CHAPTER 3

MAIN RESULTS

This chapter is divided into 3 sections. Several fixed point theorems of selfmappings in a complete metric space are given in Section 3.1. These results generalize those in [1] and [10]. In Section 3.2, several common fixed point theorems of two mappings are studied and we obtain many results which generalize those in [1] and [10]. In the last section, Section 3.3, we present some examples of applications.

3.1 Fixed Point of Selfmappings in Metric Spaces

Lemma 3.1.1 *Let (X, d) be a metric space, and let $T : X \to X$. Let $x_0 \in X$ be fixed, define $x_n = Tx_{n-1}, n \in N$. If there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

$$d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$$

then (x_n) is Cauchy in X.

Proof. Choose any $x_0 \in X$ and define the sequence (x_n) by $x_n = Tx_{n-1}, n \in N$. Then

$$d(x_n, x_{n+1}) = d(x_n, Tx_n) \leq \Phi(x_n) - \Phi(Tx_n) = \Phi(x_n) - \Phi(x_{n+1}).$$

Define $a_n = \Phi(x_n), n = 1, 2, \ldots$. It is easy to see that the sequence (a_n) is nonnegative real sequence and nonincreasing. Thus (a_n) is a convergent sequence, so it is Cauchy.

For $m, n \in N$ with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq (\Phi(x_n) - \Phi(x_{n+1})) + (\Phi(x_{n+1}) - \Phi(x_{n+2})) + \ldots + (\Phi(x_{m-1}) - \Phi(x_m)) \\ &= \Phi(x_n) - \Phi(x_m) = a_n - a_m. \end{aligned}$$

Since (a_n) is Cauchy, it implies that (x_n) is Cauchy in X.

Theorem 3.1.2 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Tx, y) + c_2 d(Tx, x)\}, \forall x \neq y \in X,$

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 < 1$ Then T has a unique fixed point.

Proof. By Lemma 3.1.1, (x_n) is Cauchy in X. Since X is complete, we have that (x_n) is convergent in X. Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T.

CaseI. There exists $m \in N$ such that $x_n = x$ for all $n > m$. Then
$0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.

CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \neq x, \forall k \in N$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), d(Tx_{n_k}, x), c_1 d(Tx, x_{n_k}) + c_2 d(Tx, x)\}.$$

By taking $k \to \infty$, we have

$$\begin{aligned} d(Tx, x) &\leq \max\{d(x, x), d(x, x), c_1 d(Tx, x) + c_2 d(Tx, x)\} \\ &\leq \max\{(c_1 + c_2) d(Tx, x)\} \\ &= (c_1 + c_2) d(Tx, x) \qquad \text{(since } c_1 + c_2 < 1). \end{aligned}$$

Hence $d(Tx, x) = 0$, so $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$. Suppose that $u \neq v$. Then

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(u, v), d(Tv, u), c_1 d(Tu, v) + c_2 d(Tu, u)\} \\ &\leq \max\{d(u, v), d(v, u), c_1 d(u, v) + c_2 d(u, u)\} \\ &\leq \max\{d(u, v), c_1 d(u, v)\} \\ &= d(u, v), \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. $\square$

Corollary 3.1.3 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \le \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Tx, y), c_2 d(Tx, x)\}, \forall x \ne y \in X,$

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 < 1$ Then T has a unique fixed point.

Proof. Since the condition (2) of Corollary 3.1.3 implies (2) of Theorem 3.1.2, the corollary is directly obtained by Theorem 3.1.2.

Theorem 3.1.4 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \le \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Ty, y) + c_2 d(Tx, y)\}, \forall x \ne y \in X,$

where $c_1 > 0$ and $0 < c_2 < 1$ Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n \in N$. By Lemma 3.1.1, (x_n) is Cauchy in X. Since X is complete, we have that (x_n) is convergent in X.
Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T.
CaseI. There exists $m \in N$ such that $x_n = x$ for all $n > m$. Then
$0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.
CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \ne x, \forall k \in N$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), d(Tx_{n_k}, x), c_1 d(Tx_{n_k}, x_{n_k}) + c_2 d(Tx, x_{n_k})\}.$$

By taking $k \to \infty$, we have

$$\begin{aligned} d(Tx, x) &\le \max\{d(x, x), d(x, x), c_1 d(x, x) + c_2 d(Tx, x)\} \\ &= c_2 d(Tx, x). \end{aligned}$$

Since $0 < c_2 < 1$, it implies that $d(Tx, x) = 0$ and hence $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$. Suppose that $u \neq v$. Then

$$\begin{aligned} d(u,v) = d(Tu,Tv) &< \max\{d(u,v), d(Tv,u), c_1 d(Tv,v) + c_2 d(Tu,v)\} \\ &\leq \max\{d(u,v), d(v,u), c_1 d(v,v) + c_2 d(u,v)\} \\ &\leq \max\{d(u,v), c_2 d(u,v)\} \\ &= d(u,v), \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. □

Corollary 3.1.5 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Ty, y), c_2 d(Tx, y)\}, \forall x \neq y \in X,$

where $c_1 < 0$ and $0 < c_2 < 1$ Then T has a unique fixed point.

Proof. Since the condition (2) of Corollary 3.1.5 implies (2) of Theorem 3.1.4, the corollary is directly obtained by Theorem 3.1.4.

Theorem 3.1.6 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Ty, y) + c_2 d(Tx, x)\}, \forall x \neq y \in X,$

where $c_1 > 0$ and $0 < c_2 < 1$ Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n \in N$. By Lemma 3.1.1, (x_n) is Cauchy in X. Since X is complete, we have that (x_n) is convergent in X.

Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T.

CaseI. There exists $m \in N$ such that $x_n = x$ for all $n > m$. Then $0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.

CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \neq x, \forall k \in N$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), d(Tx_{n_k}, x), c_1 d(Tx_{n_k}, x_{n_k}) + c_2 d(Tx, x)\}.$$

By taking $k \to \infty$, we have

$$\begin{aligned} d(Tx, x) &\leq \max\{d(x, x), d(x, x), c_1 d(x, x) + c_2 d(Tx, x)\} \\ &= c_2 d(Tx, x), \end{aligned}$$

Since $0 < c_2 < 1$, it implies that $d(Tx, x) = 0$ and hence $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$. Suppose that $u \neq v$. Then

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(u, v), d(Tv, u), c_1 d(Tv, v) + c_2 d(Tu, u)\} \\ &\leq \max\{d(u, v), d(v, u), c_1 d(v, v) + c_2 d(u, u)\} \\ &= d(u, v), \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. $\square$

Corollary 3.1.7 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Ty, y), c_2 d(Tx, x)\}, \forall x \neq y \in X$,

where $0 < c_1 < 1$ and $0 < c_2 < 1$ Then T has a unique fixed point.

Proof. Since the condition (2) of Corollary 3.1.7 implies (2) of Theorem 3.1.6, the corollary is directly obtained by Theorem 3.1.6.

Theorem 3.1.8 *Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x,Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx,Ty) < \max\{d(x,y), d(Ty,y), c_1 d(Ty,x) + c_2 d(Tx,y)\}, \forall x \neq y \in X$,

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 = 1$ Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n \in N$. By Lemma 3.1.1, (x_n) is Cauchy in X. Since X is complete, we have that (x_n) is convergent in X.
Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T.

