CHAPTER 3

MAIN RESULTS

3.1 Lemmas

The following lemmas play a crucial role in the proofs of our main result.

Let λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 - a\lambda + b$.then

$$\lambda_1 = \frac{a + \sqrt{a^2 - 4b}}{2}, \quad \lambda_2 = \frac{a - \sqrt{a^2 - 4b}}{2} \tag{3.1}$$

from (H2) and (3.1) it is clear that $\lambda_1 \geq 0$ and we have

$$-a\pi^2 - \pi^4 < b \leq \tfrac{a^2}{4}$$

$$0 \leq a^2 - 4b < a^2 - 4(-a\pi^2 - \pi^4) = (a + 2\pi^2)^2$$

$$\sqrt{a^2 - 4b} < a + 2\pi^2$$

$$-2\pi^2 < a - \sqrt{a^2 - 4b}$$

thus we have $\lambda_2 > -\pi^2$

Let $G_i(x, s)(i = 1, 2)$ be the Green's function of the following boundary value problem.

$$u''(x) - \lambda_i u(x) = -g(x), \tag{3.2}$$

$$u(0) = u(1) = 0. \tag{3.3}$$

Lemma 3.1.1 *Green's function of the problem (3.2)-(3.3) satisfy*

$$G_i(x, s) > 0, \forall x, s \in (0, 1), (i = 1, 2)$$

Proof. Set $\omega_i = \sqrt{|\lambda_i|}$.

If $\lambda_i > 0$, then $G_i(x, s)$ of the problem (3.2)-(3.3) is given by

$$G(x, s) = \begin{cases} \frac{\sinh \omega_i x \cdot \sinh \omega_i (1-s)}{\omega_i \sinh \omega_i}, & x \leq s \\ \frac{\sinh \omega_i s \cdot \sinh \omega_i (1-x)}{\omega_i \sinh \omega_i}, & s \leq x \end{cases}$$

If $\lambda_i = 0$, then $G_i(x, s)$ of the problem (3.2)-(3.3) is given by

$$G(x,s) = \begin{cases} x(1-s) \ , & x \le s \\ s(1-x) \ , & s \le x \end{cases}$$

If $-\pi^2 < \lambda_i < 0$, then $G_i(x,s)$ of the problem (3.2)-(3.3) is given by

$$G(x,s) = \begin{cases} \frac{\sin\omega_i x \cdot \sin\omega_i(1-s)}{\omega_i \sin\omega_i} \ , & x \le s \\ \frac{\sin\omega_i s \cdot \sin\omega_i(1-x)}{\omega_i \sin\omega_i} \ , & s \le x \end{cases}$$

Since $x, s \in (0,1)$ and from $G_i(x,s)$, thus $G_i(x,s) > 0 \ \forall x, s \in (0,1)$

Consider problem (3.2) with following boundary condition

$$u(0) = l, u(1) = m. \tag{3.4}$$

The solution of (3.2) and (3.4) is given by

$$u(x) = \int_0^1 G_i(x,s)g(s)ds + l(1-x) + mx \tag{3.5}$$

Lemma 3.1.2 *Let* $\lambda > -\pi^2$ *if u(x) satisfies*

$$u''(x) - \lambda u(x) \ge 0 \ , \ x \in (0,1)$$
$$u(0) \le 0 \ , \ u(1) \le 0$$
then $u(x) \le 0 \ , \ x \in [0,1]$

Proof. Set $g(x) \le 0$ in (3.2) and $l, m \le 0$ in (3.4) and from Lemma 3.1.1 we have $u(x) \le 0$.

Lemma 3.1.3 *Let* $\lambda > -\pi^2$ *if u(x) satisfies*

$$u''(x) - \lambda u(x) \le 0 \ , \ x \in (0,1)$$
$$u(0) \ge 0 \ , \ u(1) \ge 0$$
then $u(x) \ge 0 \ , \ x \in [0,1]$

Proof. Set $g(x) \ge 0$ in (3.2) and $l, m \ge 0$ in (3.4) and from Lemma 3.1.1 we have $u(x) \ge 0$.

