CHAPTER 3

MAIN RESULTS

In this chapter we consider controlling chaos and synchronization of perturbed Lü chaotic dynamical system.

3.1 The Perturbed Lü Chaotic Dynamical System

We will study the perturbed Lü chaotic dynamical system that is described by system of ordinary differential equations

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+cy \\ \dot{z} &= xy-bz+dx^2 \end{aligned} \tag{3.1}$$

where

x, y and z are the state variables.

a, b, c and d are positive real constants.

The equilibrium points of the system (3.1) are

$$E_1 = (0,0,0),\ E_2 = (\beta,\beta,c),\ E_3 = (-\beta,-\beta,c)$$

where $\beta = \sqrt{\dfrac{bc}{1+d}}$.

Theorem 3.1.1 *The equilibrium point $E_1 = (0,0,0)$ is*

(i) asymptotically stable if $a > c$ and $b > c$.

(ii) unstable if $a > c$ and $b < c$.

Proof The Jacobian matrix of the system (3.1) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ 0 & c & 0 \\ 0 & 0 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c \\ a_2 &= ab - ac - bc \\ a_3 &= -abc \\ a_1a_2 - a_3 &= (a+b)(a-c)(b-c). \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $a > c$ and $b > c$. Thus, if $a > c$ and $b > c$, then the equilibrium point $E_1 = (0, 0, 0)$ is asymptotically stable.

On the other hand, when $a > c$ and $b < c$, we have $a_1a_2 - a_3 < 0$ which does not satisfy the Routh-Hurwitz criteria and so the equilibrium point $E_1 = (0, 0, 0)$ is unstable. □

Theorem 3.1.2 *The equilibrium point $E_2 = (\beta, \beta, c)$ is*

(i) asymptotically stable if $a > 4c$.

(ii) unstable if $2c > a$, $c > b$ and $a + b > c$.

Proof The Jacobian matrix of the system (3.1) at the equilibrium point $E_2 = (\beta, \beta, c)$ is given by

$$J_2 = \begin{bmatrix} -a & a & 0 \\ -c & c & -\beta \\ \beta + 2d\beta & \beta & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_2 is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c \\ a_2 &= ab - bc + \beta^2 \\ a_3 &= 2abc \\ a_1a_2 - a_3 &= \frac{b^2d(a-c) + ab(a-3c) + abd(a-4c) + ab^2 + bc^2d}{(1+d)}. \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $a > 4c$. Thus, if $a > 4c$, then the equilibrium point $E_2 = (\beta, \beta, c)$ is asymptotically stable.

On the other hand, when $2c > a$,$c > b$ and $a + b > c$, we have $a_1a_2 - a_3 < 0$ which does not satisfy the Routh-Hurwitz criteria and so the equilibrium point $E_2 = (\beta, \beta, c)$ is unstable. □

Theorem 3.1.3 *The equilibrium point $E_3 = (-\beta, -\beta, c)$ is*

(i) asymptotically stable if $a > 4c$.

(ii) unstable if $2c > a$, $c > b$ and $a + b > c$.

Proof The Jacobian matrix of the system (3.1) at the equilibrium point $E_3 = (-\beta, -\beta, c)$ is given by

$$J_3 = \begin{bmatrix} -a & a & 0 \\ -c & c & \beta \\ -\beta - 2d\beta & -\beta & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c \\ a_2 &= ab-bc+\beta^2 \\ a_3 &= 2abc \\ a_1a_2-a_3 &= \frac{b^2d(a-c)+ab(a-3c)+abd(a-4c)+ab^2+bc^2d}{(1+d)}. \end{aligned}$$

We see that a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria when $a > 4c$. Thus, if $a > 4c$, then the equilibrium point $E_3 = (-\beta, -\beta, c)$ is asymptotically stable.

On the other hand, when $2c > a$, $c > b$ and $a+b > c$, we have $a_1a_2 - a_3 < 0$ which does not satisfy the Routh-Hurwitz criteria and so the equilibrium point $E_3 = (-\beta, -\beta, c)$ is unstable. □

Next we study the perturbed Lü chaotic dynamical system that is described by system of ordinary differential equations

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+cy \\ \dot{z} &= xy-bz+dsin(x) \end{aligned} \tag{3.2}$$

where

x, y and z are the state variables.

a, b, c and d are positive real constants.

The equilibrium points of the system (3.2) are

$$E_1 = (0,0,0),\ E_2 = (x_1, x_1, c),\ E_3 = (x_2, x_2, c)$$

where x_1 is negative real root of $g(x)$, x_2 is positive real root of $g(x)$, where $g(x) = x^2 + dsin(x) - bc$.

Theorem 3.1.4 *The equilibrium point $E_1 = (0,0,0)$ of (3.2) is*

(i) asymptotically stable if $a > c$ and $b > c$.

(ii) unstable if $a > c$ and $b < c$.

Proof The Jacobian matrix of the system (3.2) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ 0 & c & 0 \\ 0 & 0 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c \\ a_2 &= ab - ac - bc \\ a_3 &= -abc \\ a_1a_2 - a_3 &= (a+b)(a-c)(b-c). \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $a > c$ and $b > c$. Thus, if $a > c$ and $b > c$, then the equilibrium point $E_1 = (0,0,0)$ is asymptotically stable.

On the other hand, when $a > c$ and $b < c$,we have $a_1a_2 - a_3 < 0$ which does not satisfy the Routh-Hurwitz criteria and so the equilibrium point $E_1 = (0,0,0)$ is unstable. □

Theorem 3.1.5 *The equilibrium point $E_2 = (x_1, x_1, c)$ is*

(i) asymptotically stable if $a > 2c$, $b > \sqrt{x_1^2 + x_1 d cos(x_1)}$ and $b > c$.

(ii) unstable if $2c > a$, $b < \sqrt{x_1^2 + x_1 d cos(x_1)}$ and $b < c$.

Proof The Jacobian matrix of the system (3.2) at the equilibrium point $E_2 = (x_1, x_1, c)$ is given by

$$J_2 = \begin{bmatrix} -a & a & 0 \\ -c & c & -x_1 \\ x_1 + dcos(x_1) & x_1 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_2 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c \\ a_2 &= ab - bc + x_1^2 \\ a_3 &= 2x_1^2 + x_1 dcos(x_1) \\ a_1a_2 - a_3 &= ab(a - 2c) + a(b^2 - x_1^2 - x_1dcos(x_1)) + (b - c)(x_1^2 - bc). \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $a > 2c$, $b > \sqrt{x_1^2 + x_1dcos(x_1)}$ and $b > c$. Thus, if $a > 2c$, $b > \sqrt{x_1^2 + x_1dcos(x_1)}$ and $b > c$, then the equilibrium point $E_2 = (x_1, x_1, c)$ is asymptotically stable.

On the other hand, when $2c > a$, $b < \sqrt{x_1^2 + x_1dcos(x_1)}$ and $b < c$, we have $a_1a_2 - a_3 < 0$ which does not satisfy the Routh-Hurwitz criteria and so the equilibrium point $E_2 = (x_1, x_1, c)$ is unstable. □

Theorem 3.1.6 *The equilibrium point $E_3 = (x_2, x_2, c)$ is*

(i) asymptotically stable if $a > 2c$, $b > \sqrt{x_2^2 + x_2dcos(x_2)}$ and $b < c$.

(ii) unstable if $2a > c$, $b < \sqrt{x_2^2 + x_2dcos(x_2)}$, $a < \sqrt{bc}$ and $b < c$.

Proof The Jacobian matrix of the system (3.2) at the equilibrium point $E_3 = (x_2, x_2, c)$ is given by

$$J_3 = \begin{bmatrix} -a & a & 0 \\ -c & c & -x_2 \\ x_2 + dcos(x_2) & x_2 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c \\ a_2 &= ab - bc + x_2^2 \\ a_3 &= 2x_2^2 + x_2dcos(x_2) \\ a_1a_2 - a_3 &= ab(a - 2c) + a(b^2 - x_2^2 - x_2dcos(x_2)) + (c - b)(bc - x_2^2). \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $a > 2c$, $b > \sqrt{x_2^2 + x_2dcos(x_2)}$ and $b < c$. Thus, if $a > 2c$, $b > \sqrt{x_2^2 + x_2dcos(x_2)}$ and $b < c$, then the equilibrium point $E_3 = (x_2, x_2, c)$ is asymptotically stable.