CaseI. There exists $m \in N$ such that $x_n = x$ for all $n > m$. Then
$0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.

CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \neq x, \forall k \in N$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), d(Tx_{n_k}, x_{n_k}), c_1 d(Tx_{n_k}, x) + c_2 d(Tx, x_{n_k})\}.$$

By taking $k \to \infty$, we have

$$\begin{aligned} d(Tx,x) &\leq \max\{d(x,x), d(x,x), c_1 d(x,x) + c_2 d(Tx,x)\} \\ &= c_2 d(Tx,x). \end{aligned}$$

Since $0 < c_2 < 1$, it implies that $d(Tx,x) = 0$ and hence $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$. Suppose that $u \neq v$. Then

$$\begin{aligned} d(u,v) = d(Tu,Tv) &< \max\{d(u,v), d(Tv,v), c_1 d(Tv,u) + c_2 d(Tu,v)\} \\ &\leq \max\{d(u,v), d(v,v), c_1 d(v,u) + c_2 d(u,v)\} \\ &\leq \max\{d(u,v), (c_1 + c_2) d(u,v)\} \\ &= d(u,v) \qquad \text{(since } c_1 + c_2 = 1\text{)}, \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. □

Theorem 3.1.9 *Let (X,d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Ty, x), c_2 d(Tx, y)\}, \forall x \neq y \in X$,

where $0 < c_1 < 1$ *and* $0 < c_2 < 1$ *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n \in N$. By Lemma 3.1.1, (x_n) is Cauchy in X. Since X is complete, we have that (x_n) is convergent in X.
Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T.
CaseI. There exists $m \in N$ such that $x_n = x$ for all $n > m$. Then
$0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.
CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \neq x, \forall k \in N$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), d(Tx_{n_k}, x_{n_k}), c_1 d(Tx_{n_k}, x), c_2 d(Tx, x_{n_k})\}.$$

By taking $k \to \infty$, we have

$$\begin{aligned} d(Tx, x) &\leq \max\{d(x, x), d(x, x), c_1 d(x, x), c_2 d(Tx, x)\} \\ &= c_2 d(Tx, x). \end{aligned}$$

Since $0 < c_2 < 1$, it implies that $d(Tx, x) = 0$ and hence $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$. Suppose that $u \neq v$. Then

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(u, v), d(Tv, v), c_1 d(Tv, u), c_2 d(Tu, v)\} \\ &\leq \max\{d(u, v), d(v, v), c_1 d(v, u), c_2 d(u, v)\} \\ &\leq \max\{d(u, v), c_1 d(u, v), c_2 d(u, v)\} \\ &= d(u, v) \qquad \text{(since } 0 < c_1, c_2 < 1), \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. □

Theorem 3.1.10 *Let* (X, d) *be a complete metric space and let* $T : X \to X$. *Suppose that there exists a mapping* $\Phi : X \to \mathbb{R}^+$ *such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Ty, x) + c_2 d(Tx, x)\}, \forall x \neq y \in X$,

where $0 < c_1 < 1$ *and* $0 < c_2 < 1$ *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n \in N$. By Lemma 3.1.1, (x_n) is Cauchy in X. Since X is complete, we have that (x_n) is convergent in X.
Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T.

CaseI. There exists $m \in N$ such that $x_n = x$ for all $n > m$. Then
$0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.

CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \neq x, \forall k \in N$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), d(Tx_{n_k}, x_{n_k}), c_1 d(Tx_{n_k}, x) + c_2 d(Tx, x)\}.$$

By taking $k \to \infty$, we have

$$\begin{aligned} d(Tx, x) &\leq \max\{d(x, x), d(x, x), c_1 d(x, x) + c_2 d(Tx, x)\} \\ &= c_2 d(Tx, x). \end{aligned}$$

Since $0 < c_2 < 1$, it implies that $d(Tx, x) = 0$ and hence $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$. Suppose that $u \neq v$. Then

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(u, v), d(Tv, v), c_1 d(Tv, u) + c_2 d(Tu, u)\} \\ &\leq \max\{d(u, v), d(v, v), c_1 d(u, v) + c_2 d(u, u)\} \\ &\leq \max\{d(u, v), c_1 d(u, v)\} \\ &= d(u, v), \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. □

Corollary 3.1.11 *Let* (X, d) *be a complete metric space and let* $T : X \to X$. *Suppose that there exists a mapping* $\Phi : X \to \mathbb{R}^+$ *such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Ty, x), c_2 d(Tx, x)\}, \forall x \neq y \in X$,

where $0 < c_1 < 1$ *and* $0 < c_2 < 1$ *Then* T *has a unique fixed point.*

Proof. Since the condition (2) of Corollary 3.1.11 implies (2) of Theorem 3.1.10, the corollary is directly obtained by Theorem 3.1.10.

Theorem 3.1.12 *Let* (X, d) *be a complete metric space and let* $T : X \to X$. *Suppose that there exists a mapping* $\Phi : X \to \mathbb{R}^+$ *such that*

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Tx, y) + c_2 d(Tx, x)\}, \forall x \neq y \in X$,

where $c_1 > 0, c_2 > 0$ *and* $c_1 + c_2 < 1$ *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n \in N$. By Lemma 3.1.1, (x_n) is Cauchy in X. Since X is complete, we have that (x_n) is convergent in X.
Hence there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now, we show that x is a fixed point of T.
CaseI. There exists $m \in N$ such that $x_n = x$ for all $n > m$. Then
$0 = \lim_{n\to\infty} d(Tx_n, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx)$. Hence $Tx = x$.
CaseII. There exists a subsequence (x_{n_k}) such that $x_{n_k} \neq x, \forall k \in N$. By(2), we have

$$d(Tx, Tx_{n_k}) < \max\{d(x, x_{n_k}), d(Tx_{n_k}, x_{n_k}), c_1 d(Tx, x_{n_k}) + c_2 d(Tx, x)\}.$$

By taking $k \to \infty$, we have

$$\begin{aligned} d(Tx, x) &\leq \max\{d(x, x), d(x, x), c_1 d(Tx, x) + c_2 d(Tx, x)\} \\ &= (c_1 + c_2) d(Tx, x), \end{aligned}$$

so $d(Tx, x) = 0$ and hence $Tx = x$. Thus x is a fixed point of T.

Finally, we show that fixed point is unique. Let $Tu = u$ and $Tv = v$.

Suppose that $u \neq v$. Then

$$\begin{aligned} d(u,v) = d(Tu,Tv) &< \max\{d(u,v), d(Tv,v), c_1 d(Tu,v) + c_2 d(Tu,u)\} \\ &\leq \max\{d(u,v), d(v,v), c_1 d(u,v) + c_2 d(u,u)\} \\ &\leq \max\{d(u,v), c_1 d(u,v)\} \\ &= d(u,v), \end{aligned}$$

which is a contradiction, so $u = v$. Therefore fixed point of T is unique. □

Corollary 3.1.13 *Let (X,d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that*

(1) $d(x,Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx,Ty) < \max\{d(x,y), d(Ty,y), c_1 d(Tx,y), c_2 d(Tx,x)\}, \forall x \neq y \in X,$

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 < 1$ Then T has a unique fixed point.

