e^{2x}, \quad y(\pi) = 1, y'(\pi) = 0$$

are first and second-order initial value problems, respectively.

2.4.2 Boundary Value Problems

Another type of problem consists of solving a differential equation of order two or higher in which the dependent variable y or derivatives are specified at different points. A problem such as

$$y'' + p(x)y' + q(x)y = f(x) \ \ , \ \ x \in (a, b),$$

that satisfies the boundary conditions

$$a_{11}y(a) + a_{12}y'(a) + b_{11}y(b) + b_{12}y'(b) = c_1,$$

$$a_{21}y(a) + a_{22}y'(a) + b_{21}y(b) + b_{22}y'(b) = c_2. \tag{2.6}$$

is called a *boundary value problem (BVP)*. The condition in (2.6) are examples of linear boundary conditions. When $c_1 = c_2 = 0$, we say that the boundary conditions are *homogeneous*; otherwise, we refer to them as *nonhomogeneous*.

There are special types of boundary conditions that occur frequently in applications. They are

(i) Separated: $a_1y(a) + a_2y'(a) = c_1 \quad , \quad b_1y(b) + b_2y'(b) = c_2.$

(ii) Dirichlet: $y(a) = c_1 \quad , \quad y(b) = c_2.$

(iii) Neumann: $y'(a) = c_1 \quad , \quad y'(b) = c_2.$

(iv) Separated: $y(-T) = y(T) \quad , \quad y'(-T) = y'(T).$

Notice that the Dirichlet and Neumann conditions are special types of separated boundary conditions.

For boundary value problems involving third or higher-order equations, one usually specifies the same number of boundary conditions as the order of the equation. When the order is even, these boundary conditions are often separated with half the conditions at the endpoint $x = a$ and half at $x = b$.

A solution of the previous problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through the two point (a, y_0) and (b, y_1). For example,

$$y'' + 2y' = e^x, \quad y(0) = 1, y(\pi) = 1.$$

$$y^{(4)} + \lambda y = f(x), \quad y(0) = y''(0) = 0, y(L) = y''(L) = 0.$$

The second example is a fourth order eigenvalue problem that occur in continuum mechanics and, in particular, in vibration and bending problems. Where y is transverse displacement of a uniform elastic beam length L. If the beam has linear density ρ, cross-sectional moment of inertia I, Young's modulus of elasticity E, then $\lambda = K\rho/EI$, where K is a constant to be determined. And $f(x)$ is load on the beam with the simply supported at its ends.

2.5 Existence and Uniqueness Theorems

2.5.1 First-order systems

Consider the following system of n first order differential equations:

$$y_1' = f_1(x, y_1, y_2, ..., y_n)$$

$$y_2' = f_2(x, y_1, y_2, ..., y_n)$$
$$\vdots \qquad \vdots \tag{2.7}$$
$$y_n' = f_n(x, y_1, y_2, ..., y_n)$$

where y_i, $(i = 1, 2, ..., n)$ are real valued functions of the independent variable x and f_i, $(i = 1, 2, ..., n)$ are real valued functions of $x, y_1, ..., y_n$.

Theorem 2.5.1 *Let the functions $f_i(x, y_1, ..., y_n), (i = 1, 2, ..., n)$ be defined for x in an interval I and the y_r in regions D_r of the complex plane, where each D_r has at least one interior point. Let a be a point in I and, for each r, b_r be an interior point of D_r. Let there be a closed bounded sub-interval I_0 of I containing a and, for each r, a disk E_r of the form $|y_r - b_r| \leq \beta(\beta > 0)$ lying in D_r such that the following two conditions are satisfied.*
(i) For each i and any fixed values of the y_r in E_r, $f_i(x, y_1, ..., y_n)$ is a continuous function of x in I_0.
(ii) For $1 \leq i \leq n$, for all x in I_0, and for all y_r, y_r' in E_r,

$$|f_i(x, y_1', ..., y_n') - f_i(x, y_1, ..., y_n)| \leq A(|y_1' - y_1| + ... + |y_n' - y_n|), \tag{2.8}$$

where A is independent of x and the y_r and y_r'.
Then, if a is not an end-point of I_0, there is a number $h > 0$ such that the system (2.7) has a solution $\{\phi_i(x)\}(i = 1, 2, ..., n)$ which is valid for $|x - a| \leq h$ and satisfies the condition

$$\phi_i(a) = b_i \quad (i = 1, 2, ..., n). \tag{2.9}$$

Further, $\{\phi_i(x)\}$ is unique in the sense that there is no second solution $\{\phi_i(x)\}$ of (2.7) which is valid in sub-interval of $|x - a| \leq h$ and satisfies (2.9). If a is the left-or right-hand end-point of I_0, the result still holds, except that the interval $|x - a| \leq h$ is replaced by $a \leq x \leq a + h$ or $a - h \leq x \leq a$, respectively.

