CHAPTER 4

Synchronization of Perturbed Lü Chaotic Dynamical System

To begin with, the definition of chaos synchronization used in this thesis is given below.

For two nonlinear chaotic systems:

$$\dot{x} = f(t, x) \tag{4.1}$$

$$\dot{y} = g(t, y) + u(t, x, y) \tag{4.2}$$

where $x, y \in \mathbb{R}^n$, $f, g \in C^r[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $u \in C^r[\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $r \geq 1$, $\mathbb{R}^+$ is the set of non-negative real numbers.

Assume that (4.1) is the drive system, and (4.2) is the response system, $u(t, x, y)$ is the control vector. Response system and drive system are said to be *synchronic* if for

$\forall x(t_0), y(t_0) \in \mathbb{R}^n$,

$$\lim_{t \to \infty} \| x(t) - y(t) \| = 0.$$

4.1 Adaptive Synchronization of Perturbed Lü Chaotic Dynamical System

In this section we consider adaptive synchronization of perturbed Lü system. This approach can synchronize the chaotic systems with fully unmatched parameters. The synchronization problem of perturbed Lü systems with fully unknown parameters will be studied in which the adaptive controller will be introduced.

The drive system is described by

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= -xz+cy \\ \dot{z} &= xy-bz+f(x) \end{aligned} \tag{4.3}$$

Suppose that the parameters of the system (4.3) are unknown or uncertain, then the response system is given by

$$\begin{aligned} \dot{\tilde{x}} &= \hat{a}(\tilde{y}-\tilde{x})-u_1 \\ \dot{\tilde{y}} &= -\tilde{x}\tilde{z}+\hat{c}\tilde{y}-u_2 \\ \dot{\tilde{z}} &= \tilde{x}\tilde{y}-\hat{b}\tilde{z}+f(\tilde{x})-u_3 \end{aligned} \tag{4.4}$$

where $\hat{a}, \hat{b}$ and $\hat{c}$ are parameters of the response system which need to be estimated. Suppose that

$$\begin{aligned} u_1 &= k_1 e_x \\ u_2 &= (k_2+c)e_y \\ u_3 &= k_3 e_z + f(\tilde{x}) - f(x) \end{aligned} \tag{4.5}$$

where $e_x = \tilde{x}-x, e_y = \tilde{y}-y$ and $e_z = \tilde{z}-z$ and

$$\begin{aligned} \dot{\hat{a}} &= f_a = -\gamma\beta(\tilde{y}-\tilde{x})e_x \\ \dot{\hat{b}} &= f_b = \theta\tilde{z}e_z \\ \dot{\hat{c}} &= f_c = -\alpha\tilde{y}e_y \end{aligned} \tag{4.6}$$

where k_1, k_2, $k_3 \geq 0$ and γ, θ, β, $\alpha > 0$ are constants.

Theorem 4.1.1 *Suppose that $|x| \leq S < +\infty$, $|y| \leq S < +\infty$, $|z| \leq S < +\infty$, γ, θ, β, α are positive constants. When k_1, k_2 and $k_3 \geq 0$ are properly chosen so that the following matrix inequality holds,*

$$P = \begin{bmatrix} \beta(k_1+a) & -\frac{1}{2}(\beta a - S) & -S \\ -\frac{1}{2}(\beta a - S) & k_2 & 0 \\ -S & 0 & k_3+b \end{bmatrix} > 0 \tag{4.7}$$

or k_1, k_2 and k_3 can be chosen so that the following inequalities hold:

$$\begin{aligned} (i)\quad A &= \beta(k_1+a)k_2 - \frac{1}{4}(\beta a - S)^2 > 0 \\ (ii)\quad B &= A(k_3+b) - k_2S^2 > 0, \end{aligned} \tag{4.8}$$

then the two perturbed Lü systems (4.3) and (4.4) can be synchronized under the adaptive control of (4.5) and (4.6).