Proof. Since the condition (2) of Corollary 3.1.13 implies (2) of Theorem 3.1.12, the corollary is directly obtained by Theorem 3.1.12.

3.2 Common Fixed Point of Selfmappings in Metric Spaces

Theorem 3.2.1 *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx,Ty) < \max\{c_1[d(Sx,Sy) + d(Tx,Sy)], c_2[d(Tx,Sx) + d(Ty,Sy)]\}$,
$\forall x \neq y \in X$, *where* $0 \leq c_1 \leq 1/2$ *and* $0 \leq c_2 \leq 1/2$

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Proof. Since T and S satisfy the property(E.A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Suppose SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$, so $\lim_{n\to\infty} Tx_n = Sa$.
We show that $Ta = Sa$.
If there exists $n_0 \in \mathbb{N}$ such that $x_n = a \ \forall n \geq n_0$, we obtain that $Ta = Sa$.
If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq a \ \forall k \in \mathbb{N}$. By (2), we have

$$d(Tx_{n_k}, Ta) < \max\{c_1[d(Sx_{n_k}, Sa) + d(Tx_{n_k}, Sa)], c_2[d(Tx_{n_k}, Sx_{n_k}) + d(Ta, Sa)]\}.$$

Take $k \to \infty$, we have

$$\begin{aligned} d(Sa, Ta) &\leq \max\{c_1[d(Sa, Sa) + d(Sa, Sa)], c_2[d(Sa, Sa) + d(Ta, Sa)]\} \\ &= c_2 d(Ta, Sa). \end{aligned}$$

Since $0 < c_2 < 1$, it implies that $d(Ta, Sa) = 0$, hence $Ta = Sa$.

Since T and S are weakly compatible, $TSa = STa$ and $TTa = TSa = STa = SSa$.
If $Ta \neq a$, by(2), we have

$$\begin{aligned} d(Ta, TTa) &< \max\{c_1[d(Sa, STa) + d(Ta, STa)], c_2[d(Ta, Sa) + d(TTa, STa)]\} \\ &\leq \max\{c_1[d(Ta, TTa) + d(Ta, TTa)], c_2[d(Ta, Ta) + d(TTa, TTa)]\} \\ &= 2c_1 d(Ta, TTa) \\ &\leq d(Ta, TTa), \end{aligned}$$

which is a contradiction. Thus $Ta = a$, hence $Ta = Sa = a$, so a is a common fixed point of S and T. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.

Finally, we show common fixed point is unique. Let $Tv = Sv = v$ and $Tu = Su = u$. Suppose $u \neq v$. By(2), we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{c_1[d(Su, Sv) + d(Tu, Sv)], c_2[d(Tu, Su) + d(Tv, Sv)]\} \\ &\leq \max\{c_1[d(Tu, Tv) + d(Tu, Tv)], c_2[d(Tu, Tu) + d(Tv, Tv)]\} \\ &= 2c_1 d(Tu, Tv) \\ &\leq d(Tu, Tv) = d(u, v), \qquad (3.1) \end{aligned}$$

which is a contradiction, hence $u = v$. Therefore T and S have a unique common fixed point. $\square$

Taking $c_1 = c_2$ in Theorem 3.2.1, we get the following result:

Corollary 3.2.2 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < c \cdot \max\{[d(Sx, Sy) + d(Tx, Sy)], [d(Tx, Sx) + d(Ty, Sy)]\}$, $\forall x \neq y \in X$, *where* $0 \leq c \leq 1/2$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Taking $c_2 = 0$ in Theorem 3.2.1, we have the following result:

Corollary 3.2.3 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < c \cdot (d(Sx, Sy) + d(Tx, Sy)), \forall x \neq y \in X$, *where* $0 \leq c \leq 1/2$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Taking $c_1 = 0$ in Theorem 3.2.1, we have the following result:

Corollary 3.2.4 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx,Ty) < c \cdot (d(Tx,Sx) + d(Ty,Sy)), \forall x \neq y \in X,$ *where* $0 \leq c \leq 1/2$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Theorem 3.2.5 *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx,Ty) < \max\{d(Sx,Sy), c_1 d(Tx,Sy) + c_2 d(Ty,Sx), d(Tx,Sx)\}$,
$\forall x \neq y \in X,$ *where* $c_1 \geq 0$, $c_2 \geq 0$ *and* $c_1 + c_2 < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Proof. Since T and S satisfy the property(E.A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Suppose SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$,so $\lim_{n\to\infty} Tx_n = Sa$.
We show that $Ta = Sa$.
If there exists $n_0 \in \mathbb{N}$ such that $x_n = a \; \forall n \geq n_0$, we obtain that $Ta = Sa$.
If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq a \; \forall k \in \mathbb{N}$. By (2), we have

$$d(Tx_{n_k}, Ta) < \max\{d(Sx_{n_k}, Sa), c_1 d(Tx_{n_k}, Sa) + c_2 d(Ta, Sx_{n_k}), d(Tx_{n_k}, Sx_{n_k})\}.$$

Take $k \to \infty$, we have

$$\begin{aligned} d(Sa,Ta) &\leq \max\{d(Sa,Sa), c_1 d(Sa,Sa) + c_2 d(Ta,Sa), d(Sa,Sa)\} \\ &= c_2 d(Ta,Sa). \end{aligned}$$

Since $c_2 < 1$, it implies that $d(Ta,Sa) = 0$, hence $Ta = Sa$.

Since T and S are weakly compatible, $TSa = STa$ and $TTa = TSa =$

$STa = SSa$.

If $Ta \neq a$, by(2), we have

$$\begin{aligned} d(Ta, TTa) &< \max\{d(Sa, STa), c_1 d(Ta, STa) + c_2 d(TTa, Sa), d(Ta, Sa)\} \\ &\leq \max\{d(Ta, TTa), c_1 d(Ta, TTa) + c_2 d(TTa, Ta), d(Ta, Ta)\} \\ &\leq \max\{d(Ta, TTa), (c_1 + c_2) d(TTa, Ta)\} \\ &= d(Ta, TTa) \qquad \text{(since } c_1 + c_2 < 1\text{)}, \end{aligned}$$

which is a contradiction. Thus $Ta = a$, hence $Ta = Sa = a$, so a is a common fixed point of S and T. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.