Now we use property of Green's function (Lemma 3.1.1) to prove maximum principle L from (1.7)

$$L : F \to C[0,1]$$

where F defined by

$$F = \{u \in C^4[0,1] | u(0) \ge 0, u(1) \ge 0, u''(0) \le 0, u''(1) \le 0\}$$

Lemma 3.1.4 *If $u \in F$ satisfies $Lu \geq 0$, then $u(x) \geq 0 \quad , \quad x \in [0,1]$*

Proof. Consider the boundary value problem

$Lu = u^{(4)}(x) - au''(x) + bu(x) = (D^2 - \lambda_2)(D^2 - \lambda_1)u(x) \geq 0$

let $y(x) = (D^2 - \lambda_1)u(x) = u''(x) - \lambda_1 u(x)$

then $(D^2 - \lambda_2)y(x) \geq 0$ i.e. $y''(x) - \lambda_2 y(x) \geq 0$

since $\lambda_2 > -\pi^2$ by Lemma 3.1.2 we have

$$y(x) \leq 0 \quad , \quad x \in [0,1]$$

i.e. $u''(x) - \lambda_1 u(x) \leq 0$

since $\lambda_1 \geq 0$ by Lemma 3.1.3 we have

$$u(x) \leq 0 \quad , \quad x \in [0,1]$$

Lemma 3.1.5 [8] Given $a, b \in R$, the problem

$$u^{(4)}(x) - au''(x) + bu(x) = 0 \tag{3.6}$$

$$u(0) = u(1) = u''(0) = u''(1) = 0 \tag{3.7}$$

has a non-trivial solution if and only if

$$\frac{a}{(k\pi)^4} + \frac{b}{(k\pi)^2} + 1 = 0$$

for some $k \in N$

3.2 Main Theorems

Definition 3.2.1 *We say $\beta \in C^4[0,1]$ a lower solution for (1.2)−(1.3), if β satisfies*

$$\beta^{(4)}(x) \leq f(x, \beta(x), \beta''(x)), \quad x \in (0,1)$$

$$\beta(0) \leq 0 \quad , \quad \beta(1) \leq 0$$

$$\beta''(0) \geq 0 \quad , \quad \beta''(1) \geq 0$$

Definition 3.2.2 *We say $\alpha \in C^4[0,1]$ an upper solution for $(1.2) - (1.3)$, if α satisfies*

$$\alpha^{(4)}(x) \geq f(x, \alpha(x), \alpha''(x)), \quad x \in (0,1)$$

$$\alpha(0) \geq 0 \quad , \quad \alpha(1) \geq 0$$

$$\alpha''(0) \leq 0 \quad , \quad \alpha''(1) \leq 0$$

Theorem 3.2.3 *Suppose there exist β and α, lower and upper solutions, respectively, for the problem* $(1.2)-(1.3)$ *which satisfy*

$$\beta \leq \alpha \ \ , \ \ \beta'' + \lambda(\alpha - \beta) \geq \alpha'', \tag{3.8}$$

f satisfies (H1) and following the inequality

$$f(x, u_2, v) - f(x, u_1, v) \geq -b(u_2 - u_1), \tag{3.9}$$

for $\beta \leq u_1 \leq u_2 \leq \alpha \ \ , \ \ v \in R \ \ , x \in [0, 1]$.