Proof It is easy to see from (4.3) and (4.4) that the error dynamics can be obtained as follows

$$\begin{aligned} \dot{e_x} &= \hat{a}(\tilde{y}-\tilde{x}) - a(y-x) - u_1 \\ \dot{e_y} &= \hat{c}\tilde{y} - cy - \tilde{x}\tilde{z} + xz - u_2 \\ \dot{e_z} &= -\hat{b}\tilde{z} + bz + \tilde{x}\tilde{y} - xy + f(\tilde{x}) - f(x) - u_3 \end{aligned} \tag{4.9}$$

Let $e_a = \hat{a} - a, e_b = \hat{b} - b, e_c = \hat{c} - c$. Choose the following Lyapunov function:

$$V(e_x, e_y, e_z) = \frac{1}{2}(\beta e_x^2 + e_y^2 + e_z^2 + \frac{1}{\gamma}e_a^2 + \frac{1}{\theta}e_b^2 + \frac{1}{\alpha}e_c^2) \tag{4.10}$$

in which the differentiation of V along trajectories of (4.9) gives

$$
\begin{aligned}
\dot{V} &= \beta e_x\dot{e}_x + e_y\dot{e}_y + e_z\dot{e}_z + \frac{1}{\gamma}e_a\dot{e}_a + \frac{1}{\theta}e_b\dot{e}_b + \frac{1}{\alpha}e_c\dot{e}_c \\
&= \beta e_x[e_a(\tilde{y}-\tilde{x}) + a(e_y - e_x) - u_1] + e_y[e_c\tilde{y} + ce_y - \tilde{x}e_z - ze_x - u_2] \\
&\quad + e_z[\tilde{x}e_y + ye_x - \tilde{z}e_b - be_z + f(\tilde{x}) - f(x) - u_3] + \frac{1}{\gamma}e_af_a + \frac{1}{\theta}e_bf_b + \frac{1}{\alpha}e_cf_c \\
&= \beta e_xe_a(\tilde{y}-\tilde{x}) + \beta ae_x(e_y - e_x) - \beta e_xu_1 + e_ye_c\tilde{y} + ce_y^2 - ze_xe_y - e_yu_2 + ye_xe_z \\
&\quad -\tilde{z}e_be_z - be_z^2 + (f(\tilde{x}) - f(x))e_z - ezu_3 + \frac{1}{\gamma}e_af_a + \frac{1}{\theta}e_bf_b + \frac{1}{\alpha}e_cf_c \\
&= \beta e_xe_a(\tilde{y}-\tilde{x}) + \beta ae_xe_y - \beta ae_x^2 - \beta k_1e_x^2 + e_ye_c\tilde{y} + ce_y^2 - ze_xe_y - (k_2+c)e_y^2 + ye_xe_z \\
&\quad -\tilde{z}e_be_z - be_z^2 + (f(\tilde{x}) - f(x))e_z - (k_3ez + f(\tilde{x}) - f(x))e_z + \frac{1}{\gamma}e_af_a + \frac{1}{\theta}e_bf_b + \frac{1}{\alpha}e_cf_c \\
&= -\beta(k_1+a)e_x^2 - k_2e_y^2 - (k_3+b)e_z^2 + (\beta a - z)e_xe_y + ye_xe_z \\
&\quad + e_a(\frac{1}{\gamma}f_a + (\tilde{y}-\tilde{x})\beta e_x) + e_b[\frac{1}{\theta}f_b - \tilde{z}e_z] + e_c[\frac{1}{\alpha}f_c + (\tilde{y})e_y] \\
&\leq -\beta(k_1+a)e_x^2 - k_2e_y^2 - (k_3+b)e_z^2 + (\beta a - S)|e_xe_y| + S|e_xe_z| = -e^TPe
\end{aligned}
$$

where $e = \begin{bmatrix} |e_x| & |e_y| & |e_z| \end{bmatrix}^T$, P is as in (4.7). Thus the differentiation of $V(e_x, e_y, e_z)$ is negative semi definite, which implies that the origin of error system (4.9) is stable. We integrate both sides of $\dot{V}$ with respect to time which yields