Finally, we show common fixed point is unique. Let $Tv = Sv = v$ and $Tu = Su = u$. Suppose $u \neq v$. By(2), we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(Su, Sv), c_1 d(Tu, Sv) + c_2 d(Tv, Su), d(Tu, Su)\} \\ &\leq \max\{d(Tu, Tv), c_1 d(Tu, Tv) + c_2 d(Tv, Tu), d(Tu, Tu)\} \\ &\leq \max\{d(Tu, Tv), (c_1 + c_2) d(Tv, Tu)\} \\ &= d(Tu, Tv), \qquad \text{(since } c_1 + c_2 < 1\text{)}, \end{aligned}$$

which is a contradiction, hence $u = v$. Therefore T and S have a unique common fixed point. □

Taking $c_1 = c_2$ in Theorem 3.2.5, we get the following result:

Corollary 3.2.6 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c[d(Tx, Sy) + d(Ty, Sx)], d(Tx, Sx)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1/2$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Taking $c_2 = 0$ in Theorem 3.2.5, we have the following result:

Corollary 3.2.7 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Tx, Sy), d(Tx, Sx)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Taking $c_1 = 0$ in Theorem 3.2.5, we have the following result:

Corollary 3.2.8 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Ty, Sx), d(Tx, Sx)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Theorem 3.2.9 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1 d(Tx, Sy) + c_2 d(Ty, Sy), d(Tx, Sx)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c_1 < 1$ *and* $0 \leq c_2 < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Proof. Since T and S satisfy the property(E.A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Suppose SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$,so $\lim_{n\to\infty} Tx_n = Sa$.
We show that $Ta = Sa$.
If there exists $n_0 \in \mathbb{N}$ such that $x_n = a\ \forall n \geq n_0$, we obtain that $Ta = Sa$.
If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq a\ \forall k \in \mathbb{N}$. By (2), we have

$$d(Tx_{n_k}, Ta) < \max\{d(Sx_{n_k}, Sa), c_1 d(Tx_{n_k}, Sa) + c_2 d(Ta, Sa), d(Tx_{n_k}, Sx_{n_k})\}.$$

Take $k \to \infty$, we have

$$\begin{aligned} d(Sa, Ta) &\leq \max\{d(Sa, Sa), c_1 d(Sa, Sa) + c_2 d(Ta, Sa), d(Sa, Sa)\} \\ &= c_2 d(Ta, Sa). \end{aligned}$$

Since $c_2 < 1$, it implies that $d(Ta, Sa) = 0$, hence $Ta = Sa$.

Since T and S are weakly compatible, $TSa = STa$ and $TTa = TSa = STa = SSa$.
If $Ta \neq a$, by(2), we have

$$\begin{aligned} d(Ta, TTa) &< \max\{d(Sa, STa), c_1 d(Ta, STa) + c_2 d(TTa, STa), d(Ta, Sa)\} \\ &\leq \max\{d(Ta, TTa), c_1 d(Ta, TTa) + c_2 d(TTa, TTa), d(Ta, Ta)\} \\ &\leq \max\{d(Ta, TTa), c_1 d(TTa, Ta)\} \\ &= d(Ta, TTa) \qquad \text{(since } c_1 < 1\text{)}, \end{aligned}$$

which is a contradiction. Thus $Ta = a$, hence $Ta = Sa = a$, so a is a common fixed point of S and T. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.

Finally, we show common fixed point is unique. Let $Tv = Sv = v$ and

$Tu = Su = u$. Suppose $u \neq v$. By(2), we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &< \max\{d(Su,Sv), c_1 d(Tu,Sv) + c_2 d(Tv,Sv), d(Tu,Su)\} \\ &\leq \max\{d(Tu,Tv), c_1 d(Tu,Tv) + c_2 d(Tv,Tv), d(Tu,Tu)\} \\ &\leq \max\{d(Tu,Tv), c_1 d(Tv,Tu)\} \\ &= d(Tu,Tv), \qquad \text{(since } c_1 < 1\text{)}, \end{aligned}$$

which is a contradiction, hence $u = v$. Therefore T and S have a unique common fixed point. □

Taking $c_1 = c_2$ in Theorem 3.2.9, we get the following result:

Corollary 3.2.10 *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx,Ty) < \max\{d(Sx,Sy), c[d(Tx,Sy) + d(Ty,Sy)], d(Tx,Sx)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Taking $c_1 = 0$ in Theorem 3.2.9, we have the following result:

Corollary 3.2.11 *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx,Ty) < \max\{d(Sx,Sy), cd(Ty,Sy), d(Tx,Sx)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Theorem 3.2.12 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1 d(Ty, Sx) + c_2 d(Ty, Sy), d(Tx, Sx)\}$, $\forall x \neq y \in X$, *where* $c_1 \geq 0$, $c_2 \geq 0$ *and* $c_1 + c_2 < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Proof. Since T and S satisfy the property(E.A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Suppose SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$,so $\lim_{n\to\infty} Tx_n = Sa$.
We show that $Ta = Sa$.
If there exists $n_0 \in \mathbb{N}$ such that $x_n = a$ $\forall n \geq n_0$, we obtain that $Ta = Sa$.
If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq a$ $\forall k \in \mathbb{N}$. By (2), we have

$$d(Tx_{n_k}, Ta) < \max\{d(Sx_{n_k}, Sa), c_1 d(Ta, Sx_{n_k}) + c_2 d(Ta, Sa), d(Tx_{n_k}, Sx_{n_k})\}.$$

Take $k \to \infty$, we have

$$\begin{aligned} d(Sa, Ta) &\leq \max\{d(Sa, Sa), c_1 d(Ta, Sa) + c_2 d(Ta, Sa), d(Sa, Sa)\} \\ &= (c_1 + c_2) d(Ta, Sa). \end{aligned}$$

Since $c_1 + c_2 < 1$, it implies that $d(Ta, Sa) = 0$, hence $Ta = Sa$.

Since T and S are weakly compatible, $TSa = STa$ and $TTa = TSa = STa = SSa$.

If $Ta \neq a$, by(2), we have

$$\begin{aligned} d(Ta,TTa) &< \max\{d(Sa,STa), c_1d(TTa,Sa)+c_2d(TTa,STa), d(Ta,Sa)\} \\ &\leq \max\{d(Ta,TTa), c_1d(TTa,Ta)+c_2d(TTa,TTa), d(Ta,Ta)\} \\ &\leq \max\{d(Ta,TTa), c_1d(TTa,Ta)\} \\ &= d(Ta,TTa), \end{aligned}$$

which is a contradiction. Thus $Ta = a$, hence $Ta = Sa = a$, so a is a common fixed point of S and T. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.

Finally, we show common fixed point is unique. Let $Tv = Sv = v$ and $Tu = Su = u$. Suppose $u \neq v$. By(2), we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &< \max\{d(Su,Sv), c_1d(Tv,Su)+c_2d(Tv,Sv), d(Tu,Su)\} \\ &\leq \max\{d(Tu,Tv), c_1d(Tv,Tu)+c_2d(Tv,Tv), d(Tu,Tu)\} \\ &\leq \max\{d(Tu,Tv), c_1d(Tv,Tu)\} \\ &= d(Tu,Tv), \end{aligned}$$

which is a contradiction, hence $u = v$. Therefore T and S have a unique common fixed point. □

Taking $c_1 = c_2$ in Theorem 3.2.12, we get the following result:

Corollary 3.2.13 *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx,Ty) < \max\{d(Sx,Sy), c[d(Ty,Sx)+d(Ty,Sy)], d(Tx,Sx)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1/2$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Theorem 3.2.14 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1 d(Tx, Sx) + c_2 d(Ty, Sx), d(Tx, Sy)\}$,
$\forall x \neq y \in X$, *where* $c_1 \geq 0$, $0 \leq c_2 < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Proof. Since T and S satisfy the property(E.A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Suppose SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$,so $\lim_{n\to\infty} Tx_n = Sa$.
We show that $Ta = Sa$.
If there exists $n_0 \in \mathbb{N}$ such that $x_n = a \ \forall n \geq n_0$, we obtain that $Ta = Sa$.
If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq a \ \forall k \in \mathbb{N}$. By (2), we have

$$d(Tx_{n_k}, Ta) < \max\{d(Sx_{n_k}, Sa), c_1 d(Tx_{n_k}, Sx_{n_k}) + c_2 d(Ta, Sx_{n_k}), d(Tx_{n_k}, Sa)\}.$$

Take $k \to \infty$, we have

$$\begin{aligned} d(Sa, Ta) &\leq \max\{d(Sa, Sa), c_1 d(Sa, Sa) + c_2 d(Ta, Sa), d(Sa, Sa)\} \\ &= c_2 d(Ta, Sa). \end{aligned}$$

Since $c_2 < 1$, it implies that $d(Ta, Sa) = 0$, hence $Ta = Sa$.