$$f(x, u, v_2) - f(x, u, v_1) \leq a(v_2 - v_1), \tag{3.10}$$

for $v_2 + \lambda(\alpha - \beta) \geq v_1 \ \ , \ \ \alpha'' - \lambda(\alpha - \beta) \leq v_1, v_2 \leq \beta'' + \lambda(\alpha - \beta) \ \ , \ \ u \in R \ \ , x \in [0, 1]$ where a and b satisfy (H2), $\lambda_1 = \frac{a+\sqrt{a^2-4b}}{2} \ \ , \ \ \lambda_2 = \frac{a-\sqrt{a^2-4b}}{2}$ then there exist two monotone sequence $\{\beta_n\}$ and $\{\alpha_n\} \in C^4[0, 1]$, non increasing and non decreasing, respectively, with $\beta_0 = \beta$ and $\alpha_0 = \alpha$, which converge uniformly to the extremal solution in $[\beta, \alpha]$ of the problem $(1.2)-(1.3)$

Proof. Consider the problem

$$Lu = u^{(4)}(x) - au''(x) + bu(x) = f_1(x, \eta(x), \eta''(x)), \ \ x \in (0, 1) \tag{3.11}$$

with boundary condition (1.3), $\eta \in C^2[0, 1]$.

Since a, b satisfy (H2). By lemma (3.1.5) and Fredholm alternative the problem (3.11) with boundary condition (1.3) has a unique solution u.

Define $T : C^2[0, 1] \to C^4[0, 1]$ by

$$T\eta = u \tag{3.12}$$

we divide the proof into three steps.

Step 1. To show that

$$TG \subseteq G \tag{3.13}$$

Where $G = \{\eta \in C^2[0, 1] | \beta \leq \eta \leq \alpha, \alpha'' + \lambda_1(\alpha - \beta) \leq \eta'' \leq \beta'' + \lambda_1(\alpha - \beta)\}$ is a nonempty bounded closed subset in $C^2[0, 1]$.

In fact, for $\xi \in G$, set $\omega = T\xi$. From the definitions of α and G we have that

$$L\alpha(x) \geq f(x, \alpha(x), \alpha''(x)) - a\alpha''(x) + b\alpha(x) \tag{3.14}$$

$$L\omega(x) \geq f(x, \xi(x), \xi''(x)) - a\xi''(x) + b\xi(x) \tag{3.15}$$

then

$$L(\alpha - \omega)(x) \geq f(x, \alpha(x), \alpha''(x)) - f(x, \xi(x), \xi''(x)) - a(\alpha - \xi)''(x) + b(\alpha - \xi)(x) \tag{3.16}$$

from (3.9) and (3.10) we have

$$f(x, \alpha(x), v) - f(x, \xi(x), v) \geq -b(\alpha - \xi)(x)$$

$$f(x, u, \alpha''(x)) - f(x, u, \xi''(x)) \geq a(\alpha - \xi)''(x)$$

i.e.

$$L(\alpha - \omega)(x) \geq 0 \tag{3.17}$$

from α is upper solution we have

$$\alpha(0) \geq 0 \;\;,\;\; \alpha(1) \geq 0 \;\;,\;\; \alpha''(0) \leq 0 \;\;,\;\; \alpha''(1) \leq 0$$

from ω is solution of (3.11) we have

$$\omega(0) = 0 \;\;,\;\; \omega(1) = 0 \;\;,\;\; \omega''(0) = 0 \;\;,\;\; \omega''(1) = 0$$

i.e.

$$(\alpha - \omega)(0) \geq 0 \;\;,\;\; (\alpha - \omega)(1) \geq 0 \tag{3.18}$$

$$(\alpha - \omega)''(0) \leq 0 \;\;,\;\; (\alpha - \omega)''(1) \leq 0 \tag{3.19}$$

By lemma (3.1.4) we have $(\alpha - \omega)(x) \geq 0$, i.e. $\alpha(x) \geq \omega(x)$ for $x \in [0, 1]$.