$$
\begin{aligned}
\int_0^\infty \frac{dV(\tau)}{d\tau}d\tau &\leq -\int_0^\infty \beta(k_1+a)e_x^2(\tau)d\tau - \int_0^\infty k_2e_y^2(\tau)d\tau - \int_0^\infty (k_3+b)e_z^2(\tau)d\tau \\
&\quad + \int_0^\infty (\beta a - S)|e_x(\tau)e_y(\tau)|d\tau + \int_0^\infty S|e_x(\tau)e_z(\tau)|d\tau.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V(\infty) - V(0) &\leq -\int_0^\infty \beta(k_1+a)e_x^2(\tau)d\tau - \int_0^\infty k_2e_y^2(\tau)d\tau - \int_0^\infty (k_3+b)e_z^2(\tau)d\tau \\
&\quad + \int_0^\infty (\beta a - S)|e_x(\tau)e_y(\tau)|d\tau + \int_0^\infty S|e_x(\tau)e_z(\tau)|d\tau. \\
V(0) - V(\infty) &\geq +\int_0^\infty \beta(k_1+a)e_x^2(\tau)d\tau + \int_0^\infty k_2e_y^2(\tau)d\tau + \int_0^\infty (k_3+b)e_z^2(\tau)d\tau \\
&\quad - \int_0^\infty (\beta a - S)|e_x(\tau)e_y(\tau)|d\tau - \int_0^\infty S|e_x(\tau)e_z(\tau)|d\tau.
\end{aligned}
$$

Since $\dot{V}$ is negative or zero, V is either decreasing or constant which gives $V(0) \geq V(+\infty) \geq 0$. Then we obtain

$$\int_0^\infty \beta(k_1 + a)e_x^2(\tau)d\tau + \int_0^\infty k_2 e_y^2(\tau)d\tau + \int_0^\infty (k_3 + b)e_z^2(\tau)d\tau$$
$$- \int_0^\infty (\beta a - S)|e_x(\tau)e_y(\tau)|d\tau - \int_0^\infty S|e_x(\tau)e_z(\tau)|d\tau \leq V(0) < +\infty.$$

It follows that

$$\sqrt{\int_0^\infty \beta(k_1 + a)e_x^2(\tau)d\tau} < +\infty,$$
$$\sqrt{\int_0^\infty k_2 e_y^2(\tau)d\tau} < +\infty,$$
$$\sqrt{\int_0^\infty (k_3 + b)e_z^2(\tau)d\tau} < +\infty$$

which indicates, according to Definition 2.1.1, that $e_x, e_y, e_z \in L_2$. We can use (4.9) to show that $\dot{e_x}, \dot{e_y}, \dot{e_z} \in L_\infty$. By Proposition 2.1.2 we obtain the errors system (4.9) tend to zero as $t \to +\infty$. Therefore, the response system (4.4) is synchronizing with the drive system (4.3) under the controller (4.5) and a parameter estimation update law (4.6), provided that the condition (4.8) are satisfied. $\square$

Numerical Simulations

The numerical simulations are carried out using the fourth-order Runge-Kutta method. The initial conditions of the drive and response systems are $(6, 4, -8)$ and $(-5, 4, 5)$. The parameters of the drive system are $a = 36$, $b = 3$ and $c = 20$. $f(x) = x^2$.

In order to choose the control parameters, $S > |x|$, $S > |y|$ and $S > |z|$ must be estimated. Through simulations, we obtain $S \approx 60$. Then choose $\gamma = \alpha = \beta = \theta = 1$, and then choose $k_1 = 30$, $k_2 = 90$, $k_3 = 60$ which satisfy (4.8) and the initial values of the parameters $\hat{a}$, $\hat{b}$ and $\hat{c}$ are all chosen to be 0, the response system synchronizes with the drive system as shown in Fig. 4.1-4.3.