See [9] for more details.

Theorem 2.5.2 *Let $\partial f_i/\partial y_r$ exist for $r = 1, 2, ..., n$ and be continuous for all x in I_0, y_1 in $E_1, ...,$ y_n in E_n, where E_i is the disk $\{y_i| \ \ |y_i - b_i| \leq \beta\}$. Then (2.8) is satisfied with some constant A.*

See [9] for more details.

2.5.2 Differential equations of order n

Now, consider the nth order differential equations.

$$y^{(n)}(x) = f(x, y(x), y'(x), ..., y^{(n-1)}(x)). \tag{2.10}$$

The first-order system is following one:

$$\begin{aligned} y_0'(x) &= y_1(x) \\ y_1'(x) &= y_2(x) \\ &\vdots \\ y_{n-2}'(x) &= y_{n-1}(x) \\ y_{n-1}'(x) &= f(x, y_0(x), y_1(x), ..., y_{(n-1)}(x)). \end{aligned} \tag{2.11}$$

If $\{\phi_i(x)\}(i = 0, 1, ..., n-1)$ is a solution of (2.11), the first $n-1$ equations in (2.11) show in turn that $\phi_i(x) = \phi_0^{(i)}(x)(i = 1, 2, ..., n-1)$, while the last one then shows that

$$\phi_0^{(n)}(x) = f(x, \phi_0(x), \phi_0'(x), ..., \phi_0^{(n-1)}(x)).$$

Hence $\phi_0(x)$ is solution of (2.10). Thus any solution $\{\phi_i(x)\}$ of (2.11) give rise in this way to a solution $\phi(x)$ of (2.10) and the connection between the two solutions is

$$\phi_i(x) = \phi_0^{(i)}(x)(i = 0, 1, ..., n-1).$$

Conversely, if $\phi(x)$ is a solution of (2.10), then clearly $\{\phi_i(x)\}(i = 0, 1, ..., n-1)$ is a solution of (2.11). We can therefore apply Theorem 2.3.1 to the system (2.11) to obtain the corresponding theorem for (2.10) as follows.

Theorem 2.5.3 *Let the functions $f(x, y_0, ..., y_{n-1})$ be defined for x in an interval I and the y_r in regions D_r of the complex plane, where each D_r has at least one interior point. Let a be a point in I and, for each r, b_r be an interior point of D_r. Let there be a closed bounded sub-interval I_0 of I containing a and, for each r, a* *disk E_r of the form $|y_r - b_r| \leq \beta(\beta > 0)$ lying in D_r such that the following two conditions are satisfied.*

(i) For any fixed values of the y_r in E_r, $f(x, y_0, ..., y_{n-1})$ is a continuous function of x in I_0.

(ii) For all x in I_0, and for all y_r, y'_r in E_r,

$$|f(x, y'_0, ..., y'_{n-1}) - f(x, y_0, ..., y_{n-1})| \leq A(|y'_0 - y_0| + ... + |y'_{n-1} - y_{n-1}|),$$

where A is independent of x and the y_r and y'_r.

Then, if a is not an end-point of I_0, there is a number $h > 0$ such that the equation (2.10) has a unique solution $\{\phi_i(x)\}$ which is valid for $|x - a| \leq h$ and satisfies the condition

$$\phi_i(a) = b_i \quad (i = 0, 1, ..., n-1). \tag{2.12}$$

If a is the left-or right-hand end-point of I_0, the result still holds, except that the interval $|x - a| \leq h$ is replaced by $a \leq x \leq a + h$ or $a - h \leq x \leq a$, respectively.

See [9] for more details.

2.5.3 Homogeneous linear differential equations

After the previous general theory, we consider the linear differential equation and start with the homogeneous equation

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = 0 \ , \tag{2.13}$$

where the assumptions about the coefficients $a_r(x)$ will always be that they are continuous in an interval I and $a_0(x) \neq 0$ for any x in I.