From (3.17) we have

$$(D^2 - \lambda_2)(D^2 - \lambda_1)(\alpha - \omega)(x) \geq 0$$

let $y(x) = (D^2 - \lambda_1)(\alpha - \omega)(x) = (\alpha - \omega)''(x) - \lambda_1(\alpha - \omega)(x)$ then $(D^2 - \lambda_2)y(x) \geq 0$ i.e. $y''(x) - \lambda_2 y(x) \geq 0$. Since $\lambda_1 \geq 0, \lambda_2 > -\pi^2$ and $(\alpha - \omega)(x) \in F$, by Lemma (3.1.2) we obtain

$$y(x) = (\alpha - \omega)''(x) - \lambda_1(\alpha - \omega)(x) \le 0 \quad , \quad x \in [0,1]$$

i.e.

$$\alpha''(x) - \lambda_1(\alpha - \omega)(x) \le \omega''(x) \quad , \quad x \in [0,1]$$

$$\alpha''(x) - \lambda_1(\alpha - \beta)(x) \le \omega''(x) \quad , \quad x \in [0,1]$$

Similarly we can prove that $\beta(x) \le \omega(x)$ for $x \in [0,1]$.

$$L\beta(x) \le f(x, \beta(x), \beta''(x)) - a\beta''(x) + b\beta(x) \tag{3.20}$$

then

$$L(\omega - \beta)(x) \ge f(x, \xi(x), \xi''(x)) - f(x, \beta(x), \beta''(x)) - a(\xi - \beta)''(x) + b(\xi - \beta)(x) \tag{3.21}$$

from (3.9) and (3.10) we have

$$f(x, \xi(x), v) - f(x, \beta(x), v) \ge -b(\xi - \beta)(x)$$

$$f(x, u, \xi''(x)) - f(x, u, \beta''(x)) \ge a(\xi - \beta)''(x)$$

i.e.

$$L(\omega - \beta)(x) \ge 0 \tag{3.22}$$

from β is lower solution we have

$$\beta(0) \le 0 \quad , \quad \beta(1) \le 0 \quad , \quad \beta''(0) \ge 0 \quad , \quad \beta''(1) \ge 0$$

then we obtain

$$(\omega - \beta)(0) \ge 0 \quad , \quad (\omega - \beta)(1) \ge 0 \tag{3.23}$$

$$(\omega - \beta)''(0) \le 0 \quad , \quad (\omega - \beta)''(1) \le 0 \tag{3.24}$$

By Lemma (3.1.4) we have $(\omega - \beta)(x) \ge 0$, i.e. $\omega(x) \ge \beta(x)$ for $x \in [0,1]$.

From (3.22) we have

$$(D^2 - \lambda_2)(D^2 - \lambda_1)(\omega - \beta)(x) \ge 0$$

let $y(x) = (D^2 - \lambda_1)(\omega - \beta)(x) = (\omega - \beta)''(x) - \lambda_1(\omega - \beta)(x)$ then $(D^2 - \lambda_2)y(x) \ge 0$ i.e. $y''(x) - \lambda_2 y(x) \ge 0$. Since $\lambda_1 \ge 0, \lambda_2 > -\pi^2$ and $(\omega - \beta)(x) \in F$, by Lemma (3.1.2) we obtain

$$y(x) = (\omega - \beta)''(x) - \lambda_1(\omega - \beta)(x) \leq 0 \quad , \quad x \in [0,1]$$

i.e.

$$\omega''(x) \leq \beta''(x) + \lambda_1(\omega - \beta)(x) \quad , \quad x \in [0,1]$$

$$\omega''(x) \leq \beta''(x) + \lambda_1(\alpha - \beta)(x) \quad , \quad x \in [0,1]$$

Thus (3.13) holds.

Step 2. Let $u_1 = T\eta_1, u_2 = T\eta_2$, where $\eta_1, \eta_2 \in G$ satisfy $\eta_1 \leq \eta_2$ and$\eta_2''(x) \leq \eta_1''(x) + \lambda_1(\alpha - \beta)(x)$. We obtain

$$Lu_1(x) = f_1(x, \eta_1(x), \eta_1''(x)) = f(x, \eta_1(x), \eta_1''(x)) - a\eta_1''(x) + b\eta_1(x)$$