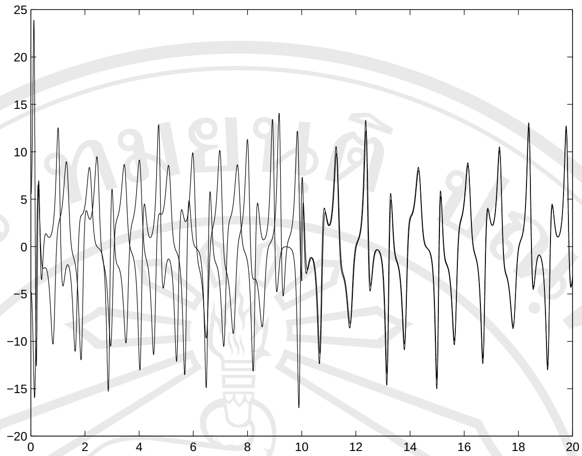

Figure 4.1: The states x, $\tilde{x}$ of the coupled perturbed Lü system of equations with the adaptive control is activated at the time $t = 10$.

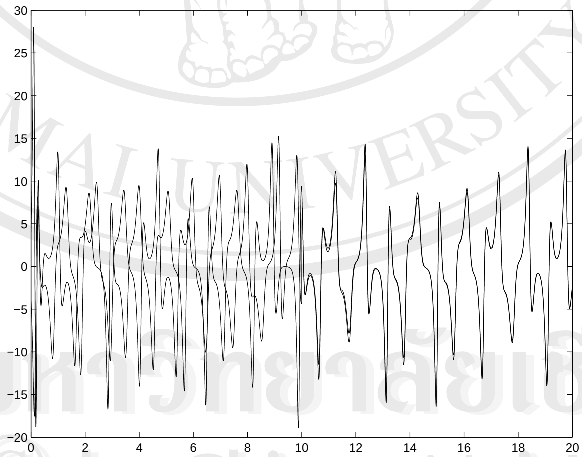

Figure 4.2: The states y, $\tilde{y}$ of the coupled perturbed Lü system of equations with the adaptive control is activated at the time $t = 10$.

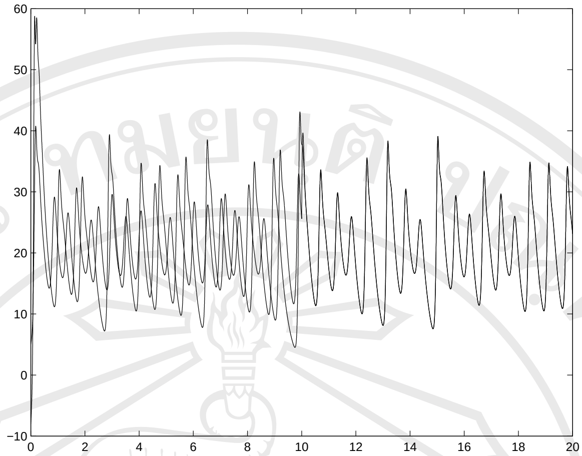

Figure 4.3: The states z, $\tilde{z}$ of the coupled perturbed Lü system of equations with the adaptive control is activated at the time $t = 10$.

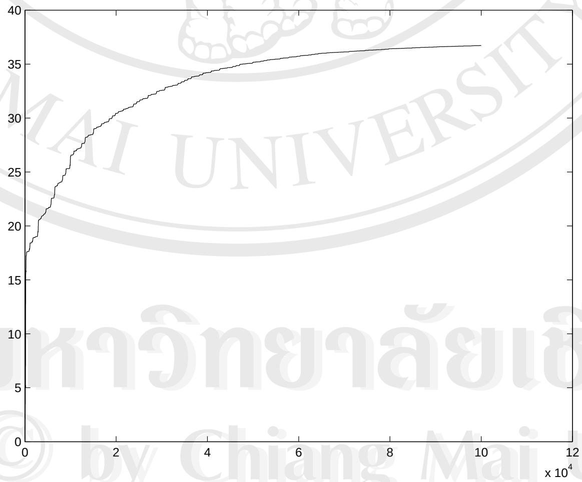

Figure 4.4: Changing parameters: $\hat{a}$.