Property 2.5.4 *If $\phi_1(x)$ and $\phi_2(x)$ are two solutions of (2.13), then $c_1\phi_1(x) + c_2\phi_2(x)$, where c_1 and c_2 are constants, is also a solution.*

Property 2.5.5 *If $\phi(x)$ is a solution of (2.13) such that, for some point x_0 in I, $\phi^{(i)}(x_0) = 0 \ \ (i = 0, 1, ..., n-1)$, then $\phi(x) = 0$ for all x in I.*

Definition 2.5.6 A finite set of functions $f_m(x)$ $(m = 1, 2, ..., M)$ is defined in an interval J is said to *linearly independent* in J if the equation

$$\alpha_1 f_1(x) + \cdots + \alpha_M f_M(x) = 0 \quad , \tag{2.14}$$

where the r_m are complex constants, holds for all x in J only when $\alpha_1 = \ldots = \alpha_M = 0$. If the equation is possible for values of the α_m not all zero, then the set is said to be *linearly dependent* in J.

Definition 2.5.7 A set of n solution of (2.13) is said to be a *fundamental set* for (2.13) if it is linearly independent in I.

Lemma 2.5.8 *Let b_{ij} $(i, j = 1, 2, ..., n)$ be any real or complex numbers and let a be any point in I. For each j, let $\phi_j(x)$ be the solution of (2.13) which satisfies the initial conditions*

$$\phi_j^{(i-1)}(a) = b_{(ij)} \quad (i = 1, 2, ..., n). \tag{2.15}$$

Then a necessary and sufficient condition that the $\phi_j(x)$ form a fundamental set for (2.13) is that $det(b_{(ij)}) \neq 0$.

See [9] for more details.

Theorem 2.5.9 *Let $\phi(x)$ be any solution of (2.13). Then there are unique constants c_j such that*

$$\phi(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x) \quad (i = 1, 2, ..., n) \tag{2.16}$$

for all x in I.

See [9] for more details.

2.5.4 The Wronskian

Definition 2.5.10 Let the functions $f_m(x)$ $(m = 1, 2, ..., M)$ be defined in an interval J, each having $M-1$ derivatives in J. Then the $M \times M$ determinant

$$\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_M(x) \\ f_1'(x) & f_2'(x) & \cdots & f_M'(x) \\ \vdots & & & \vdots \\ f_1^{(M-1)}(x) & f_2^{(M-1)}(x) & \cdots & f_M^{(M-1)}(x) \end{vmatrix}$$

is called *Wronskian* of $f_1(x), \ldots, f_M(x)$ and is denoted by $W(f_1, \ldots, f_M)(x)$.

Theorem 2.5.11 *Let $\phi(x)$ be any solution of (2.13). Then there are unique constants c_j such that*

$$\phi(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x) \quad (i = 1, 2, ..., n) \tag{2.17}$$

for all x in I.

See [9] for more details.

Theorem 2.5.12 *Let $\phi_1(x), \ldots, \phi_n(x)$ be n solution of (2.13). Then $W(\phi_1, \ldots, \phi_n)(x)$ is either not zero for any x in I or zero for x in I. The first case occurs when $\phi_1(x), \ldots, \phi_n(x)$ form a fundamental set for (2.13) and the second when they do not.*

See [9] for more details.

2.5.5 Inhomogeneous linear differential equations

Consider the inhomogeneous linear differential equations

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_n(x)y(x) = b(x) \quad , \tag{2.18}$$

where the assumptions about the coefficients $a_r(x)$ will always be that they are continuous in an interval I, $a_0(x) \neq 0$ for any x in I and $b(x)$ is assumed to be continuous in I.

Property 2.5.13 *Let $\psi_1(x)$ and $\psi_2(x)$ be two solutions of (2.18). Then $\psi_1(x) - \psi_2(x)$ is a solution of the corresponding homogeneous equation (2.13).*

Theorem 2.5.14 *Let $\phi_1(x), \ldots, \phi_n(x)$ form a fundamental set for (2.13) and let $\psi_0(x)$ be a solution of (2.18). Then, If $\psi(x)$ is any solution of (2.18), there are unique constants $c_1, \ldots, c_n$ such that*

$$\psi(x) = c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \psi_0(x). \tag{2.19}$$

See [9] for more details.