$$Lu_2(x) = f_1(x, \eta_2(x), \eta_2''(x)) = f(x, \eta_2(x), \eta_2''(x)) - a\eta_2''(x) + b\eta_2(x)$$

then we have

$$L(u_2 - u_1)(x) \geq f(x, \eta_2(x), \eta_2''(x)) - f(x, \eta_1(x), \eta_1''(x)) - a(\eta_2 - \eta_1)''(x) + b(\eta_2 - \eta_1)(x) \tag{3.25}$$

from (3.9) and (3.10) we have

$$f(x, \eta_2(x), v) - f(x, \eta_1(x), v) \geq -b(\eta_2 - \eta_1)(x)$$

$$f(x, u, \eta_2''(x)) - f(x, u, \eta_1''(x)) \geq a(\eta_2 - \eta_1)''(x)$$

i.e.

$$L(u_2 - u_1)(x) \geq 0. \tag{3.26}$$

Since u_1 and u_2 are solutions of (3.11) and (1.3), thus u_1 and u_2 satisfy

$$(u_2 - u_1)(0) = 0 \quad , \quad (u_2 - u_1)(1) = 0$$

$$(u_2 - u_1)''(0) = 0 \quad , \quad (u_2 - u_1)''(1) = 0$$

By Lemma (3.1.4) we have $(u_2 - u_1)(x) \geq 0$, i.e. $u_1 \leq u_2$ for $x \in [0,1]$.

From (3.26) we have

$$(D^2 - \lambda_2)(D^2 - \lambda_1)(u_2 - u_1)(x) \geq 0$$

let $y(x) = (D^2 - \lambda_1)(u_2 - u_1)(x) = (u_2 - u_1)''(x) - \lambda_1(u_2 - u_1)(x)$
then $(D^2 - \lambda_2)y(x) \geq 0$ i.e. $y''(x) - \lambda_2 y(x) \geq 0$. Since $\lambda_1 \geq 0, \lambda_2 > -\pi^2$ and $(u_2 - u_1)(x) \in F$, by Lemma (3.1.2) we obtain

$$y(x) = (u_2 - u_1)''(x) - \lambda_1(u_2 - u_1)(x) \le 0 \quad , \quad x \in [0,1]$$

i.e.

$$u_2''(x) \le u_1''(x) + \lambda_1(u_2 - u_1)(x) \quad , \quad x \in [0,1]$$
$$u_2''(x) \le u_1''(x) + \lambda_1(\alpha - \beta)(x) \quad , \quad x \in [0,1].$$

Thus this step we can show that

$$u_1(x) \le u_2(x) \quad , \quad u_2''(x) \le u_1''(x) + \lambda_1(\alpha - \beta)(x) \tag{3.27}$$

From step 1 and step 2 if we choose $\eta_1, \eta_2 \in G$ satisfy $\beta \le \eta_1 \le \eta_2 \le \alpha$, $\eta_2''(x) \le \eta_1''(x) + \lambda_1(\alpha - \beta)(x)$ and let $u_1 = T\eta_1, u_2 = T\eta_2$ then we obtain

$$\beta(x) \le u_1(x) \le u_2(x) \le \alpha(x),$$

$$\alpha''(x) - \lambda_1(\alpha - \beta)(x) \le u_1''(x), u_2''(x) \le \beta''(x) + \lambda_1(\alpha - \beta)(x),$$

$$u_2''(x) \le u_1''(x) + \lambda_1(\alpha - \beta)(x).$$

Step 3. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are obtained by recurrence:

$$\alpha_n = T\alpha_{n-1} \quad , \quad \beta_n = T\beta_{n-1} \quad , \quad \alpha_0 = \alpha, \beta_0 = \beta; n = 1, 2, ... \tag{3.28}$$

From the results of step 1 and step 2, we get

$$\beta = \beta_0 \le \beta_1 \le \cdots \le \beta_n \le \cdots \le \alpha_n \le \cdots \le \alpha_1 \le \alpha_0 = \alpha, \tag{3.29}$$

$$\alpha'' - \lambda_1(\alpha - \beta) \le \alpha_n'', \beta_n'' \le \beta'' + \lambda_1(\alpha - \beta). \tag{3.30}$$