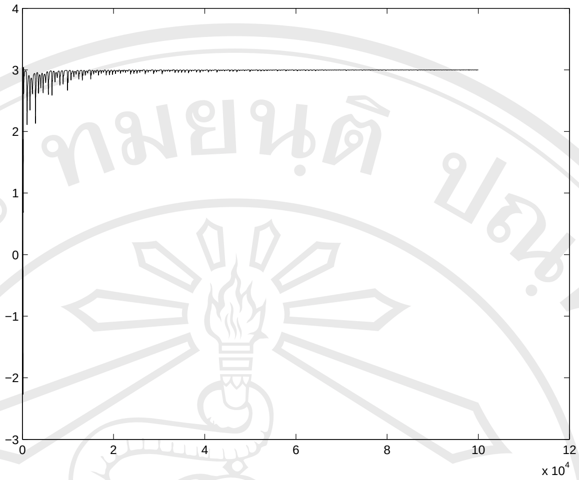

Figure 4.5: Changing parameters: $\hat{b}$.

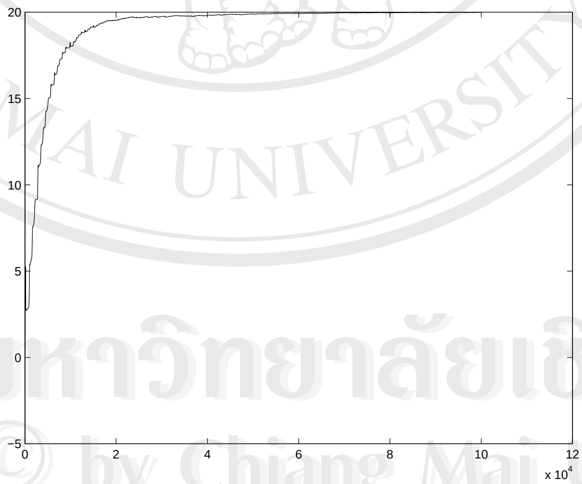

Figure 4.6: Changing parameters: $\hat{c}$.

4.2 Synchronization via Pecora and Carroll method

In this section we will apply one of the most popular method for synchronizing systems which was introduced by Pecora and Carroll (PC)[**2**]. Let us consider an autonomous chaotic dynamical system

$$\dot{u} \;=\; f(u),$$

where $u = (u_1, u_2, ..., u_n)^T$ is an n-dimensional state vector, with f defining a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. The method of Pecora and Carroll decomposes the dynamical system $\dot{u} = f(u)$ into two subsystems

$$\begin{aligned} \dot{v} &= g(v, w), \\ \dot{w} &= h(v, w), \end{aligned}$$

where $v = (v_1, v_2, ..., v_m)^T$ and $w = (w_{m+1}, w_{m+2}, ..., w_n)^T$ such that any second subsystem (response)

$$\dot{w'} \;=\; h(v, w'),$$

with the same driving v but with different variable w' synchronizes ($w \to w'$ as $t \to +\infty$) with the original w-subsystem. The $v - system$ and the $w - system$ are called the *drive* and the *response* systems, respectively. Let us build a PC drive-response configuration with a drive system given by the Perturbed Lü system (with three state variables denoted by subscript 1) and with a response system given by the subspace containing the (x, z) variables. We will use the chaotic signal y_1 to drive the response subsystem whose variables are denoted by subscript 2. The drive system is described by system

$$\begin{aligned} \dot{x}_1 &= a(y_1 - x_1) \\ \dot{y}_1 &= -x_1 z_1 + c y_1 \\ \dot{z}_1 &= x_1 y_1 - b z_1 + f(x_1) \end{aligned} \tag{4.11}$$

The response system is given by

$$\begin{aligned} \dot{x}_2 &= a(y_1 - x_2) \\ \dot{z}_2 &= x_2 y_1 - b z_2 + f(x_2) \end{aligned} \tag{4.12}$$

Define the synchronization errors by

$e_x = x_2 - x_1$ and $e_z = z_2 - z_1$.

Using this notation and subtracting system (4.11) from system (4.12), we obtain the error system

$$\begin{aligned} \dot{e}_x &= -a e_x, \\ \dot{e}_z &= (y_1 + \frac{f(x_2) - f(x_1)}{x_2 - x_1}) e_x - b e_z. \end{aligned} \tag{4.13}$$

The linear system of synchronization error (4.13) has two negative eigenvalues $\lambda_1 = -a$ and $\lambda_2 = -b$ and this implies that the zero solution of system (4.13) satisfies

$\|e_x\| \to 0$ as $t \to +\infty$ and $\|e_z\| \to 0$ as $t \to +\infty$. Hence, the zero solution of the error system (4.13) is asymptotically stable and then the response system (4.12) with $y-$drive configuration achieves synchonization.