On the other hand, when $2a > c$, $b < \sqrt{x_2^2 + x_2dcos(x_2)}$, $a < \sqrt{bc}$ and $b < c$, we have $a_1a_2 - a_3 < 0$ which does not satisfy the Routh-Hurwitz criteria and so the equilibrium point $E_3 = (x_2, x_2, c)$ is unstable. □

3.1.1 Numerical Simulations

Numerical experiments are carried out to investigate perturbed Lü chaotic dynamical system by using fourth-order Runge-Kutta method with time step 0.001. In Fig. 3.1-3.3, the parameters a, b, c and d are chosen as $a = 36$, $b = 3$, $c = 20$ and $d = 1$. The initial states are taken as $x = 10$, $y = 1$ and $z = 8$. Fig. 3.1 shows the behavior of the states x, y and z of the system (3.1) with time in xy-plane. Fig. 3.2 shows the behavior of the states x, y and z

of the system (3.1) with time in xz-plane. Fig. 3.3 shows the behavior of the states x, y and z of the system (3.1) with time in yz-plane. In Fig. 3.4-3.6, the parameters a, b, c and d are chosen as $a = 40$, $b = 0.1$, $c = 40 - 10sin(2)$ and $d = 1$. The initial states are taken as $x = 10$, $y = 1$ and $z = 8$. Fig. 3.4 shows the behavior of the states x, y and z of the system (3.2) with time in xy-plane. Fig. 3.5 shows the behavior of the states x, y and z of the system (3.2) with time in xz-plane. Fig. 3.6 shows the behavior of the states x, y and z of the system (3.2) with time in yz-plane. In Fig. 3.7-3.9, the parameters a, b, c and d are chosen as $a = 1.2$, $b = 1.5$, $c = \frac{4-sin(2)}{1.5}$ and $d = 1$. The initial states are taken as $x = 10$, $y = 1$ and $z = 8$. Fig. 3.7 shows the behavior of the states x, y and z of the system (3.2) with time in xy-plane. Fig. 3.8 shows the behavior of the states x, y and z of the system (3.2) with time in xz-plane. Fig. 3.9 shows the behavior of the states x, y and z of the system (3.2) with time in yz-plane.

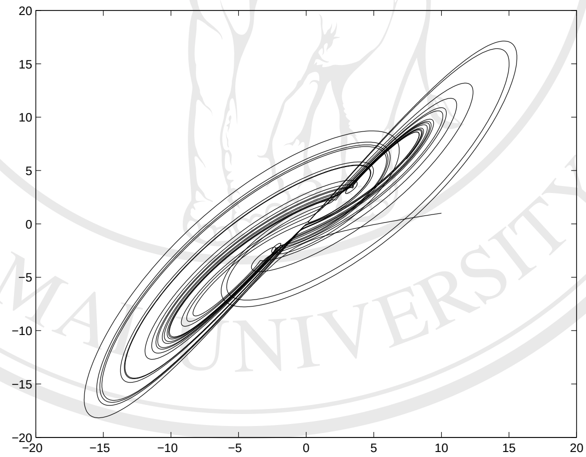

Figure 3.1: The chaotic attractor of perturbed Lü chaotic dynamical system (3.1) in the xy-plane where $a = 36$, $b = 3$, $c = 20$ and $d = 1$.

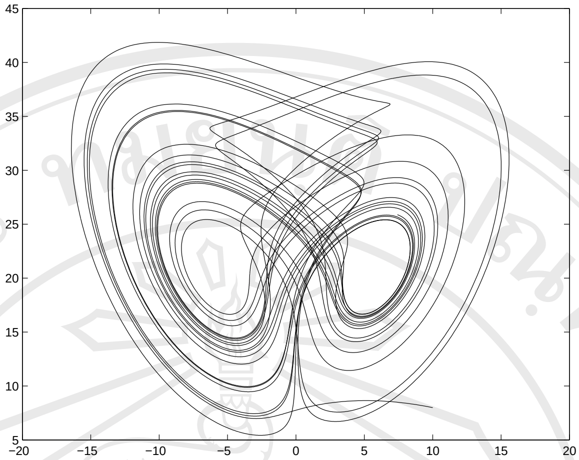

Figure 3.2: The chaotic attractor of perturbed Lü chaotic dynamical system (3.1) in the xz-plane where $a = 36$, $b = 3$, $c = 20$ and $d = 1$.

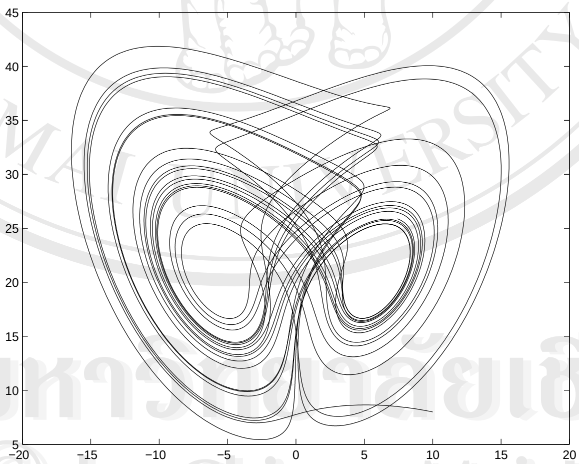

Figure 3.3: The chaotic attractor of perturbed Lü chaotic dynamical system (3.1) in the yz-plane where $a = 36$, $b = 3$, $c = 20$ and $d = 1$.

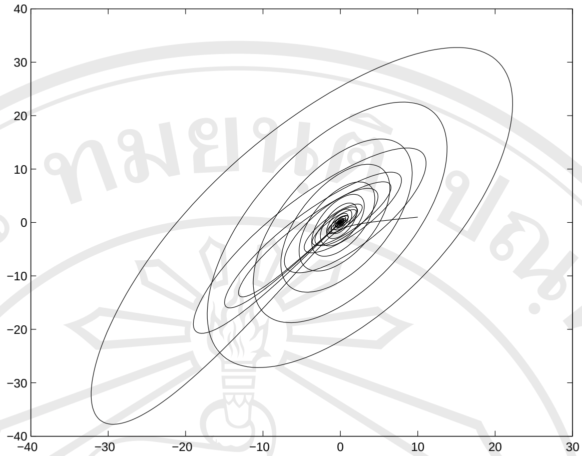

Figure 3.4: The chaotic attractor of perturbed Lü chaotic dynamical system (3.2) in the xy-plane where $a = 40$, $b = 0.1$, $c = 40 - 10sin(2)$ and $d = 1$.

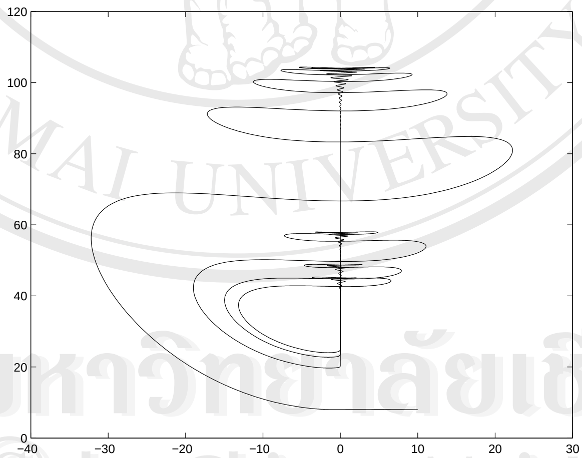

Figure 3.5: The chaotic attractor of perturbed Lü chaotic dynamical system (3.2) in the xz-plane where $a = 40$, $b = 0.1$, $c = 40 - 10sin(2)$ and $d = 1$.

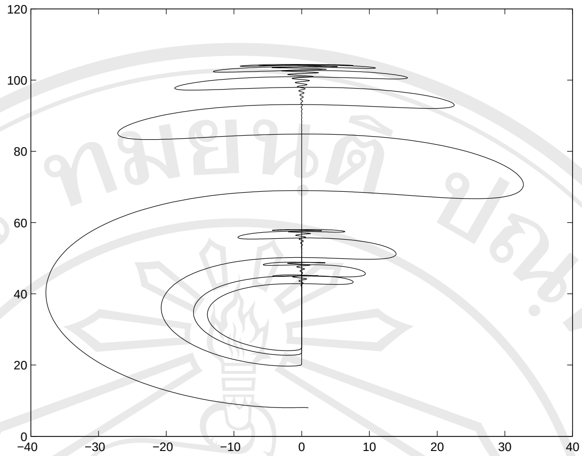

Figure 3.6: The chaotic attractor of perturbed Lü chaotic dynamical system (3.2) in the yz-plane where $a = 40$, $b = 0.1$, $c = 40 - 10sin(2)$ and $d = 1$.