Moreover from the definition of T, we have

$$\alpha_n^{(4)}(x) - a\alpha_n''(x) + b\alpha_n(x) = f_1(x, \alpha_{n-1}(x), \alpha_{n-1}''(x)),$$

i.e.,

$$\begin{aligned} \alpha_n^{(4)}(x) &= f_1(x, \alpha_{n-1}(x), \alpha_{n-1}''(x)) + a\alpha_n''(x) - b\alpha_n(x) \\ &\le f_1(x, \alpha_{n-1}(x), \alpha_{n-1}''(x)) + a[\beta'' + \lambda_1(\alpha - \beta)](x) - b\beta(x), \end{aligned} \tag{3.31}$$

$$\alpha_n(0) = \alpha_n(1) = \alpha_n''(0) = \alpha_n''(1) = 0. \tag{3.32}$$

From (3.29),(3.30) and (3.31), since $\alpha, \beta \in C^4[0,1]$ and continuity of f_1, we have that there exists $M1_{\alpha,\beta} > 0$ depending only on α and β (but not on n or x) such that

$$|\alpha_n^{(4)}(x)| \leq M1_{\alpha,\beta} \quad , \quad \forall x \in [0,1] \tag{3.33}$$

Using the boundary condition (3.32)and Rolle's theorem, we get that for each $n \in N$, there exist $\xi_n \in (0,1)$ such that

$$\alpha_n^{(3)}(\xi_n) = 0 \tag{3.34}$$

This together with (3.33) we have

$$|\alpha_n^{(3)}(x)| = \left|\alpha_n^{(3)}(\xi_n) + \int_{\xi_n}^{x} \alpha_n^{(4)}(s)ds\right| \leq M1_{\alpha,\beta} \tag{3.35}$$

and from (3.30) we have that there exists $M2_{\alpha,\beta} > 0$ depending only on α and β (but not on n or x) such that

$$|\alpha_n''(x)| \leq M2_{\alpha,\beta} \quad , \quad \forall x \in [0,1] \tag{3.36}$$

Using the boundary condition (3.32) and Rolle's theorem, we get that for each $n \in N$, there exist $\zeta_n \in (0,1)$ such that

$$\alpha_n''(\zeta_n) = 0 \tag{3.37}$$

This together with (3.36) we have

$$|\alpha_n'(x)| = \left|\alpha_n'(\zeta_n) + \int_{\zeta_n}^{x} \alpha''\zeta_n(s)ds\right| \leq M2_{\alpha,\beta} \tag{3.38}$$

Thus, from (3.29), (3.33), (3.35), (3.36) and (3.38), we know that $\{\alpha_n\}$ bounded in $C^4[0,1]$

Similarly, we obtain

$$\beta_n^{(4)}(x) - a\beta_n''(x) + b\beta_n(x) = f_1(x, \beta_{n-1}(x), \beta_{n-1}''(x))$$

i.e.,

$$\begin{aligned}\beta_n^{(4)}(x) &= f_1(x, \beta_{n-1}(x), \beta_{n-1}''(x)) + a\beta_n''(x) - b\beta_n(x) \\ &\leq f_1(x, \beta_{n-1}(x), \beta_{n-1}''(x)) + a[\beta'' + \lambda_1(\alpha - \beta)](x) - b\beta(x) \quad (3.39)\end{aligned}$$

$$\beta_n(0) = \beta_n(1) = \beta_n''(0) = \beta_n''(1) = 0 \tag{3.40}$$

From (3.29),(3.30) and (3.39), since $\alpha, \beta \in C^4[0,1]$ and continuity of f_1, we have that there exists $M3_{\alpha,\beta} > 0$ depending only on α and β (but not on n or x) such that

$$|\beta_n^{(4)}(x)| \leq M3_{\alpha,\beta} \quad , \quad \forall x \in [0,1] \tag{3.41}$$