Remark 1. Only one choice induces the appearance of chaos synchronization, namely (x, z) driven by y. For the other possible choices (x, y) driven by z or (y, z) driven by x, the Pecora and Carroll scheme did not succeed in achieving synchronization.

Numerical Simulations

The numerical simulations are carried out using the fourth-order Runge-Kutta method. The initial conditions of the drive and response systems are $(10, 1, 8)$ and $(-5, -10)$. The parameters of the drive system are $a = 36$, $b = 3$ and $c = 20$. $f(x) = x^2$. The response system synchronizes with the drive system as shown in Fig. 4.7 and 4.8.

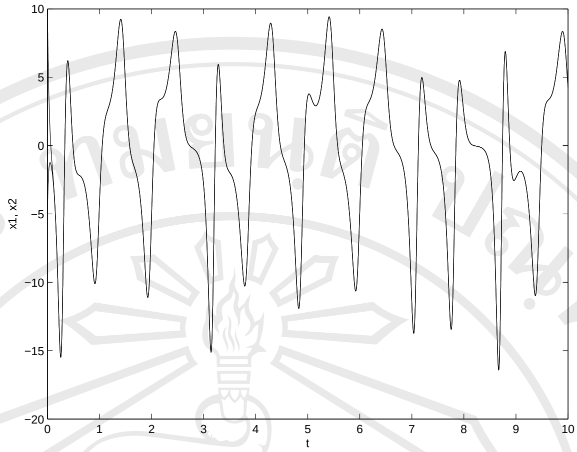

Figure 4.7: The states x_1, x_2 of the coupled perturbed Lü system of equations under Pecora and Carroll method.

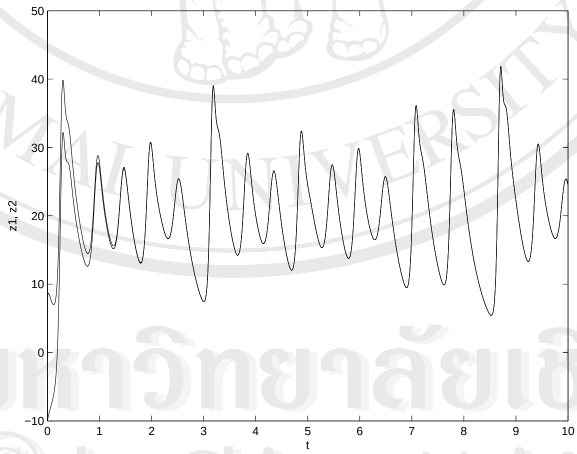

Figure 4.8: The states z_1, z_2 of the coupled perturbed Lü system of equations under Pecora and Carroll method.

4.3 Synchronization via one-way coupling

We take two identical chaotic systems $\dot{u} = f(u)$ and $\dot{u}' = f(u')$ and introduce a coupling term $\delta u = (u - u')$ into the second equation, which leads to the coupled system

$$\begin{aligned}\dot{u} &= f(u),\\ \dot{u}' &= f(u') + k(u - u'),\end{aligned}$$

where k is a tuning parameter that controls the strength of the feedback into the coupled system. We note that this kind of coupling does not change the solution to the autonomous uncoupled system $\dot{u} = f(u)$ and further, as $\delta u \to 0$, the driving system and the responding system essentially become uncoupled. Our objective in this section is to study the chaos synchronization of the Perturbed Lü system by applying one-way coupling technique. In this method, the behavior of the response system is dependent on the behavior of another identical drive system where the latter is not influenced by the behavior of the response system. We have two Perturbed Lü systems where the drive system (4.11) with three state variables denoted by subscript 1 drives the response system which has identical equations denoted by subscript 2. However, the initial conditions on the drive system is different from that of the response system. The Perturbed Lü response systems are described by

$$\begin{aligned}\dot{x}_2 &= a(y_2 - x_2)\\ \dot{y}_2 &= -x_2 z_2 + c y_2 - k(y_2 - y_1) \qquad (4.14)\\ \dot{z}_2 &= x_2 y_2 - b z_2 + f(x_2)\end{aligned}$$

where k is the coupling strength. Introducing the error variables

$e_x = x_2 - x_1$, $e_y = y_2 - y_1$ and $e_z = z_2 - z_1$.