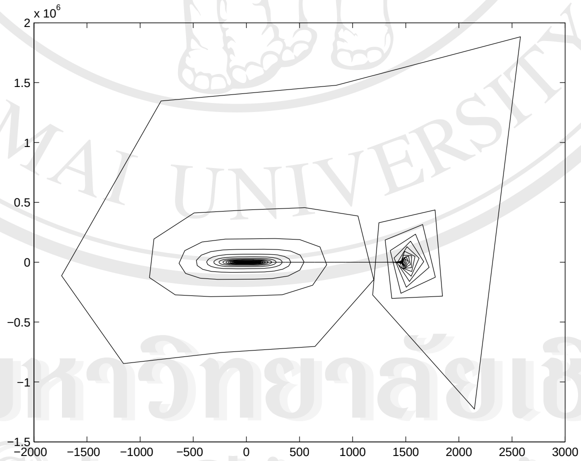

Figure 3.7: The chaotic attractor of perturbed Lü chaotic dynamical system (3.2) in the xy-plane where $a = 1.2$, $b = 1.5$, $c = \frac{4-sin(2)}{1.5}$ and $d = 1$.

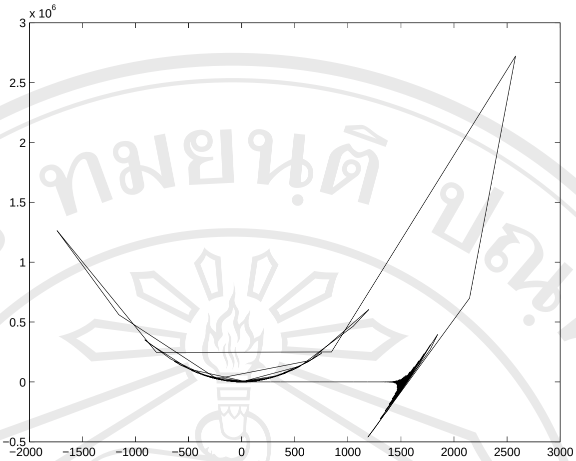

Figure 3.8: The chaotic attractor of perturbed Lü chaotic dynamical system (3.2) in the xz-plane where $a = 1.2$, $b = 1.5$, $c = \frac{4-sin(2)}{1.5}$ and $d = 1$.

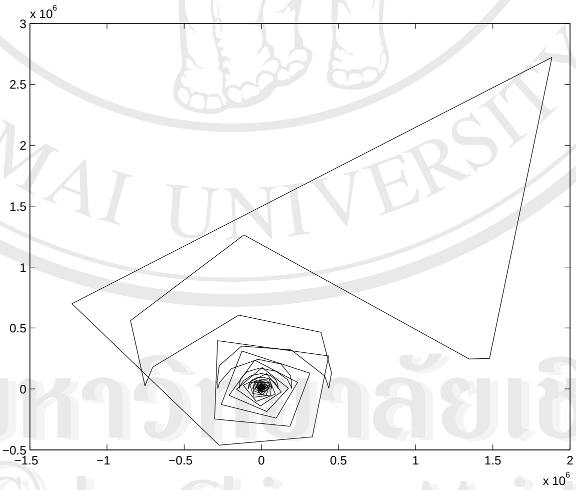

Figure 3.9: The chaotic attractor of perturbed Lü chaotic dynamical system (3.2) in the yz-plane where $a = 1.2$, $b = 1.5$, $c = \frac{4-sin(2)}{1.5}$ and $d = 1$.

3.2 Controlling Chaos of Perturbed Lü System to Equilibrium Point

In this section, the chaos of system (3.1) and system (3.2) are controlled to one of three equilibrium points of the system. Feedback and bounded feedback controls are applied to achieve this goal. We shall study the case when equilibrium points of (3.1) and (3.2) are unstable.

3.2.1 Feedback Control Method

The goal of linear feedback control is to control the chaotic behavior of the system (3.1) and system (3.2) to one of three unstable equilibrium points (E_1, E_2 or E_3). For system (3.1), we assume that the controlled system is given by

$$\begin{aligned} \dot{x} &= a(y-x)+u_1 \\ \dot{y} &= -xz+cy+u_2 \\ \dot{z} &= xy-bz+dx^2+u_3, \end{aligned}$$

where u_1, u_2 and u_3 are controllers that satisfy the following control law

$$\begin{aligned} \dot{x} &= a(y-x)-k_{11}(x-\bar{x}) \\ \dot{y} &= -xz+cy-k_{22}(y-\bar{y}) \\ \dot{z} &= xy-bz+dx^2-k_{33}(z-\bar{z}), \end{aligned} \tag{3.3}$$

where $E=(\bar{x},\bar{y},\bar{z})$ is an equilibrium point of system (3.1).

Stability of the Equilibrium Point $E_1=(0,0,0)$

In this case $E=E_1$ and the controlled system (3.3) is in the form of

$$\begin{aligned}\dot{x} &= a(y-x) - k_{11}x \\ \dot{y} &= -xz + (c-k_{22})y \\ \dot{z} &= xy - bz + dx^2 - k_{33}z.\end{aligned} \tag{3.4}$$

Theorem 3.2.1 *The equilibrium point $E_1 = (0,0,0)$ is asymptotically stable if $k_{11} = 0$, $k_{33} > 0$ and $k_{22} > c$.*

Proof The Jacobian matrix of the system (3.4) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ 0 & c-k_{22} & 0 \\ 0 & 0 & -b-k_{33} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c + k_{22} + k_{33} \\ a_2 &= (a+b+k_{33})(k_{22}-c) + ab + ak_{33} \\ a_3 &= (ab + ak_{33})(k_{22}-c) \\ a_1a_2 - a_3 &= (b + (k_{22}-c) + k_{33})((a+b+k_{33})(k_{22}-c) + ab + ak_{33}) \\ &= +a^2(k_{22}-c) + a^2b + a^2k_{33}. \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k_{11} = 0$, $k_{33} > 0$ and $k_{22} > c$. Thus, if $k_{11} = 0$, $k_{33} > 0$ and $k_{22} > c$, then the equilibrium point $E_1 = (0,0,0)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_2 = (\beta, \beta, c)$

In this case $E = E_2$ and the controlled system (3.3) is in the form of

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}(x-\beta) \\ \dot{y} &= -xz + cy - k_{22}(y-\beta) \\ \dot{z} &= xy - bz + dx^2 - k_{33}(z-c). \end{aligned} \tag{3.5}$$

Theorem 3.2.2 *The equilibrium point* $E_2 = (\beta, \beta, c)$ *is asymptotically stable if* $k_{11}, k_{33} > 0$ *and* $k_{22} > c$.

Proof The Jacobian matrix of the system (3.5) at the equilibrium point $E_2 = (\beta, \beta, c)$ is given by

$$J_2 = \begin{bmatrix} -a-k_{11} & a & 0 \\ -c & c-k_{22} & -\beta \\ \beta+2d\beta & \beta & -b-k_{33} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_2 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c+k_{11}+k_{22}+k_{33} \\ a_2 &= (b+k_{11}+k_{33})(k_{22}-c) + a(b+k_{22}+k_{33}) + b(k_{11}+k_{22}) + k_{11}k_{33} + \beta^2 \\ a_3 &= (k_{22}-c)(bk_{11}+k_{11}k_{33}) + a(bk_{22}+k_{22}k_{33}) + (2a+2ad+k_{11})\beta^2 \\ a_1a_2 - a_3 &= [(a+b-c+k_{22}+k_{33})(b+k_{11}+k_{33}) + k_{11}^2 + ab + (a(b+k_{22}+k_{33}) \\ &= +b(k_{11}+k_{22}) + k_{11}k_{33} + \beta^2)](k_{22}-c) + ab(a-c) + k_{11}(ab+ak_{22}+ak_{33} \\ &= +bk_{11}+bk_{22}+k_{11}k_{33}) + a(ak_{22}+ak_{33}+bk_{11}+k_{11}k_{33}) + b(ab+ak_{33} \\ &= +bk_{11}+bk_{22}+k_{11}k_{33}+\beta^2) + k_{33}(ab+ak_{33}+bk_{11}+bk_{22}+k_{11}k_{33}+\beta^2). \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k_{11}, k_{33} > 0$, and $k_{22} > c$. Thus, if $k_{11}, k_{33} > 0$, and $k_{22} > c$, then the equilibrium point $E_2 = (\beta, \beta, c)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_3 = (-\beta, -\beta, c)$

In this case $E = E_3$ and the controlled system (3.3) is in the form of

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}(x+\beta) \\ \dot{y} &= -xz + cy - k_{22}(y+\beta) \\ \dot{z} &= xy - bz + dx^2 - k_{33}(z-c). \end{aligned} \tag{3.6}$$

Theorem 3.2.3 *The equilibrium point* $E_3 = (-\beta, -\beta, c)$ *is asymptotically stable if* $k_{11}, k_{33} > 0$ *and* $k_{22} > c$.