2.6 Variation of Parameters for Inhomogeneous linear second order differential equation

In this section we present a general method, called variation of parameters, for finding a particular solution. This method applies even when the coefficients of the differential equation are function of x, provided we know a fundamental solution set for the corresponding homogeneous linear equation.

Consider the inhomogeneous linear second order differential equation

$$L[y](x) := y'' + p(x)y' + q(x)y = g(x), \tag{2.20}$$

where the coefficient of y'' is taken to be 1, and let $\{y_1(x), y_2(x)\}$ be a fundamental solution set for the corresponding homogeneous equation

$$L[y] = 0 \quad .$$

Then we know that the solutions to this homogeneous equation are given by

$$y_h(x) = c_1y_1(x) + c_2y_2(x) \quad , \tag{2.21}$$

where c_1 and c_2 are constants. To find a particular solution to the nonhomogeneous equation, the idea behind variation of parameters is to replace the constants in (2.21) by functions of x. That is, we seek a solution of (2.20) of the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \quad . \tag{2.22}$$

Because we have introduced two unknown functions, $v_1(x)$ and $v_2(x)$, it is reasonable to expect that we can impose two equations on these functions. Naturally, one of these equations should come from (2.20). Let's therefore plug $y_p(x)$ given by (2.22) into (2.20). To accomplish this, we must first compute $y_p'(x)$ and $y_p''(x)$. From (2.22), we obtain

$$y_p' = (v_1'y_1 + v_2'y_2) + (v_1y_1' + v_2y_2').$$

To simplify the computation and to avoid second order derivatives for the unknowns v_1, v_2 in the expression for y_p'', we impose the requirement

$$v_1'y_1 + v_2'y_2 = 0 \quad ; \tag{2.23}$$

thus the formula for y_p' becomes

$$y_p' = v_1y_1' + v_2y_2' \quad . \tag{2.24}$$

and so

$$y_p'' = v_1'y_1' + v_2'y_2' + v_1'y_1'' + v_2'y_2'' \quad . \tag{2.25}$$

Now, substituting y_p, y_p' and y_p'', as given in (2.22),(2.24) and (2.25) into (2.20), we find

$$\begin{aligned} g &= L[y_p] \\ &= (v_1'y_1' + v_2'y_2' + v_1'y_1'' + v_2'y_2'') + p(v_1y_1' + v_2y_2') + q(v_1y_1 + v_2y_2) \\ &= (v_1'y_1' + v_2'y_2') + v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) \\ &= (v_1'y_1' + v_2'y_2') + v_1L[y_1] + v_2L[y_2] \quad . \end{aligned} \tag{2.26}$$

Since y_1 and y_2 are solutions to the homogeneous equation, we have $L[y_1] = L[y_2] = 0$. Thus (2.26) reduce to

$$v_1'y_1' + v_2'y_2' = g \quad . \tag{2.27}$$

To summarize, if we can find v_1 and v_2 that satisfy both (2.23) and (2.27), that is,

$$v_1'y_1 + v_2'y_2 = 0 \quad , v_1'y_1' + v_2'y_2' = g \quad , \tag{2.28}$$

then y_p given by (2.22) will be a particular solution to (2.20). To determine v_1 and v_2, we first solve the linear system (2.28) for v_1' and v_2'. Algebraic manipulation of Cramer's rule immediately gives

$$v_1'(x) = \frac{-g(x)y_2(x)}{W[y_1, y_2](x)} \quad , \quad v_2'(x) = \frac{g(x)y_1(x)}{W[y_1, y_2](x)}$$

where $W[y_1, y_2](x)$, which occurs in the denominator, is the Wronskian of $y_1(x)$ and $y_2(x)$. Notice that this Wronskian is never zero because $\{y_1(x), y_2(x)\}$ is a fundamental solution set. Upon

$$v_1(x) = \int \frac{-y_2(s)g(s)}{W[y_1, y_2](s)} ds \quad , \quad v_2(x) = \int \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds \tag{2.29}$$

2.7 Green's functions

In this section we derive an integral representation for the solution to a nonhomogeneous boundary value problem. Namely, we show that the solution can be expressed in the form

$$y(x) = \int_a^b G(x, s)f(s)ds, \tag{2.30}$$

where the kernel function, $G(x, s)$, is called a *Green's function*, or an *influence function*. This representation is known to exist for very general problem. However, we will consider only the regular Sturm-Liouvill boundary value problems

$$L[y](x) = -f(x) \quad , \quad x \in (a, b) \tag{2.31}$$

$$a_1y(a) + a_2y'(a) = 0 \tag{2.32}$$

$$b_1y(b) + b_2y'(b) = 0 \tag{2.33}$$

where operator defined by

$$L = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \tag{2.34}$$

where a_1, a_2, b_1, b_2 are real and $p(x), p'(x), q(x)$ are continuous on $[a, b]$ with $p(x) > 0$ on $[a, b]$. While it is not necessary, it is customary in the present context to write $-f(x)$, rather than $f(x)$, as the nonhomogeneous term in equation (2.31).