Since T and S are weakly compatible, $TSa = STa$ and $TTa = TSa = STa = SSa$.
If $Ta \neq a$, by(2), we have

$$\begin{aligned} d(Ta, TTa) &< \max\{d(Sa, STa), c_1 d(Ta, Sa) + c_2 d(TTa, Sa), d(Ta, STa)\} \\ &\leq \max\{d(Ta, TTa), c_1 d(Ta, Ta) + c_2 d(TTa, Ta), d(Ta, TTa)\} \\ &\leq \max\{d(Ta, TTa), c_2 d(TTa, Ta)\} \\ &= d(Ta, TTa), \end{aligned}$$

which is a contradiction. Thus $Ta = a$, hence $Ta = Sa = a$, so a is a common fixed point of S and T. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.

Finally, we show common fixed point is unique. Let $Tv = Sv = v$ and $Tu = Su = u$. Suppose $u \neq v$. By(2), we have

$$\begin{aligned} d(u,v) = d(Tu,Tv) &< \max\{d(Su,Sv), c_1 d(Tu,Su) + c_2 d(Tv,Su), d(Tu,Sv)\} \\ &\leq \max\{d(Tu,Tv), c_1 d(Tu,Tu) + c_2 d(Tv,Tu), d(Tu,Tv)\} \\ &\leq \max\{d(Tu,Tv), c_2 d(Tu,Tv)\} \\ &= d(Tu,Tv), \end{aligned}$$

which is a contradiction, hence $u = v$. Therefore T and S have a unique common fixed point. □

Taking $c_1 = c_2$ in Theorem 3.2.14, we get the following result:

Corollary 3.2.15 *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx,Ty) < \max\{d(Sx,Sy), c[d(Tx,Sx) + d(Ty,Sx)], d(Tx,Sy)\}$, $\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Taking $c_1 = 0$ in Theorem 3.2.14, we have the following result:

Corollary 3.2.16 *Let S and T be two weakly compatible selfmappings of a metric space (X,d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Ty, Sx), d(Tx, Sy)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX *or* TX *is a complete subspace of* X, *then* T *and* S *have a unique commom fixed point.*

Taking $c_2 = 0$ in Theorem 3.2.14, we have the following result:

Corollary 3.2.17 *Let* S *and* T *be two weakly compatible selfmappings of a metric space* (X, d) *such that*

(1) T *and* S *satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Tx, Sx), d(Tx, Sy)\}$,
$\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX *or* TX *is a complete subspace of* X, *then* T *and* S *have a unique commom fixed point.*

Theorem 3.2.18 *Let* S *and* T *be two weakly compatible selfmappings of a metric space* (X, d) *such that*

(1) T *and* S *satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1 d(Ty, Sx) + c_2 d(Ty, Sy), d(Tx, Sy)\}$,
$\forall x \neq y \in X$, *where* $c_1 \geq 0$, $c_2 \geq 0$ *and* $c_1 + c_2 < 1$.

(3) $TX \subset SX$.

If SX *or* TX *is a complete subspace of* X, *then* T *and* S *have a unique commom fixed point.*

Proof. Since T and S satisfy the property(E.A), there exists a sequence (x_n) in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Suppose SX is complete. Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$,so $\lim_{n\to\infty} Tx_n = Sa$.
We show that $Ta = Sa$.
If there exists $n_0 \in \mathbb{N}$ such that $x_n = a$ $\forall n \geq n_0$, we obtain that $Ta = Sa$.
If there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \neq a$ $\forall k \in \mathbb{N}$. By (2), we have

$$d(Tx_{n_k}, Ta) < \max\{d(Sx_{n_k}, Sa), c_1 d(Ta, Sx_{n_k}) + c_2 d(Ta, Sa), d(Tx_{n_k}, Sa)\}.$$

Take $k \to \infty$, we have

$$\begin{aligned} d(Sa, Ta) &\leq \max\{d(Sa, Sa), c_1 d(Ta, Sa) + c_2 d(Ta, Sa), d(Sa, Sa)\} \\ &= (c_1 + c_2) d(Ta, Sa). \end{aligned}$$

Since $c_1 + c_2 < 1$, it implies that $d(Ta, Sa) = 0$, hence $Ta = Sa$.

Since T and S are weakly compatible, $TSa = STa$ and $TTa = TSa = STa = SSa$.
If $Ta \neq a$, by(2), we have

$$\begin{aligned} d(Ta, TTa) &< \max\{d(Sa, STa), c_1 d(TTa, Sa) + c_2 d(TTa, STa), d(Ta, STa)\} \\ &\leq \max\{d(Ta, TTa), c_1 d(TTa, Ta) + c_2 d(TTa, TTa), d(Ta, TTa)\} \\ &\leq \max\{d(Ta, TTa), c_1 d(Ta, TTa)\} \\ &= d(Ta, TTa), \end{aligned}$$

which is a contradiction. Thus $Ta = a$, hence $Ta = Sa = a$, so a is a common fixed point of S and T. The proof is similar when TX is assumed to be a complete subspace of X since $TX \subset SX$.

Finally, we show common fixed point is unique. Let $Tv = Sv = v$ and $Tu = Su = u$. Suppose $u \neq v$. By(2), we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &< \max\{d(Su, Sv), c_1 d(Tv, Su) + c_2 d(Tv, Sv), d(Tu, Sv)\} \\ &\leq \max\{d(Tu, Tv), c_1 d(Tv, Tu) + c_2 d(Tv, Tv), d(Tu, Tv)\} \\ &\leq \max\{d(Tu, Tv), c_1 d(Tu, Tv)\} \\ &= d(Tu, Tv), \end{aligned}$$

which is a contradiction, hence $u = v$. Therefore T and S have a unique common fixed point. $\square$

Taking $c_1 = c_2$ in Theorem 3.2.18, we get the following result:

Corollary 3.2.19 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c[d(Ty, Sx) + d(Ty, Sy)], d(Tx, Sy)\}$, $\forall x \neq y \in X$, *where* $0 \leq c < 1/2$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Taking $c_1 = 0$ in Theorem 3.2.18, we have the following result:

Corollary 3.2.20 *Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that*

(1) *T and S satisfy the property(E.A),*

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Ty, Sy), d(Tx, Sy)\}$, $\forall x \neq y \in X$, *where* $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