Proof The Jacobian matrix of the system (3.6) at the equilibrium point $E_3 = (-\beta, -\beta, c)$ is given by

$$J_3 = \begin{bmatrix} -a-k_{11} & a & 0 \\ -c & c-k_{22} & \beta \\ -\beta-2d\beta & -\beta & -b-k_{33} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c + k_{11} + k_{22} + k_{33} \\ a_2 &= (b + k_{11} + k_{33})(k_{22} - c) + a(b + k_{22} + k_{33}) + b(k_{11} + k_{22}) + k_{11}k_{33} + \beta^2 \\ a_3 &= (k_{22} - c)(bk_{11} + k_{11}k_{33}) + a(bk_{22} + k_{22}k_{33}) + (2a + 2ad + k_{11})\beta^2 \\ a_1a_2 - a_3 &= [(a + b - c + k_{22} + k_{33})(b + k_{11} + k_{33}) + k_{11}^2 + ab + (a(b + k_{22} + k_{33}) \\ &= +b(k_{11} + k_{22}) + k_{11}k_{33} + \beta^2)](k_{22} - c) + ab(a - c) + k_{11}(ab + ak_{22} + ak_{33} \end{aligned}$$

$$= \quad +bk_{11} + bk_{22} + k_{11}k_{33}) + a(ak_{22} + ak_{33} + bk_{11} + k_{11}k_{33}) + b(ab + ak_{33}$$
$$= \quad +bk_{11} + bk_{22} + k_{11}k_{33} + \beta^2) + k_{33}(ab + ak_{33} + bk_{11} + bk_{22} + k_{11}k_{33} + \beta^2).$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k_{11}, k_{33} > 0$, and $k_{22} > c$. Thus, if $k_{11}, k_{33} > 0$, and $k_{22} > c$, then the equilibrium point $E_3 = (-\beta, -\beta, c)$ is asymptotically stable. □

For system (3.2), we assume that the controlled system is given by

$$\begin{aligned} \dot{x} &= a(y-x) + u_1 \\ \dot{y} &= -xz + cy + u_2 \\ \dot{z} &= xy - bz + dsin(x) + u_3, \end{aligned}$$

where u_1, u_2 and u_3 are controllers that satisfy the following control law

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}(x-\bar{x}) \\ \dot{y} &= -xz + cy - k_{22}(y-\bar{y}) \\ \dot{z} &= xy - bz + dsin(x) - k_{33}(z-\bar{z}), \end{aligned} \tag{3.7}$$

where $E = (\bar{x}, \bar{y}, \bar{z})$ is an equilibrium point of system (3.2).

Stability of the Equilibrium Point $E_1 = (0, 0, 0)$

In this case $E = E_1$ and the controlled system (3.7) is in the form of

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}x \\ \dot{y} &= -xz + (c-k_{22})y \\ \dot{z} &= xy - bz + dsin(x) - k_{33}z. \end{aligned} \tag{3.8}$$

Theorem 3.2.4 *The equilibrium point* $E_1 = (0, 0, 0)$ *is asymptotically stable if* $k_{11} = 0$, $k_{33} > 0$ *and* $k_{22} > c$.

Proof The Jacobian matrix of the system (3.8) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ 0 & c-k_{22} & 0 \\ 0 & 0 & -b-k_{33} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c+k_{22}+k_{33} \\ a_2 &= (a+b+k_{33})(k_{22}-c)+ab+ak_{33} \\ a_3 &= (ab+ak_{33})(k_{22}-c) \\ a_1a_2-a_3 &= (b+(k_{22}-c)+k_{33})((a+b+k_{33})(k_{22}-c)+ab+ak_{33}) \\ &= +a^2(k_{22}-c)+a^2b+a^2k_{33}. \end{aligned}$$

We see that a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria when $k_{11}=0$, $k_{33}>0$ and $k_{22}>c$. Thus, if $k_{11}=0$, $k_{33}>0$ and $k_{22}>c$, then the equilibrium point $E_1=(0,0,0)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_2 = (x_1, x_1, c)$

In this case $E = E_2$ and the controlled system (3.7) is in the form of

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}(x-x_1) \\ \dot{y} &= -xz + cy - k_{22}(y-x_1) \qquad (3.9) \\ \dot{z} &= xy - bz + dsin(x) - k_{33}(z-c). \end{aligned}$$

Theorem 3.2.5 *The equilibrium point $E_2 = (x_1, x_1, c)$ is asymptotically stable if $k_{11}, k_{33} > 0$ and $k_{22} > c$.*

Proof The Jacobian matrix of the system (3.9) at the equilibrium point $E_2 = (x_1, x_1, c)$ is given by

$$J_2 = \begin{bmatrix} -a-k_{11} & a & 0 \\ -c & c-k_{22} & -x_1 \\ x_1+dcos(x_1) & x_1 & -b-k_{33} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_2 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned}
a_1 &= a+b-c+k_{11}+k_{22}+k_{33} \\
a_2 &= (b+k_{11}+k_{33})(k_{22}-c)+a(b+k_{22}+k_{33})+b(k_{11}+k_{22})+k_{11}k_{33}+x_1^2 \\
a_3 &= (k_{22}-c)(bk_{11}+k_{11}k_{33})+a(bk_{22}+k_{22}k_{33})+(2x_1^2+x_1dcos(x_1))a+k_{11}x_1^2 \\
a_1a_2-a_3 &= [(a+b-c+k_{22}+k_{33})(b+k_{11}+k_{33})+k_{11}^2+(a(b+k_{22}+k_{33}) \\
&= +b(k_{11}+k_{22})+k_{11}k_{33}+x_1^2)](k_{22}-c)+a(ab-2x_1^2)+a(bk_{22}-x_1dcos(x_1)) \\
&= +k_{11}(ab+ak_{22}+ak_{33}+bk_{11}+bk_{22}+k_{11}k_{33})+a(ak_{22}+ak_{33}+bk_{11} \\
&= +k_{11}k_{33})+b(ab+ak_{33}+bk_{11}+bk_{22}+k_{11}k_{33}+x_1^2)+k_{33}(ab+ak_{33}+bk_{11} \\
&= +bk_{22}+k_{11}k_{33}+x_1^2).
\end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k_{11}, k_{33} > 0$, and $k_{22} > c$. Thus, if $k_{11}, k_{33} > 0$, and $k_{22} > c$, then the equilibrium point $E_2 = (x_1, x_1, c)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_3 = (x_2, x_2, c)$

In this case $E = E_3$ and the controlled system (3.7) is in the form of

$$\begin{aligned} \dot{x} &= a(y-x) - k_{11}(x-x_2) \\ \dot{y} &= -xz + cy - k_{22}(y-x_2) \\ \dot{z} &= xy - bz + dsin(x) - k_{33}(z-c). \end{aligned} \tag{3.10}$$

Theorem 3.2.6 *The equilibrium point $E_3 = (x_2, x_2, c)$ is asymptotically stable if $k_{11}, k_{33} > 0$ and $k_{22} > c$.*

Proof The Jacobian matrix of the system (3.10) at the equilibrium point $E_3 = (x_2, x_2, c)$ is given by

$$J_3 = \begin{bmatrix} -a-k_{11} & a & 0 \\ -c & c-k_{22} & -x_2 \\ x_2 + dcos(x_2) & x_2 & -b-k_{33} \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c+k_{11}+k_{22}+k_{33} \\ a_2 &= (b+k_{11}+k_{33})(k_{22}-c) + a(b+k_{22}+k_{33}) + b(k_{11}+k_{22}) + k_{11}k_{33} + x_2^2 \\ a_3 &= (k_{22}-c)(bk_{11}+k_{11}k_{33}) + a(bk_{22}+k_{22}k_{33}) + (2x_2^2 + x_2dcos(x_1))a + k_{11}x_2^2 \\ a_1a_2 - a_3 &= [(a+b-c+k_{22}+k_{33})(b+k_{11}+k_{33}) + k_{11}^2 + (a(b+k_{22}+k_{33}) \\ &= +b(k_{11}+k_{22}) + k_{11}k_{33} + x_2^2)](k_{22}-c) + a(ab-2x_2^2) + a(bk_{22} - x_2dcos(x_2)) \\ &= +k_{11}(ab+ak_{22}+ak_{33}+bk_{11}+bk_{22}+k_{11}k_{33}) + a(ak_{22}+ak_{33}+bk_{11} \\ &= +k_{11}k_{33}) + b(ab+ak_{33}+bk_{11}+bk_{22}+k_{11}k_{33}+x_2^2) + k_{33}(ab+ak_{33}+bk_{11} \\ &= +bk_{22}+k_{11}k_{33}+x_2^2). \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k_{11}, k_{33} > 0$, and $k_{22} > c$. Thus, if $k_{11}, k_{33} > 0$, and $k_{22} > c$, then the equilibrium point $E_3 = (x_2, x_2, c)$ is asymptotically stable. □