Throughout this section we assume that the corresponding homogeneous problem ($f(x) \equiv 0$) has only the trivial solution, or equivalently, we assume that $\lambda = 0$ is not an eigenvalue. We will follow from the *Fredholm alternative* following.

Fredholm alternative, If the corresponding homogeneous problem ($f(x) \equiv 0$) of problem (2.31)-(2.33) has only the trivial solution and $\lambda = 0$ is not an eigenvalue then the nonhomogeneous problem (2.31)-(2.33) has a solution. This solution is unique, since the corresponding homogeneous problem has a unique solution.

We will use the method of variation of parameters to obtain an integral representation for the solution to (2.31)-(2.33). To begin, choose two nontrivial solutions y_1 and y_2 to the homogeneous equation $L[y] = 0$. Pick y_1 so that it satisfies first boundary condition (2.32) and y_2 it satisfies second boundary condition (2.33). The existence of y_1 and y_2 follows from the existence and uniqueness theorem for initial value problems, since one can choose initial conditions at a (respectively b) so that the boundary condition (2.32) (respectively (2.33)) is satisfied and the solution is nontrivial.

Before we can use y_1 and y_2 with the method of variation of parameters, we must verify that y_1 and y_2 are linearly independent. Note that if y_1 and y_2 were linearly dependent, then one would be a constant multiple of the other, say $y_1 = cy_2$. Since y_2 satisfies (2.33), it follows that y_1 also satisfies (2.33).

To apply the formulas from the variation of parameters method, we first write the nonhomogeneous equation (2.31) in the standard form

$$y'' + \frac{p'}{p}y' + \frac{q}{p}y = -\frac{f}{p}$$

A particular solution to this equation is then given by

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \tag{2.35}$$

where

$$c_1'(x) = \frac{f(x)y_2(x)}{p(x)W[y_1, y_2](x)}, c_2'(x) = \frac{f(x)y_1(x)}{p(x)W[y_1, y_2](x)} \tag{2.36}$$

and

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2,$$

$W[y_1, y_2]$ is the Wronskian of y_1 and y_2. Since we are free to pick the constants in the antiderivatives for c_1' and c_2', it will turn out to be convenient to choose

$$c_1(x) = \int_x^b \frac{-y_2(s) f(s)}{p(s) W[y_1, y_2](s)} ds \tag{2.37}$$

and

$$c_2(x) = \int_a^x \frac{-y_1(s) f(s)}{p(s) W[y_1, y_2](s)} ds \tag{2.38}$$

Substituting into (2.35), we obtain the following solution

$$\begin{aligned} y(x) &= y_1(x) \int_x^b \frac{-y_2(s) f(s)}{p(s) W[y_1, y_2](s)} ds + y_2(x) \int_a^x \frac{-y_1(s) f(s)}{p(s) W[y_1, y_2](s)} ds \\ &= \int_a^b G(x, s) f(s) ds \end{aligned} \tag{2.39}$$

where

$$G(x, s) = \begin{cases} -y_2(x) \frac{y_1(s)}{p(s) W[y_1, y_2](s)} & , a \le s \le x \\ -y_1(x) \frac{y_2(s)}{p(s) W[y_1, y_2](s)} & , x \le s \le b \end{cases}$$

Using the fact that y_1 and y_2 satisfy the equation $L[y] = 0$, one can show that

$$p(x) W[y_1, y_2](x) = C \quad , \quad x \in [a, b] \tag{2.40}$$

where C is a constant. Hence $G(x, s)$ has the the simpler form

$$G(x, s) = \begin{cases} -y_2(x) y_1(s)/C & , a \le s \le x \\ -y_1(x) y_2(s)/C & , x \le s \le b \end{cases} \tag{2.41}$$

where C is given by equation (2.40). The function $G(x, s)$ is called the **Green's function** for the problem (2.31)-(2.33).