CHAPTER 2

PRELIMINARIES

The aim of this chapter is to give some definitions, notations and results of metric spaces, fixed point of selfmappings and common fixed point of selfmappings in metric spaces which will be used in the later chapters.

2.1 Metric Spaces

Definition 2.1.1 (cf. [8]) *A* metric space *is a pair* (X, d), *where* X *is a set and* d *is a metric on* X *(or distance function on* X*),that is, a real valued function defined on* $X \times X$ *such that for all* $x, y, z \in X$ *we have:*

(1) $d(x, y) \geq 0$

(2) $d(x, y) = 0$ *if and only if* $x = y$

(3) $d(x, y) = d(y, x)$ *(symmetry)*

(4) $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality).*

Definition 2.1.2 (cf. [8]) *A sequence* (x_n) *in a metric space* $X = (X, d)$ *is said to be* convergent *if there is an* $x \in X$ *such that*

$$\lim_{n\to\infty} d(x_n, x) = 0$$

x *is called the* limit *of* (x_n) *and we write*

$$\lim_{n\to\infty} x_n = x$$

$$\text{or, simple, } x_n \to x$$

we say that (x_n) converges to x *.If* (x_n) *is not convergent, it is said to be* divergent*.*

Proposition 2.1.3 (cf. [8]) *Let $X = (X, d)$ be a metric space. Then:*

(a) A convergent sequence in X is bounded and its limit is unique.

(b) If $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.

Definition 2.1.4 (cf. [8]) *A sequence (x_n) in a metric space $X = (X, d)$ is said to be* Cauchy *if for every $\epsilon > 0$ there is an $N(\epsilon) \in N$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N(\epsilon)$. The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).*

Theorem 2.1.5 (cf. [8]) *Every convergent sequence in a metric space is a Cauchy sequence.*

2.2 Fixed Point of Selfmappings in Metric Space

In this section, we give definitions and some results of fixed point in metric spaces.

Definition 2.2.1 (cf. [8]) *A* fixed point *of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself (is"kept fixed"by T), that is, $Tx = x$, the image Tx coincides with x.*

Definition 2.2.2 (cf. [8]) *Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a* contraction *on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Theorem 2.2.3 (cf. [8]) *Suppose that X is a complete metric space and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point.*

Theorem 2.2.4 (cf. [7]) *Let M be a complete metric space and suppose $f : M \to M$ satisfies $d(f(x), f(y)) \leq \alpha(d(x, y))d(x, y)$ for each $x, y \in M$,where $\alpha : [0, \infty) \to [0, 1)$ is monotonically decreasing. Then f has a unique fixed point $\overline{x}$,and $\{f^n(x)\}$ converges to $\overline{x}$ for each $x \in M$.*

Theorem 2.2.5 (cf. [3]) *Let (M, d) be a complete metric space and $f : M \to M$. If there exists a lower semicontinuous function ψ mapping M into the nonnegative numbers such that*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)), x \in M,$$

then f has a fixed point.

Theorem 2.2.6 (cf. [10]) *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), c_1 d(x, Ty) + c_2 d(y, Tx)\}, \forall x \neq y \in X,$

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 = 1$ Then T has a unique fixed point.

2.3 Common Fixed Point of Selfmappings in Metric Spaces

In this section, we give some definitions and theorems concerning common fixed point of selfmappings in metric spaces.

Definition 2.3.1 (cf. [10]) *Two selfmappings T and S of a metric space (X, d) are said to be* commuting *if and only if $STx = TSx, \forall x \in X$*

Definition 2.3.2 (cf. [1]) *Let T and S be two selfmappings of a metric space (X, d). Then S and T are said to be* compatible *if*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$.

Definition 2.3.3 (cf. [1]) *Two selfmapping T and S of a metric space (X, d) are said to be* weakly compatible *if they commute at there coicidence point; i.e, if $Tu = Su$ for some $u \in X$, then $STu = TSu$.*

It is easy to see that two compatible maps are weakly compatible.

Definition 2.3.4 (cf. [1]) *Let S and T be two selfmappings of a metric space (X, d). We say that T and S* satisfy property (E.A) *if there exists a sequence (x_n) such that*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Example 2.3.5 (1) *Let $X = [0, \infty)$. Define $T, S : X \to X$ by $Tx = \frac{x}{4}$ and $Sx = \frac{3x}{4}, \forall x \in X$. Consider the sequence $x_n = \frac{1}{n}$, clearly*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = 0.$$

Then S and T satisfy property(E.A).

(2) *Let $X = [2, \infty)$. Define $T, S : X \to X$ by $Tx = x+1$ and $Sx = 2x+1, \forall x \in X$. Suppose that S and T satisfy property (E.A). Then there exists in X a sequence (x_n) satisfying*

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$. Therefore $\lim_{n\to\infty} x_n = t - 1$ and $\lim_{n\to\infty} x_n = \frac{t-1}{2}$. Then $t = 1$, which is a contradiction since $1 \notin X$. Hence T and S do not satisfy property(E.A). □

Theorem 2.3.6 (cf. [1]) *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), [d(Tx, Sx)+d(Ty, Sy)]/2, [d(Ty, Sx)+d(Tx, Sy)]/2\}$, *for all $x \neq y \in X$,*

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique common fixed point.

Theorem 2.3.7 (cf. [4]) *Let S and T be two commuting mappings of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Sx, Sy) \leq c \cdot \max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Tx, Sy), d(Ty, Sx)\}$$

for all $x, y \in X$, where $0 \leq c < 1$. If the range of T contains the range of S and if T is continuous, then S and T have a unique common fixed point.

Theorem 2.3.8 (cf. [10]) *Let (X, d) be a complete metric space and let $S, T : X \to X$ are commuting mappings satisfying the inequality*

$$d(Sx, Sy) \leq F(\max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy), d(Ty, Sx)\}), \forall x, y \in X$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function such that $F(t) < t$ for each $t > 0$.
If $SX \subset TX$ and T is continuous then S and T have a unique common fixed point. Moreover, if x is the common fixed point of S and T, then for any $x_0 \in X$, $Sx_n \to x$ and $Tx_n \to x$ where (x_n) is the sequence given by $Sx_n = Tx_{n+1}$, $n = 0, 1, 2, \ldots$