Numerical Simulations

Numerical experiments are carried out to investigate controlled systems by using fourth-order Runge-Kutta method with time step 0.001. In Fig. 3.10-3.12, the parameters a, b, c and d are chosen as $a = 36$, $b = 3$, $c = 20$ and $d = 1$ to ensure the existence of chaos in the absence of control. The initial states are taken as $x = 10$, $y = 1$ and $z = 8$. The equilibrium point $E_1 = (0, 0, 0)$ of the system (3.1) is stabilized for $k_{11} = 0$, $k_{22} = 25$ and $k_{33} = 1$. Fig. 3.10 shows the behavior of the states x, y and z of the controlled system (3.4) with time. The control is active at $t = 10$. The equilibrium point $E_2 = (\sqrt{30}, \sqrt{30}, 20)$ of the system (3.1) is stabilized for $k_{11} = 1$, $k_{22} = 22$ and $k_{33} = 3$. Fig. 3.11 shows the behavior of the states x,y and z of the controlled system (3.5) with time. The control is active at $t = 10$. The equilibrium point $E_3 = (-\sqrt{30}, -\sqrt{30}, 20)$ of the system (3.1) is stabilized for $k_{11} = 1$, $k_{22} = 22$ and $k_{33} = 3$. Fig. 3.12 shows the behavior of the states x, y and z of the controlled system (3.6) with time. The control is active at $t = 10$. In Fig. 3.13, the parameters a, b, c and d are chosen as $a = 40$, $b = 0.1$, $c = 40 - 10sin(2)$ and $d = 1$ to ensure the existence of chaos in the absence of control. The initial states are taken as $x = 10$, $y = 1$ and $z = 8$. The equilibrium point $E_2 = (-2, -2, 40 - 10sin(2))$ of the system (3.2) is stabilized for $k_{11} = 1$, $k_{22} = 32$ and $k_{33} = 2$. Fig. 3.13 shows the behavior of the states x, y and z of the controlled system (3.9) with time. The control is active at $t = 10$. In Fig. 3.14, the parameters a, b, c and d are chosen as $a = 1.2$, $b = 1.5$, $c = \frac{4-sin(2)}{1.5}$ and $d = 1$ to ensure the existence of chaos in the absence of control. The initial states are taken as $x = 0.1$, $y = 0.2$ and $z = 0.3$. The equilibrium point $E_3 = (x_2, x_2, \frac{4-sin(2)}{1.5})$ of the system (3.2) is stabilized for $k_{11} = 1$, $k_{22} = 4$ and $k_{33} = 2$ where x_2 is positive real root of $g(x)$, where $g(x) = x^2 + sin(x) - 4 + sin(2)$. Fig. 3.14 shows the behavior of the states x, y and z of the controlled system (3.10) with time. The control is active at $t = 10$.

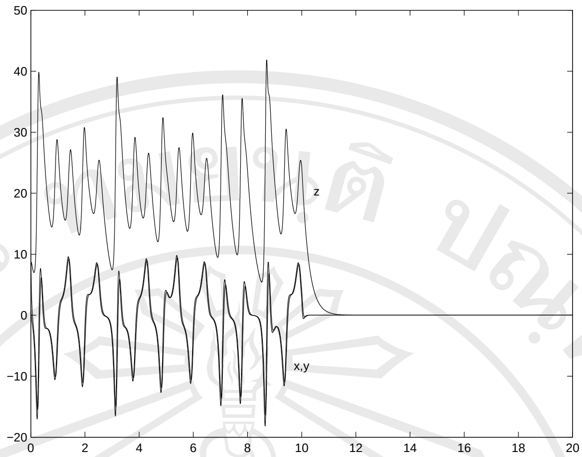

Figure 3.10: The time responses for the states x, y and z of the controlled system (3.4) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 0$, $k_{22} = 25$ and $k_{33} = 1$.

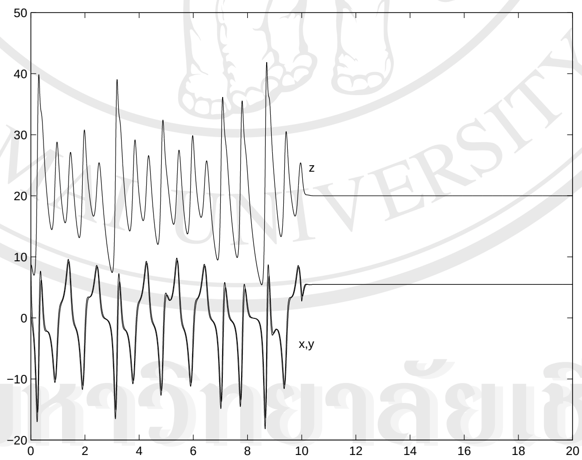

Figure 3.11: The time responses for the states x, y and z of the controlled system (3.5) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 1$, $k_{22} = 22$ and $k_{33} = 3$.

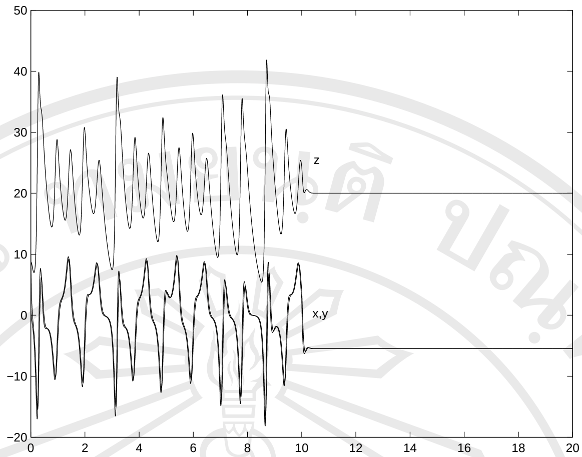

Figure 3.12: The time responses for the states x, y and z of the controlled system (3.6) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 1$, $k_{22} = 22$ and $k_{33} = 3$.

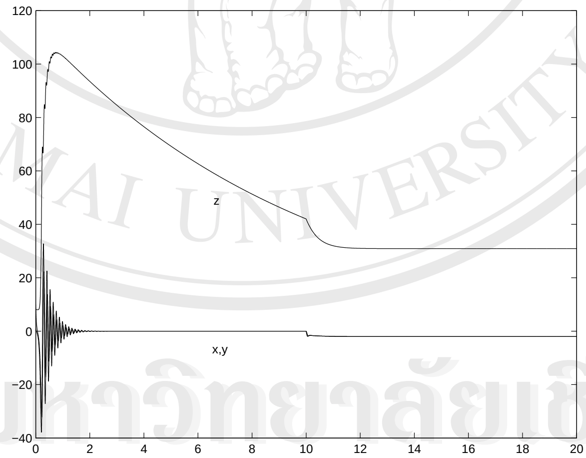

Figure 3.13: The time responses for the states x, y and z of the controlled system (3.9) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 1$, $k_{22} = 32$ and $k_{33} = 2$.

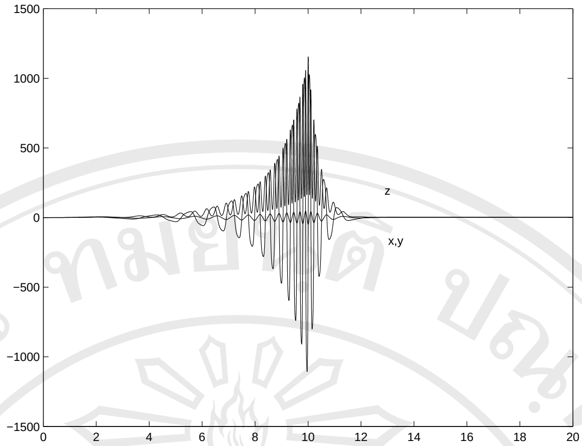

Figure 3.14: The time responses for the states x, y and z of the controlled system (3.10) before and after control activation with time. The control is activated at $t = 10$, $k_{11} = 1$, $k_{22} = 4$ and $k_{33} = 2$.