Theorem 3.2.21 *Let (X, d) be a complete metric space and let $S, T : X \to X$ are commuting mappings satisfying the inequality*

$$d(Sx, Sy) \leq F(\max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy) + d(Ty, Sx)\}), \forall x, y \in X \tag{3.2}$$

where $F:\mathbb{R}^+\to\mathbb{R}^+$ *is a nondecreasing continuous function such that* $F(t)<t$ *for each* $t>0$*. If* $SX\subset TX$ *and* T *is continuous then* S *and* T *have a unique common fixed point.*

proof. Let $x_0\in X$, chose $x_1\in X$ such that $Sx_0=Tx_1$. This can be done since $SX\subset TX$. In general, having chosen x_n choose x_{n+1} such that $Sx_n=Tx_{n+1}$.
We shall show that

$$d(Sx_n,Sx_{n+1})\le F(d(Sx_{n-1},Sx_n)). \tag{3.3}$$

$$d(Sx_n,Sx_{n+1})\le d(Sx_{n-1},Sx_n). \tag{3.4}$$

By (3.2),we have

$$\begin{aligned} d(Sx_n,Sx_{n+1}) &\le F(\max\{d(Tx_n,Tx_{n+1}),d(Tx_n,Sx_n),d(Tx_{n+1},Sx_{n+1})+d(Tx_{n+1},Sx_n)\}) \\ &\le F(\max\{d(Sx_{n-1},Sx_n),d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})+d(Sx_n,Sx_n)\}) \\ &\le F(\max\{d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})\}). \end{aligned}$$

If $0\le d(Sx_{n-1},Sx_n)<d(Sx_n,Sx_{n+1})$, then $d(Sx_n,Sx_{n+1})\le F(d(Sx_n,Sx_{n+1}))$ $<d(Sx_n,Sx_{n+1})$ which is a contradiction. Hence $d(Sx_{n-1},Sx_n)\ge d(Sx_n,Sx_{n+1})$ and $d(Sx_n,Sx_{n+1})\le F(d(Sx_{n-1},Sx_n))$. Thus (3.3) and (3.4) are satisfied. Thus the sequence $(d(Sx_n,Sx_{n+1}))_{n=0}^\infty$ is a nonincreasing sequence of positive real number and therefore has a limit $L\ge 0$. We claim that $L=0$. Suppose $L>0$, by taking $n\to\infty$ in(3.3) and continuity of F, we have

$$L=\lim_{n\to\infty} d(Sx_n,Sx_{n+1})\le\lim_{n\to\infty} F(d(Sx_{n-1},Sx_n))=F(L)<L,$$

which is a contradiction, hence $L=0$. Thus $\lim_{n\to\infty} d(Sx_n,Sx_{n+1})=0$
Next, we show that $(Sx_n)_{n=0}^\infty$ is a Cauchy sequence in X.To show this, suppose not.Then there exist $\epsilon>0$ and strictly increasing sequences of positive integer (m_k) and (n_k) with $m_k>n_k\ge k$ such that

$$d(Sx_{m_k},Sx_{n_k})\ge\epsilon. \tag{3.5}$$

Assume that for each k, m_k is the smallest number greater than n_k for which (3.5) holds. By (3.4) and (3.5)

$$\begin{aligned}\epsilon \le d(Sx_{m_k}, Sx_{n_k}) &\le d(Sx_{m_k}, Sx_{m_k-1}) + d(Sx_{m_k-1}, Sx_{n_k})\\ &\le d(Sx_{m_k}, Sx_{m_k-1}) + \epsilon\\ &\le d(Sx_k, Sx_{k-1}) + \epsilon.\end{aligned}$$

This implies $\lim_{n\to\infty} d(Sx_{m_k}, Sx_{n_k}) = \epsilon$.
By triangle inequality and (3.4), we have

$$\begin{aligned}d(Sx_{m_k}, Sx_{n_k}) &\le d(Sx_{m_k}, Sx_{m_k+1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k})\\ &\le d(Sx_{m_k}, Sx_{m_k-1}) + d(Sx_{m_k+1}, Sx_{n_k+1}) + d(Sx_{n_k-1}, Sx_{n_k})\\ &\le 2d(Sx_k, Sx_{k-1}) + d(Sx_{m_k+1}, Sx_{n_k+1}). \qquad (3.6)\end{aligned}$$

By (3.2) and (3.4) we have

$$\begin{aligned}d(Sx_{m_k+1}, Sx_{n_k+1}) &\le F(\max\{d(Tx_{m_k+1}, Tx_{n_k+1}), d(Tx_{m_k+1}, Sx_{m_k+1}), d(Tx_{n_k+1}, Sx_{n_k+1}) +\\ &\qquad d(Tx_{n_k+1}, Sx_{m_k+1})\})\\ &\le F(\max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{m_k}, Sx_{m_k+1}), d(Sx_{n_k}, Sx_{n_k+1}) +\\ &\qquad d(Sx_{n_k}, Sx_{m_k+1})\})\\ &\le F(\max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{n_k}, Sx_{n_k+1}) + d(Sx_{n_k}, Sx_{m_k+1})\}).\end{aligned}$$

Since $d(Sx_{n_k}, Sx_{m_k+1}) \le d(Sx_{m_k+1}, Sx_{m_k}) + d(Sx_{m_k}, Sx_{n_k})$, so by (3.2) and (3.4) we have

$$\begin{aligned}d(Sx_{m_k+1}, Sx_{n_k+1}) &\le F(\max\{d(Sx_{m_k}, Sx_{n_k}), d(Sx_{n_k}, Sx_{n_k+1}) + d(Sx_{m_k+1}, Sx_{m_k}) +\\ &\qquad d(Sx_{m_k}, Sx_{n_k})\})\\ &\le F(d(Sx_{n_k}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k})).\end{aligned}$$

Hence by (3.3),(3.5) and (3.6), we have

$$\begin{aligned}d(Sx_{m_k}, Sx_{n_k}) &\le 2d(Sx_k, Sx_{k-1}) + F(d(Sx_{n_k}, Sx_{n_k+1}) + d(Sx_{n_k+1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k}))\\ &\le 2d(Sx_k, Sx_{k-1}) + F(d(Sx_{n_k-1}, Sx_{n_k}) + d(Sx_{n_k-1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k}))\\ &\le 2d(Sx_k, Sx_{k-1}) + F(2d(Sx_{n_k-1}, Sx_{n_k}) + d(Sx_{m_k}, Sx_{n_k}))\\ &\le 2d(Sx_k, Sx_{k-1}) + F(2d(Sx_{k-1}, Sx_k) + d(Sx_{m_k}, Sx_{n_k})).\end{aligned}$$

By taking $k \to \infty$ in above inequality, we have $\epsilon \leq F(\epsilon) < \epsilon$ which is a contradiction. Hence $(Sx_n)_{n=0}^{\infty}$ is a Cauchy sequence in X. Since X is a complete metric space, there exists $t \in X$ such that$\lim_{n\to\infty} Sx_n = t$. Also $\lim_{n\to\infty} Tx_n = t$.

Since T is continuous, we have $\lim_{n\to\infty} T^2x_n = Tt$ and $\lim_{n\to\infty} TSx_n = Tt$. So $\lim_{n\to\infty} STx_n = Tt$ because T and S are commute. We now have

$d(STx_n, Sx_n) \leq F(\max\{d(T^2x_n, Tx_n), d(T^2x_n, STx_n), d(Tx_n, Sx_n)+d(Tx_n, STx_n)\})$.