Using the boundary condition (3.40) and Rolle's theorem, we get that for each $n \in N$, there exist $\phi_n \in (0,1)$ such that

$$\beta_n^{(3)}(\phi_n) = 0 \tag{3.42}$$

This together with (3.41) we have

$$|\beta_n^{(3)}(x)| = \left|\beta_n^{(3)}(\phi_n) + \int_{\xi_n}^{x} \beta_n^{(4)}(s)ds\right| \leq M3_{\alpha,\beta} \tag{3.43}$$

and from (3.30) we have that there exists $M4_{\alpha,\beta} > 0$ depending only on α and β (but not on n or x) such that

$$|\beta_n''(x)| \leq M4_{\alpha,\beta} \quad , \quad \forall x \in [0,1] \tag{3.44}$$

Using the boundary condition (3.40) and Rolle's theorem, we get that for each $n \in N$, there exist $\psi_n \in (0,1)$ such that

$$\beta_n''(\psi_n) = 0 \tag{3.45}$$

This together with (3.44) we have

$$|\beta_n'(x)| = \left|\beta_n'(\zeta_n) + \int_{\psi_n}^{x} \beta''\psi_n(s)ds\right| \leq M4_{\alpha,\beta} \tag{3.46}$$

Thus, from (3.29), (3.41), (3.43), (3.44) and (3.46), we know that $\{\beta_n\}$ bounded in $C^4[0,1]$

Now, by using the fact that $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded in $C^4[0,1]$, we can conclude that $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly to the extremal solutions in [0,1] of the problem (1.2) – (1.3) □

Let $\lambda_1 = \lambda_2 = 0$, then $a = b = 0$. In this case, Theorem (3.2.3) reduces to the following corollary, as in [12] Theorem (3.1).

Corollary 3.2.4 *Suppose there exist β and α, lower and upper solutions, respectively, for the problem* (1.2) − (1.3) *which satisfy*

$$\beta \leq \alpha \quad , \quad \beta'' \geq \alpha'', \tag{3.47}$$

$a = b = 0$, f satisfies (H1) and following the inequality

$$f(x, u_2, v) - f(x, u_1, v) \geq 0, \tag{3.48}$$

for $\beta \leq u_1 \leq u_2 \leq \alpha \quad , \quad v \in R \quad , x \in [0, 1]$.

$$f(x, u, v_2) - f(x, u, v_1) \leq 0, \tag{3.49}$$

for $\alpha'' \leq v_1 \leq v_2 \leq \beta'' \quad , \quad u \in R \quad , x \in [0, 1]$

then there exist two monotone sequence $\{\beta_n\}$ and $\{\alpha_n\} \in C^4[0, 1]$, non increasing and non decreasing, respectively, with $\beta_0 = \beta$ and $\alpha_0 = \alpha$, which converge uniformly to the extremal solution in $[\beta, \alpha]$ of the problem (1.2) − (1.3)

Example 3.2.5 Consider the boundary value problem

$$u^{(4)}(x) + \frac{\pi^2}{2} u''(x) = e^{u(x)} \sin(\pi x),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

We can see that $a = -\frac{\pi^2}{2}$ and $b = 0$ satisfy (H2), $\lambda_1 = 0$ and $\lambda_2 = -\frac{\pi^2}{2}$. We try to choose $\alpha(x) = \sin(\pi x)$ for the upper solution and check by definition, we have

$$\alpha(0) = \alpha(1) = \alpha''(0) = \alpha''(1) = 0 \quad ; \alpha^{(4)}(x) = \pi^4 \sin(\pi x),$$

$$f(x, \alpha(x), \alpha''(x)) = \left(\frac{\pi^4}{2} + e^{\sin(\pi x)}\right) \sin(\pi x).$$