3.2.2 Bounded Feedback Control Method

In this section, we control chaos with bounded controller that vanishes after the stabilization is achieved.

Stability of the Equilibrium Point $E_1 = (0, 0, 0)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.1) as follows:

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= -xz + cy + u(t) \\ \dot{z} &= xy - bz + dx^2 \end{aligned} \tag{3.11}$$

where $u(t) = -kay, k > 0$.

Theorem 3.2.7 *The equilibrium point* $E_1 = (0, 0, 0)$ *is asymptotically stable if* $k > \dfrac{c}{a}$.

Proof The Jacobian matrix of the system (3.11) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0 \\ 0 & c-ka & 0 \\ 0 & 0 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b+(ka-c) \\ a_2 &= (ka-c)(a+b)+ab \\ a_3 &= ab(ka-c) \\ a_1a_2-a_3 &= (a+b)(ab+(a+b)(ka-c)+(ka-c)^2). \end{aligned}$$

We see that a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria when $k > \frac{c}{a}$. Thus, if $k > \frac{c}{a}$, then the equilibrium point $E_1 = (0,0,0)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_2 = (\beta, \beta, c)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.1) as follows:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+cy+u(t) \\ \dot{z} &= xy-bz+dx^2 \end{aligned} \qquad (3.12)$$

where $u(t) = -k(a(y-\beta)), k > 0$.

Theorem 3.2.8 *The equilibrium point $E_2 = (\beta, \beta, c)$ is asymptotically stable if $k > 2$.*

Proof The Jacobian matrix of the system (3.12) at the equilibrium point $E_2 = (\beta, \beta, c)$ is given by

$$J_2 = \begin{bmatrix} -a & a & 0 \\ -c & c-ka & -\beta \\ \beta + 2d\beta & \beta & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_2 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c + ka \\ a_2 &= ab - bc + \beta^2 + kab + ka^2 \\ a_3 &= 2abc + ka^2 b \\ a_1a_2 - a_3 &= bc(c-b) + (ka^2 + 2kab)(a-c) + ab(k^2a - 4c) + a^2b + (a+b-c)\beta^2 \\ &= +k^2a^3 + ab^2(1+k) + ka\beta^2. \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k > 2$. Thus, if $k > 2$, then the equilibrium point $E_2 = (\beta, \beta, c)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_3 = (-\beta, -\beta, c)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.1) as follows:

$$\begin{aligned} \dot{x} &= a(y - x) \\ \dot{y} &= -xz + cy + u(t) \\ \dot{z} &= xy - bz + dx^2 \end{aligned} \qquad (3.13)$$

where $u(t) = -k(a(y+\beta)), k > 0$.

Theorem 3.2.9 *The equilibrium point $E_3 = (-\beta, -\beta, c)$ is asymptotically stable if $k > 2$.*

Proof The Jacobian matrix of the system (3.13) at the equilibrium point $E_3 = (-\beta, -\beta, c)$ is given by

$$J_3 = \begin{bmatrix} -a & a & 0 \\ -c & c-ka & \beta \\ -\beta-2d\beta & -\beta & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b-c+ka \\ a_2 &= ab-bc+\beta^2+kab+ka^2 \\ a_3 &= 2abc+ka^2b \\ a_1a_2-a_3 &= bc(c-b)+(ka^2+2kab)(a-c)+ab(k^2a-4c)+a^2b+(a+b-c)\beta^2 \\ &= +k^2a^3+ab^2(1+k)+ka\beta^2. \end{aligned}$$

We see that a_1 and $a_1a_2-a_3$ satisfy the Routh-Hurwitz criteria when $k > 2$. Thus, if $k > 2$, then the equilibrium point $E_3 = (-\beta, -\beta, c)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_1 = (0,0,0)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.2) as follows:

$$\begin{aligned}\dot{x} &= a(y-x)\\ \dot{y} &= -xz+cy+u(t) \qquad (3.14)\\ \dot{z} &= xy-bz+dsin(x)\end{aligned}$$

where $u(t) = -kay, k > 0$.

Theorem 3.2.10 *The equilibrium point $E_1 = (0,0,0)$ is asymptotically stable if $k > \dfrac{c}{a}$.*

Proof The Jacobian matrix of the system (3.14) at the equilibrium point $E_1 = (0,0,0)$ is given by

$$J_1 = \begin{bmatrix} -a & a & 0\\ 0 & c-ka & 0\\ 0 & 0 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_1 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a+b+(ka-c)\\ a_2 &= (ka-c)(a+b)+ab\\ a_3 &= ab(ka-c)\\ a_1a_2-a_3 &= (a+b)(ab+(a+b)(ka-c)+(ka-c)^2). \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k > \frac{c}{a}$. Thus, if $k > \frac{c}{a}$, then the equilibrium point $E_1 = (0,0,0)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_2 = (x_1, x_1, c)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.2) as follows:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= -xz + cy + u(t) \\ \dot{z} &= xy - bz + dsin(x) \end{aligned} \tag{3.15}$$

where $u(t) = -k(a(y - x_1)), k > 0$.

Theorem 3.2.11 *The equilibrium point* $E_2 = (x_1, x_1, c)$ *is asymptotically stable if* $k > \frac{2x_1 + dcos(x_1)}{x_1}$.

Proof The Jacobian matrix of the system (3.15) at the equilibrium point $E_2 = (x_1, x_1, c)$ is given by

$$J_2 = \begin{bmatrix} -a & a & 0 \\ -c & c - ka & -x_1 \\ x_1 + dcos(x_1) & x_1 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_2 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c + ka \\ a_2 &= ab - bc + x_1^2 + kab + ka^2 \\ a_3 &= 2x_1^2 + x_1 dcos(x_1) + ka^2 b \\ a_1a_2 - a_3 &= bc(c-b) + (ka^2 + 2kab)(a-c) + ab(k^2a - 2c) + a(kx_1^2 - 2x_1^2 - x_1 dcos(x_1)) \\ &= +k^2a^3 + ab^2(1+k) + x_1^2(a+b-c) + a^2b. \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k > \frac{2x_1 + dcos(x_1)}{x_1}$. Thus, if $k > \frac{2x_1 + dcos(x_1)}{x_1}$, then the equilibrium point $E_2 = (x_1, x_1, c)$ is asymptotically stable. □

Stability of the Equilibrium Point $E_3 = (x_2, x_2, c)$

In order to stabilize this equilibrium point by bounded feedback control, the control is chosen for system (3.2) as follows:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= -xz + cy + u(t) \\ \dot{z} &= xy - bz + dsin(x) \end{aligned} \tag{3.16}$$

where $u(t) = -k(a(y - x_2)), k > 0$.

Theorem 3.2.12 *The equilibrium point* $E_3 = (x_2, x_2, c)$ *is asymptotically stable if* $k > \frac{2x_2 + dcos(x_2)}{x_2}$.

Proof The Jacobian matrix of the system (3.16) at the equilibrium point $E_3 = (x_2, x_2, c)$ is given by

$$J_3 = \begin{bmatrix} -a & a & 0 \\ -c & c - ka & -x_2 \\ x_2 + dcos(x_2) & x_2 & -b \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix J_3 has the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where

$$\begin{aligned} a_1 &= a + b - c + ka \\ a_2 &= ab - bc + x_2^2 + kab + ka^2 \\ a_3 &= 2x_2^2 + x_2 dcos(x_2) + ka^2 b \\ a_1a_2 - a_3 &= bc(c-b) + (ka^2 + 2kab)(a-c) + ab(k^2a - 2c) + a(kx_2^2 - 2x_2^2 - x_2dcos(x_2)) \\ &= +k^2a^3 + ab^2(1+k) + x_2^2(a+b-c) + a^2b. \end{aligned}$$

We see that a_1 and $a_1a_2 - a_3$ satisfy the Routh-Hurwitz criteria when $k > \frac{2x_2 + dcos(x_2)}{x_2}$. Thus, if $k > \frac{2x_2 + dcos(x_2)}{x_2}$, then the equilibrium point $E_3 = (x_2, x_2, c)$ is asymptotically stable. □