By taking $n \to \infty$, we have

$$\begin{aligned} d(Tt, t) &\leq F(\max\{d(Tt, t), d(Tt, Tt), d(t, t) + d(t, Tt)\}) \\ &\leq F(d(Tt, t)). \end{aligned}$$

This implies $d(Tt, t) = 0$, hence $Tt = t$.
By (3.1), we have

$$d(St, Sx_n) \leq F(\max\{d(Tt, Tx_n), d(Tt, St), d(Tx_n, Sx_n) + d(Tx_n, St)\}).$$

By taking $n \to \infty$, we have

$$\begin{aligned} d(St, t) &\leq F(\max\{d(Tt, t), d(Tt, St), d(t, t) + d(t, St)\}) \\ &\leq F(d(t, St)). \end{aligned}$$

This implies $St = t$. Hence t is a commom fixed point of S and T.

Finally, we show that common fixed point of T and S is unique.
Let $Sw = Tw = w$ and $Sv = Tv = v$, then by (3.1)

$$\begin{aligned} d(w, v) = d(Sw, Sv) &\leq F(\max\{d(Tw, Tv), d(Tw, Sw), d(Tv, Sv) + d(Tv, Sw)\}) \\ &\leq F(d(w, v)). \end{aligned}$$

This implies $w = v$. Therefore S and T have a unique common fixed point. □

Corollary 3.2.22 *Let (X, d) be a complete metric space and let $S, T : X \to X$ are commuting mappings satisfying the inequality*

$$d(Sx, Sy) \leq c \cdot \max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy) + d(Ty, Sx)\}), \forall x, y \in X,$$

where $0 \leq c < 1$. If $SX \subset TX$ and T is continuous then S and T have a unique common fixed point.

Proof. Define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(t) = ct$ for all $t \in \mathbb{R}^+$. Then F is satisfied the condition in Theorem 3.2.21. Hence the corollary is obtained directly by Theorem 3.2.21. $\square$

Corollary 3.2.23 *Let S be selfmapping of a complete metric space (X, d) satisfying the inequality*

$$d(Sx, Sy) \leq F(\max\{d(x, y), d(x, Sx), d(y, Sy) + d(y, Sx)\}), \forall x, y \in X$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function such that $F(t) < t$ for each $t > 0$. Then S has a unique fixed point.

Proof. Let T be the identity mapping in Theorem 3.2.21. Then all conditions of Theorem 3.2.21 are satisfied and so S has a unique fixed point. $\square$

Corollary 3.2.24 *Let S be selfmapping of a complete metric space (X, d) satisfying the inequality*

$$d(Sx, Sy) \leq c \cdot (\max\{d(x, y), d(x, Sx), d(y, Sy) + d(y, Sx)\}), \forall x, y \in X$$

where $0 \leq c < 1$. Then S have a unique fixed point.

Proof. Define $F : \mathbb{R}^+ \to \mathbb{R}^+$ by $F(t) = ct$ for all $t \in \mathbb{R}^+$. and Let T be the identity mapping in Theorem 3.2.21. Then all conditions of Theorem 3.2.21 are satisfied and so S has a unique fixed point. $\square$

3.3 Examples of Applications

The theorem then yields existence and uniqueness theorems for differential and integral equations, as we shall see.

Example 3.3.1 Application to Ordinary Differential Equation

Let consider an explicit ordinary differential equation of the first order

$$x' = f(t, x). \tag{3.7}$$

An initial value problem for such an equation consists of the equation and an initial condition

$$x(t_0) = x_0 \tag{3.8}$$

where t_0 and x_0 are given real numbers. Let f be continuous on a rectangle

$$R = \{(t,x)||t - t_0 \leq a, |x - x_0| \leq b\}$$

and thus bounded on R, say

$$|f(t,x)| \leq c \tag{3.9}$$

for all $(t,x) \in R$.

Suppose that f satisfies a Lipschitz condition on R with respect to its second argument, that is, there is a constant k (Lipschitz constant) such that for $(t,x),(t,y) \in R$

$$|f(t,x) - f(t,y)| \leq k|x - y|. \tag{3.10}$$

Then the initial value problem (1) has a unique solution. This solution exist on interval $[t_0 - \beta, t_0 + \beta]$, where

$$\beta < \min\{a, \frac{b}{c}, \frac{1}{k}\}. \tag{3.11}$$

Proof Let $C(J)$ be the metric space of all real-valued continuous functions on the interval $J = [t_0 - \beta, t_0 + \beta]$ with metric d defined by

$$d(x,y) = \max_{t \in J} |x(t) - y(t)|. \tag{3.12}$$

$C(J)$ is complete, Let $\tilde{C}$ be the subspace of $C(J)$ consisting of all those function $x \in C(J)$ that satisfy

$$|x(t) - x_0| \leq c\beta. \tag{3.13}$$

It is not difficult to see that $\tilde{C}$ is closed in $C(J)$, so that $\tilde{C}$ is complete.
By integration we see that (1) can be written $x = Tx$, where $T : \tilde{C} \to \tilde{C}$ is defined by

$$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau. \tag{3.14}$$

Indeed, T is defined for all $x \in \tilde{C}$, because $c\beta < b$ by (3.11) , so that if $x \in \tilde{C}$, then $\tau \in J$ and $(\tau, x(\tau)) \in R$, and the integral in (3.14) exist since f is continuous on R. To see that T maps $\tilde{C}$ into itself, we can use (3.14) and (3.9), obtaining

$$|Tx(t) - x_0| = \left|\int_{t_0}^{t} f(\tau, x(\tau))\right| d\tau \leq c|t - t_0| \leq c\beta.$$

We show that T satisfying strict contractive condition on $\tilde{C}$. By the Lipschitz condition (3.10),

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left|\int_{t_0}^{t} f(\tau, x(\tau)) - f(\tau, y(\tau)) d\tau\right| \\ &\leq |t - t_0| \max_{t \in J} k|x(\tau) - y(\tau)| \\ &\leq k\beta d(x, y). \end{aligned}$$

Since the last expression does not depend on t, we can take the maximum on the left and have

$$d(Tx, Ty) \leq \alpha d(x, y) \qquad \text{where } \alpha = k\beta.$$

From (3.11) we see that $\alpha = k\beta < 1$, so that

$$d(Tx, Ty) \leq \alpha d(x, v) \leq c \cdot \max\{d(x, y), d(x, Tx), d(y, Ty) + d(y, Tx)\},$$

where $0 < c < 1, \forall x, y \in \bar{C}$. Thus implies that T has a unique fixed point $x \in \tilde{C}$, that is, a continuous function x on J satisfying $x = Tx$. So we have by (3.14)

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) d\tau.$$

□

Example 3.3.2 Application to Integral Equation

An integral equation of the form

$$x(t) - \mu \int_a^b k(t, \tau) x(\tau) d\tau = v(t) \tag{3.15}$$

is called a Fredholm equation of the second kind. Here,$[a, b]$ is a given interval. x is a function on $[a, b]$ which is unknown. μ is a parameter. The kernel k of the equation is a given function on the square $G = [a, b] \times [a, b]$ and v is a given function on $[a, b]$.