For $x \in [0, 1]$, we obtain $\alpha^{(4)}(x) \geq f(x, \alpha(x), \alpha''(x))$. Thus $\alpha(x) = \sin(\pi x)$ be the upper solution. Similarly, It is easy to check that $\beta(x) = 0$ be the lower solution. And we have,

$$(\beta(x) = 0) \leq (\alpha(x) = \sin(\pi x)) \quad , \quad x \in [0, 1]$$

satisfies (3.16), for $\beta \le u_1 \le u_2 \le \alpha$, $v \in R$, $x \in [0,1]$.

$$\begin{aligned} f(x,u_2,v) - f(x,u_1,v) &= (e^{(u_2)} - e^{(u_2)})\sin(\pi x) \\ &\ge 0 \\ &\ge -b(u_2 - u_1) \end{aligned}$$

satisfies (3.17), for $v_2 + \lambda(\alpha - \beta) \ge v_1$, $\alpha'' - \lambda(\alpha - \beta) \le v_1, v_2 \le \beta'' + \lambda(\alpha - \beta)$, $u \in R, x \in [0,1]$.

$$\begin{aligned} f(x,u,v_2) - f(x,u,v_1) &= -\frac{\pi^2}{2}(v_2 - v_1) \\ &= a(v_2 - v_1), \end{aligned}$$

satisfies (3.18). Thus by Theorem (3.2.3) this problem has at least one solution u, which satisfies $0 \le u \le \sin(\pi x)$

Example 3.2.6 Consider the boundary value problem

$$u^{(4)}(x) + \frac{\pi^2}{2}u''(x) - \frac{\pi^4}{4}u(x) = x(1-x) + \pi^4\sin(\pi x) - \pi^4(u(x))^2,$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

We can see that $a = -\frac{\pi^2}{2}$ and $b = -\frac{\pi^4}{4}$ satisfy (H2), $\lambda_1 = \frac{-1+\sqrt{5}}{4}$ and $\lambda_2 = \frac{-1-\sqrt{5}}{4}$. We choose $\alpha(x) = \sin(\pi x)$ for the upper solution and check by definition, we have

$$\alpha(0) = \alpha(1) = \alpha''(0) = \alpha''(1) = 0 \ ; \alpha^{(4)}(x) = \pi^4\sin(\pi x),$$
$$f(x,\alpha(x),\alpha''(x)) = \frac{3\pi^4}{4}\sin(\pi x) + x(1-x).$$

For $x \in [0,1]$, we obtain $\alpha^{(4)}(x) \ge f(x,\alpha(x),\alpha''(x))$. Thus $\alpha(x) = \sin(\pi x)$ be the upper solution. Similarly, It is easy to check that $\beta(x) = 0$ be the lower solution. And we have,

$$(\beta(x) = 0) \le (\alpha(x) = sin(\pi x)) \ , \ x \in [0,1]$$

satisfies (3.16), for $\beta \le u_1 \le u_2 \le \alpha$, $v \in R$, $x \in [0,1]$.

$$\begin{aligned} f(x,u_2,v) - f(x,u_1,v) &= [\frac{\pi^4}{4} + \pi^4 - (u_2 + u_2)]\sin(\pi x) \\ &\ge [\frac{\pi^4}{4} + \pi^4 - 2]\sin(\pi x) \\ &\ge -b(u_2 - u_1) \end{aligned}$$

satisfies (3.17), for $v_2 + \lambda(\alpha - \beta) \geq v_1$, $\alpha'' - \lambda(\alpha - \beta) \leq v_1, v_2 \leq \beta'' + \lambda(\alpha - \beta)$, $u \in R, x \in [0, 1]$.

$$\begin{aligned} f(x, u, v_2) - f(x, u, v_1) &= -\frac{\pi^2}{2}(v_2 - v_1) \\ &= a(v_2 - v_1), \end{aligned}$$

satisfies (3.18). Thus by Theorem (3.2.3) this problem has at least one solution u, which satisfies $0 \leq u \leq sin(\pi x)$ $\square$