Numerical Simulations

We will show a series of numerical experiments by using the fourth-order Runge-Kutta method with step size 0.001. In Fig. 3.15-3.23, the parameters a, b, c and d are chosen as $a = 36$, $b = 3$, $c = 20$ and $d = 1$. The control is active at $t = 10$ for all simulations. In the first numerical experiment, we intend to control the chaos to equilibrium point $E_1 = (0, 0, 0)$ of system (3.1). Fig. 3.15-3.17 shows the time response of the states x, y and z of system (3.11) time for $k = 1$. The initial condition are $x = 10$, $y = 1$ and $z = 8$. In the second numerical experiment, we intend to control the chaos to equilibrium point $E_2 = (\sqrt{30}, \sqrt{30}, 20)$ of system (3.1). Fig. 3.18-3.20 shows the time response of the states x, y and z of system (3.12) with time for $k = 3$. The initial condition are $x = 10$, $y = 1$ and $z = 8$. In the third numerical experiment, we intend to control the chaos to equilibrium point $E_3 = (-\sqrt{30}, -\sqrt{30}, 20)$ of system (3.1). Fig. 3.21-3.23 shows the time response of the states x, y and z of system (3.13) with time for $k = 3$. The initial condition are $x = 10$, $y = 1$ and $z = 8$. In the fourth numerical experiment, we intend to control the chaos to equilibrium point $E_2 = (-2, -2, 40 - 10sin(2))$ of system (3.2). Fig. 3.24-3.26 shows the time response of the states x, y and z of system (3.15) with time for $k = 3$. In Fig. 3.24-3.26, the parameters a, b, c and d are chosen as $a = 40$, $b = 0.1$, $c = 40 - 10sin(2)$ and $d = 1$. The control is active at $t = 10$ for all simulations. The initial condition are $x = 10$, $y = 1$ and $z = 8$. In the fifth numerical experiment, we intend to control the chaos to equilibrium point $E_3 = (x_2, x_2, \frac{4-sin(2)}{1.5})$ of system (3.2) where x_2 is positive real root of $g(x)$, where $g(x) = x^2 + sin(x) - 4 + sin(2)$. Fig. 3.27-3.29 shows the time response of the states x, y and z of system (3.16) with time for $k = 3$. In Fig. 3.27-3.29, the parameters a, b, c and d are chosen as $a = 1.2$, $b = 1.5$, $c = \frac{4-sin(2)}{1.5}$ and $d = 1$. The control is active at $t = 10$ for all simulations. The initial condition are $x = 0.5$, $y = 1$ and $z = 1.5$.

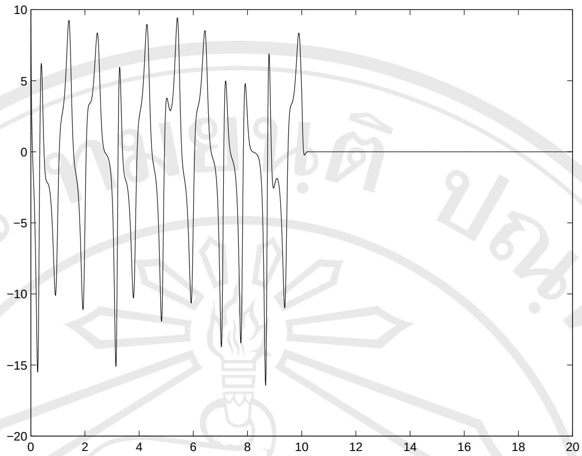

Figure 3.15: The state x of the controlled system (3.11) responses with time before and after control activation. The control is activated at $t = 10$, $k = 1$.

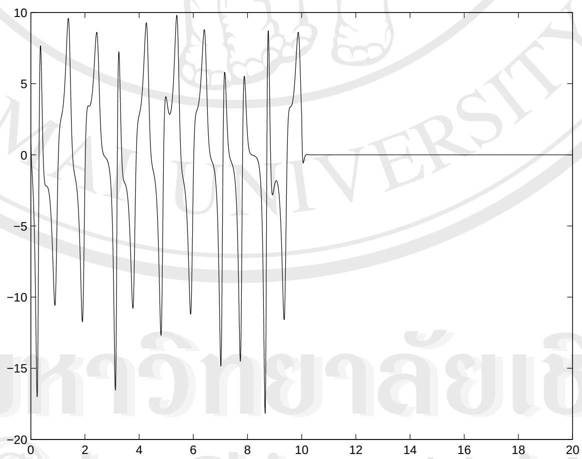

Figure 3.16: The state y of the controlled system (3.11) responses with time before and after control activation. The control is activated at $t = 10$, $k = 1$.

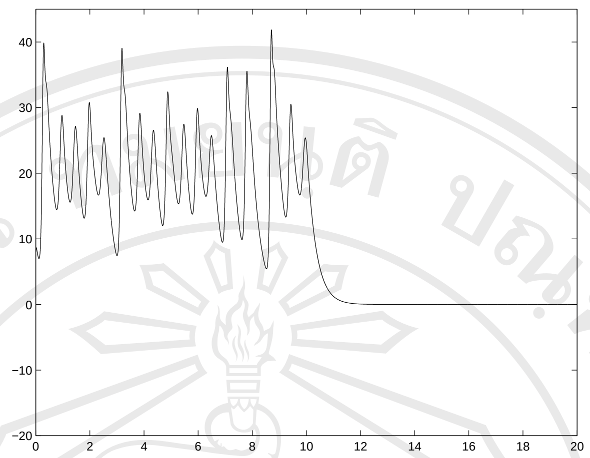

Figure 3.17: The state z of the controlled system (3.11) responses with time before and after control activation. The control is activated at $t = 10$, $k = 1$.

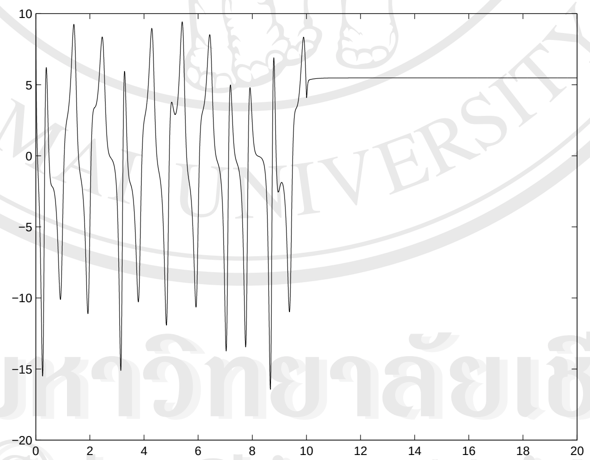

Figure 3.18: The state x of the controlled system (3.12) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

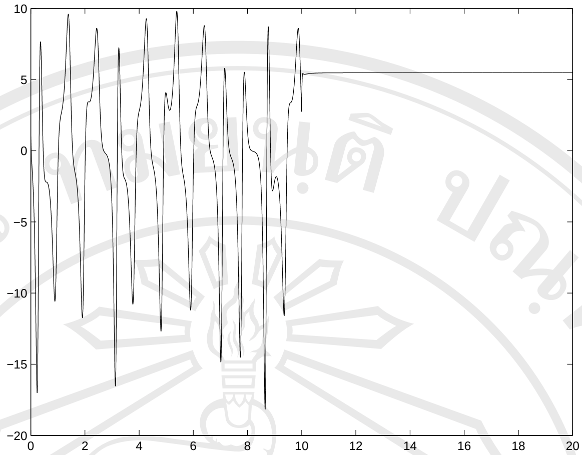

Figure 3.19: The state y of the controlled system (3.12) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

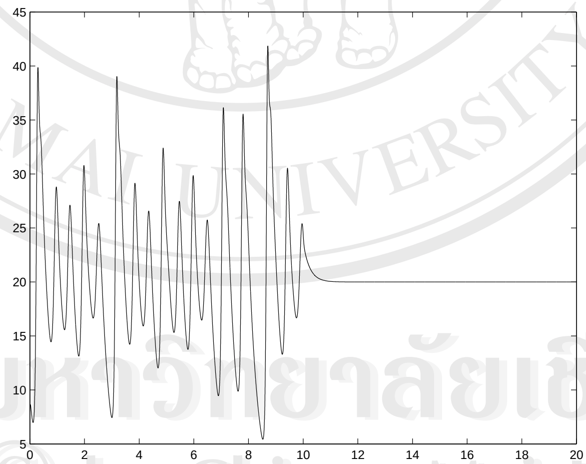

Figure 3.20: The state z of the controlled system (3.12) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