we consider (3.15) on $C[a, b]$, the space of all continuous functions defined on the interval $J = [a, b]$ with metric d given by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|. \tag{3.16}$$

For apply this theorem it is important to note that $C[a, b]$ is complete. We assume that $v \in C[a, b]$ and k is continuous on G. Then k is a bounded function on G, say,

$$|k(t, \tau)| \leq c \tag{3.17}$$

for all $(t, \tau) \in G$ Obviously,(3.15)) can be written $x = Tx$ where

$$Tx(t) = v(t) + \mu \int_a^b k(t, \tau) x(\tau) d\tau. \tag{3.18}$$

Since v and k are continuous, formular (3.18) defines an operator $T : C[a, b] \rightarrow C[a, b]$. We now impose a restriction on μ such that T becomes a contraction. From (3.16)to (3.18) we have

$$\begin{aligned} d(Tx, Ty) &= max|Tx(t) - Ty(t)| \\ &= |\mu| \max_{t \in J} \left| \int_a^b k(t, \tau)[x(\tau) - y(\tau) d\tau \right| \\ &\leq |\mu| \max_{t \in J} \int_a^b |\, k(t, \tau)| \left| x(\tau) - y(\tau) \right| d\tau \\ &\leq |\mu| \; c \max_{t \in J} |x(\sigma) - y(\sigma)| \int_a^b d\tau \\ &= |\mu| \; c \; d(x, y) \, (b - a). \end{aligned}$$

This can be written $d(Tx, Ty) \leq \alpha d(x, y)$, where $\alpha = |\mu| c(b - a)$,

$$|\mu| < \frac{1}{c(b - a)}.$$

So that, $d(Tx, Ty) < \alpha d(x, y) < c \cdot \max\{d(x, y), d(x, Tx), d(y, Ty) + d(y, Tx)\}$ where $0 < c < 1$, $\forall x, y \in C[a, b]$. Thus implies that T has a unique fixed point $x \in C$, that is, a continuous function x on $[a, b]$ satisfying $x = Tx$. So we have by (3.18) $\quad x(t) = v(t) + \mu \int_a^b k(t, \tau) x(\tau) d\tau.$ □

Example 3.3.3

Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$. Define $T : X \to X$ by $Tx = \frac{1}{5}(x^3 + x^2 + 1), \forall x \in X$, and define $\phi : X \to \mathbb{R}^+$ by

$$\phi(x) = \begin{cases} -\frac{3}{2}x + 3 & 0 \le x \le 0.210756 \\ \\ 3x + 1 & 0.210756 < x \le 1. \end{cases}$$

Then $d(x,Tx) = |x - Tx| = |x - \frac{1}{5}(x^3 + x^2 + 1)|$ and

Case I $\quad 0 \le x \le 0.210756.$

$$\begin{aligned} \phi(x) - \phi(Tx) &= \big(-\frac{3}{2}x + 3\big) - \big[-\frac{3}{2}\bigg(\frac{1}{5}(x^3 + x^2 + 1)\bigg) + 3\big] \\ &= -\frac{3}{2}x + 3 + \frac{3}{2}\bigg(\frac{1}{5}(x^3 + x^2 + 1)\bigg) - 3 \\ &= \frac{3}{2}\big[\frac{1}{5}(x^3 + x^2 + 1) - x\big] \\ &= \frac{3}{2}\big|x - \frac{1}{5}(x^3 + x^2 + 1)\big|, \end{aligned}$$

so $d(x,Tx) \le \phi(x) - \phi(Tx)$ where $0 \le x \le 0.210756.$

Case II $\quad 0.210756 < x \le 1.$

$$\begin{aligned} \phi(x) - \phi(Tx) &= (3x + 1) - \big(3\big[\frac{1}{5}(x^3 + x^2 + 1)\big] + 1\big) \\ &= 3x - 3\big(\frac{1}{5}(x^3 + x^2 + 1)\big) \\ &= 3\big|x - \frac{1}{5}(x^3 + x^2 + 1)\big|, \end{aligned}$$

so $d(x,Tx) \le \phi(x) - \phi(Tx)$ where $0.210756 < x \le 1.$

Hence $d(x,Tx) \le \phi(x) - \phi(Tx), \forall x \in X$. And for $x \neq y \in X$ we have

$$\begin{aligned} d(Tx,Ty) = |Tx - Ty| &= \big|\frac{1}{5}(x^3 + x^2 + 1) - \frac{1}{5}(y^3 + y^2 + 1)\big| \\ &= \frac{1}{5}\big|x^3 - y^3 + x^2 - y^2\big| \\ &= \frac{1}{5}|x - y|\big|x^2 + xy + y^2 + x + y\big| \\ &< |x - y| = d(x,y), \end{aligned}$$

so $d(Tx, Ty) < d(x, y)$. Thus T satisfies the condition (2) of theorem 3.1.2. By Theorem 3.1.2 T has a fixed point. Let $x_0 = 0$ and let $x_n = Tx_{n-1}$, $n \in N$.
We obtain that

x_n	$T(x_n)$	$\lvert x_n - T(x_n)\rvert$
$x_1 = 0.200000000000$	0.209600000000	0.00960000000
$x_2 = 0.209600000000$	0.210628068147	0.001028068147
$x_3 = 0.210628068147$	0.210741705054	0.000113636907
$x_4 = 0.210741705054$	0.210754308163	0.000012603109
$x_5 = 0.210754308163$	0.210755706453	0.000001398290
$x_6 = 0.210755706453$	0.210755861597	0.000000155144
$x_7 = 0.210755861597$	0.210755878811	0.000000017214
$x_8 = 0.210755878811$	0.210755880721	0.000000001910
$x_9 = 0.210755880721$	0.210755880933	0.000000000212
$x_{10} = 0.210755880933$	0.210755880956	0.000000000023

By using MATLAB , the fixed point of T is approximated 0.210756 .

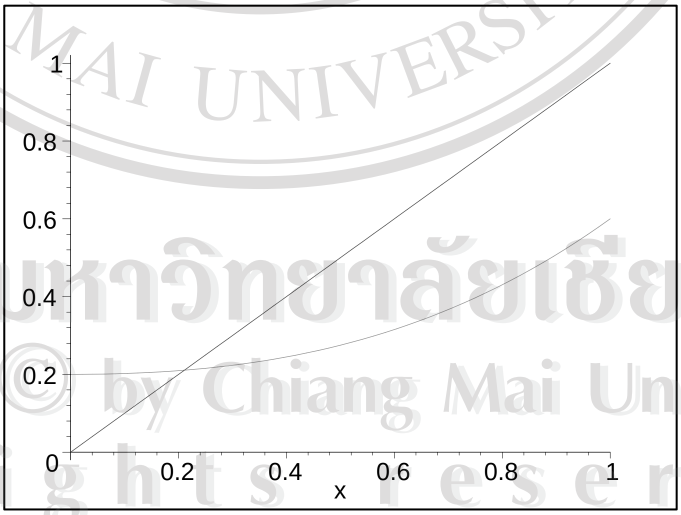

Figure 3.1: The relation of graph between $y = \frac{1}{5}(x^3 + x^2 + 1)$ and $y = x$.

$\square$