CHAPTER 4

CONCLUSION

This research extends the results obtained by D.El Moutawaki [1] and B.Rodjanadid [10] . The new strict contractive conditions are given for selfmappings in complete metric space X to guarantee that selfmappings on X have a unique common fixed point and the iterative methods for computing their common fixed point are given. The followings are the main result of this study.

1. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Tx, y) + c_2 d(Tx, x)\}, \forall x \neq y \in X$,

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 < 1$ Then T has a unique fixed point.

2. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X$,

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where $c_1 > 0$ and $0 < c_2 < 1$ Then T has a unique fixed point.

4. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, x), c_1 d(Ty, y), c_2 d(Tx, y)\}, \forall x \neq y \in X,$

where $c_1 > 0$ and $0 < c_2 < 1$ Then T has a unique fixed point.

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where $c_1 > 0$ and $0 < c_2 < 1$ Then T has a unique fixed point.

6. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

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where $c_1 > 0$ and $0 < c_2 < 1$ Then T has a unique fixed point.

7. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

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where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 = 1$ Then T has a unique fixed point.

8. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

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where $0 < c_1 < 1$ and $0 < c_2 < 1$ Then T has a unique fixed point.

9. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Ty, x) + c_2 d(Tx, x)\}, \forall x \neq y \in X,$

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(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Ty, x), c_2 d(Tx, x)\}, \forall x \neq y \in X,$

where $0 < c_1 < 1$ and $0 < c_2 < 1$ Then T has a unique fixed point.

11. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Tx, y) + c_2 d(Tx, x)\}, \forall x \neq y \in X,$

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 < 1$ Then T has a unique fixed point.

12. Let (X, d) be a complete metric space and let $T : X \to X$. Suppose that there exists a mapping $\Phi : X \to \mathbb{R}^+$ such that

(1) $d(x, Tx) \leq \Phi(x) - \Phi(Tx), \forall x \in X,$

(2) $d(Tx, Ty) < \max\{d(x, y), d(Ty, y), c_1 d(Tx, y), c_2 d(Tx, x)\}, \forall x \neq y \in X,$

where $c_1 > 0, c_2 > 0$ and $c_1 + c_2 < 1$ Then T has a unique fixed point.

13. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(1) T and S satisfy the property(E.A),

(2) $d(Tx,Ty) < \max\{c_1[d(Sx,Sy)+d(Tx,Sy)] + c_2[d(Tx,Sx)+d(Ty,Sy)]\}$, $\forall x \neq y \in X$, where $0 \leq c_1 \leq 1/2$ and $0 \leq c_2 \leq 1/2$

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

14. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(1) T and S satisfy the property(E.A),

(2) $d(Tx,Ty) < c \cdot \max\{[d(Sx,Sy)+d(Tx,Sy)], [d(Tx,Sx)+d(Ty,Sy)]\}$, $\forall x \neq y \in X$, where $0 \leq c \leq 1/2$.

(3) $TX \subset SX$.

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(1) T and S satisfy the property(E.A),

(2) $d(Tx,Ty) < c \cdot (d(Sx,Sy)+d(Tx,Sy)), \forall x \neq y \in X$, where $0 \leq c \leq 1/2$.

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17. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

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(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1 d(Tx, Sy) + c_2 d(Ty, Sx), d(Tx, Sx)\}$,
$\forall x \neq y \in X$, where $c_1 \geq 0$, $c_2 \geq 0$ and $c_1 + c_2 < 1$.

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If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

18. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(1) T and S satisfy the property(E.A),

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c[d(Tx, Sy) + d(Ty, Sx)], d(Tx, Sx)\}$,
$\forall x \neq y \in X$, where $0 \leq c < 1/2$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

19. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(1) T and S satisfy the property(E.A),

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Tx, Sy), d(Tx, Sx)\}$,
$\forall x \neq y \in X$, where $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

20. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(1) T and S satisfy the property(E.A),

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Ty, Sx), d(Tx, Sx)\}$,
$\forall x \neq y \in X$, where $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

21. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(1) T and S satisfy the property(E.A),

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1d(Tx, Sy) + c_2d(Ty, Sy), d(Tx, Sx)\}$,
$\forall x \neq y \in X$, where $0 \leq c_1 < 1$ and $0 \leq c_2 < 1$.

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$\forall x \neq y \in X$, where $c_1 \geq 0$, $c_2 \geq 0$ and $c_1 + c_2 < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

25. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

(1) T and S satisfy the property(E.A),

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c[d(Ty, Sx) + d(Ty, Sy)], d(Tx, Sx)\}$,
$\forall x \neq y \in X$, where $0 \leq c < 1/2$.

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If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

26. Let S and T be two weakly compatible selfmappings of a metric space (X, d) such that

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(2) $d(Tx, Ty) < \max\{d(Sx, Sy), c_1 d(Tx, Sx) + c_2 d(Ty, Sx), d(Tx, Sy)\}$, $\forall x \neq y \in X$, where $c_1 \geq 0$, $0 \leq c_2 < 1$.

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$\forall x \neq y \in X$, where $0 \leq c < 1/2$.

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If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

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(2) $d(Tx, Ty) < \max\{d(Sx, Sy), cd(Ty, Sy), d(Tx, Sy)\}$,
$\forall x \neq y \in X$, where $0 \leq c < 1$.

(3) $TX \subset SX$.

If SX or TX is a complete subspace of X, then T and S have a unique commom fixed point.

33. Let (X, d) be a complete metric space and let $S, T : X \to X$ are commuting mappings satisfying the inequality

$$d(Sx, Sy) \leq F(\max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy) + d(Ty, Sx)\}), \forall x, y \in X \tag{4.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function such that $F(t) < t$ for each $t > 0$.
If $SX \subset TX$ and T is continuous then S and T have a unique common fixed point.

34. Let (X, d) be a complete metric space and let $S, T : X \to X$ are commuting mappings satisfying the inequality

$$d(Sx, Sy) \leq c \cdot \max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy) + d(Ty, Sx)\}), \forall x, y \in X,$$

where $0 \leq c < 1$. If $SX \subset TX$ and T is continuous then S and T have a unique common fixed point.

35. Let S be selfmapping of a complete metric space (X, d) satisfying the inequality

$$d(Sx, Sy) \leq F(\max\{d(x, y), d(x, Sx), d(y, Sy) + d(y, Sx)\}), \forall x, y \in X$$

where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function such that $F(t) < t$ for each $t > 0$. Then S has a unique fixed point.

36. Let S be selfmapping of a complete metric space (X, d) satisfying the inequality

$$d(Sx, Sy) \leq c \cdot (\max\{d(x, y), d(x, Sx), d(y, Sy) + d(y, Sx)\}), \forall x, y \in X$$

Then S has a unique fixed point.