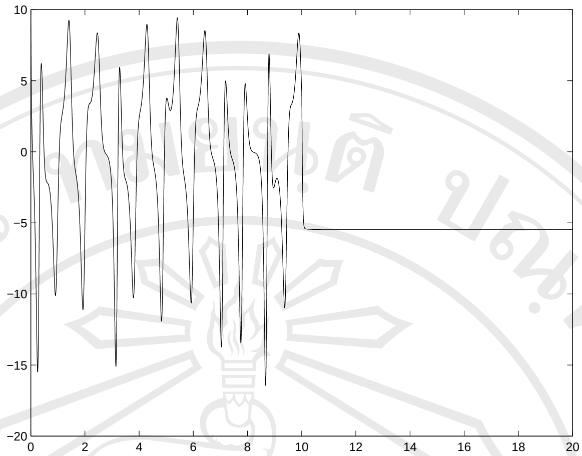

Figure 3.21: The state x of the controlled system (3.13) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

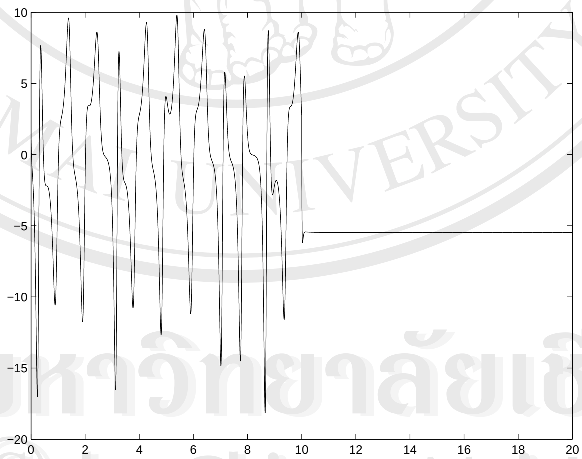

Figure 3.22: The state y of the controlled system (3.13) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

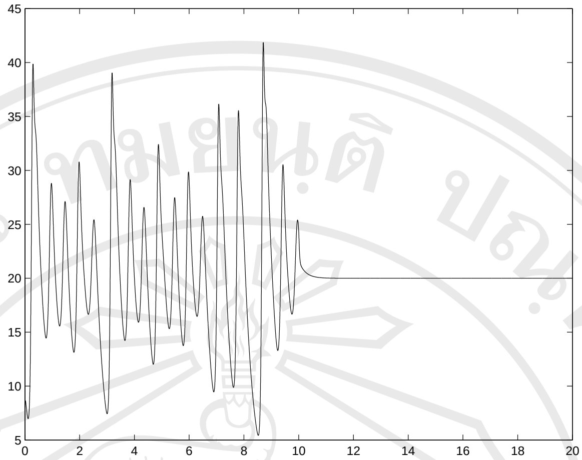

Figure 3.23: The state z of the controlled system (3.13) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

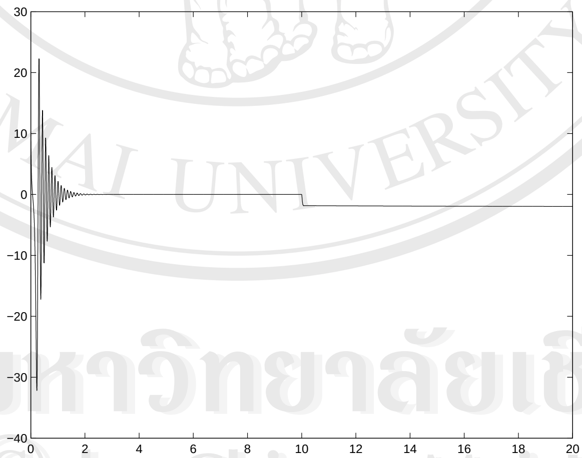

Figure 3.24: The state x of the controlled system (3.15) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

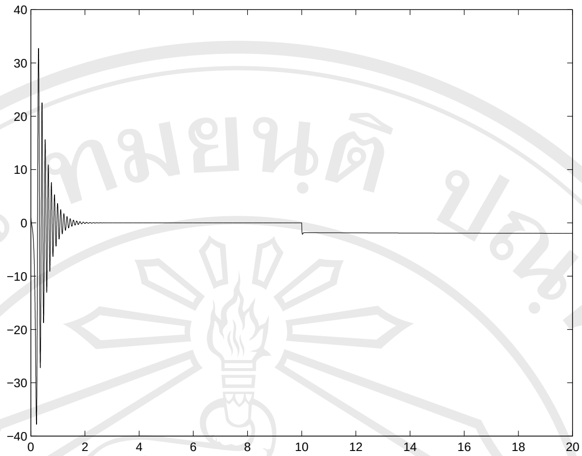

Figure 3.25: The state y of the controlled system (3.15) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

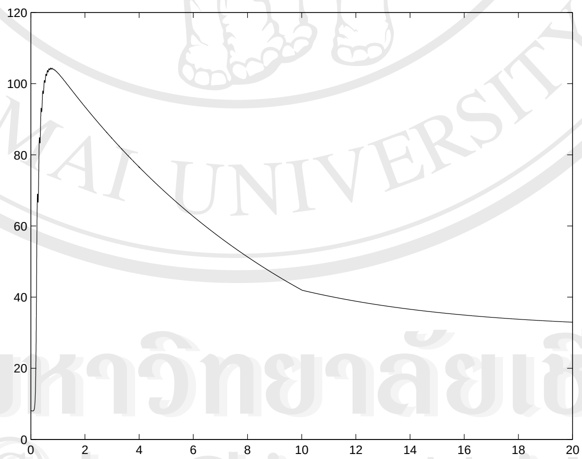

Figure 3.26: The state z of the controlled system (3.15) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

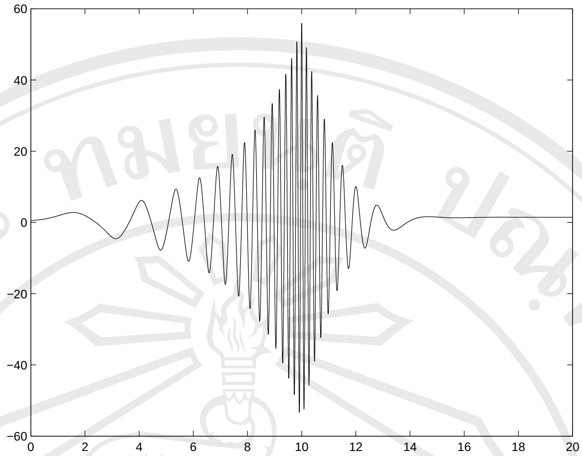

Figure 3.27: The state x of the controlled system (3.16) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

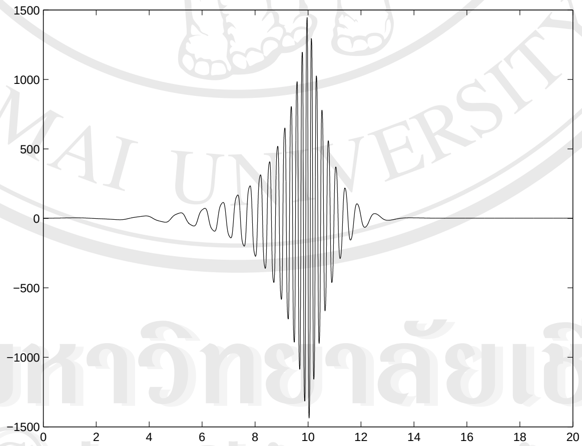

Figure 3.28: The state y of the controlled system (3.16) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.

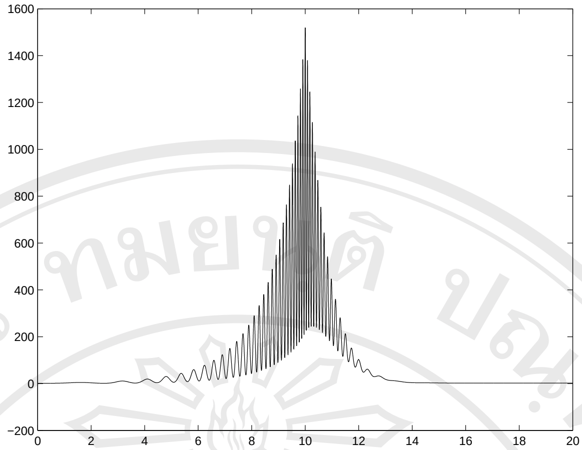

Figure 3.29: The state z of the controlled system (3.16) responses with time before and after control activation. The control is activated at $t = 10$, $k = 3$.