Using the previous notations, the error dynamical system is obtained by subtracting system (4.11) from system (4.14) and has the form

$$\begin{aligned}
\dot{e}_x &= -a(e_y - e_x), \\
\dot{e}_y &= (c-k)e_y - x_2 e_z - z_1 e_x, \qquad (4.15)\\
\dot{e}_z &= x_2 e_y + y_1 e_x - b e_z + f(x_2) - f(x_1).
\end{aligned}$$

Theorem 4.3.1 *The two Perturbed Lü system (4.11)and (4.14) are synchronized if the feedback control parameter k is chosen so that $k \geq max(k_1, k_2)$ where,*

$$k_1 = min_\mu \left(c + \frac{(\mu a + S)^2}{4a\mu} \right), \quad k_2 = min_\mu \left(c + \frac{b(\mu a + S)^2}{4ab\mu - (S + \left| \frac{f(x_2)-f(x_1)}{x_2-x_1} \right| \Big|_S)^2} \right)$$

and

$$\mu > \frac{(S + \left| \frac{f(x_2)-f(x_1)}{x_2-x_1} \right| \Big|_S)^2}{4ab}, \qquad (4.16)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is satisfied Lipschitz condition, $|f(x_2) - f(x_1)| \leq c|x_2 - x_1|$, $\forall x_1, x_2 \in \mathbb{R}$, c is positive constant, or differentiable.

Proof. In order to achieve the complete synchronization of the drive and response systems (4.11)and (4.14) we will prove that the zero solution of the error dynamical system (4.15) is asymptotically stable. Let us define a Lyapunov function for system (4.15) in the form

$$V(e_x, e_y, e_z) = \frac{1}{2}(\mu e_x^2 + e_y^2 + e_z^2) \qquad (4.17)$$

Then V is positive definite and its derivative with respect to (4.15) is given by

$$\begin{aligned}
\dot{V} &= e_x \dot{e}_x + e_y \dot{e}_y + e_z \dot{e}_z \\
&= \mu e_x (a(e_x - e_y)) + e_y((c-k)e_y - x_2 e_z - z_1 e_x) \\
&\quad + e_z(x_2 e_y + y_1 e_x - b e_z + f(x_2) - f(x_1)) \\
&= \mu a e_x e_y - \mu a e_x^2 + (c-k)e_y^2 - z_1 e_x e_y + y_1 e_x e_z - b e_z^2 + (f(x_2) - f(x_1))e_z
\end{aligned}$$

$$\begin{aligned}
&= -\mu a e_x^2 - (k-c)e_y^2 - b e_z^2 + (\mu a - z_1)e_x e_y + (y_1 + \frac{f(x_2)-f(x_1)}{x_2-x_1})e_x e_z \\
&\leq -\mu a e_x^2 - (k-c)e_y^2 - b e_z^2 + (\mu a + S)e_x e_y + (S + |\frac{f(x_2)-f(x_1)}{x_2-x_1}|\Big|_S)e_x e_z \\
&= -e^T P e,
\end{aligned}$$

where $e = [|e_x| \ |e_y| \ |e_z| \ ]^T$ and

$$P = \begin{bmatrix} \mu a & -\frac{(\mu a+S)}{2} & -\frac{1}{2}(S + \left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right|\Big|_S) \\ -\frac{(\mu a+S)}{2} & k-c & 0 \\ -\frac{1}{2}(S + \left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right|\Big|_S) & 0 & b \end{bmatrix}. \quad (4.18)$$

The matrix P is positive definite if $k \geq \hat{k}$ where $\hat{k} = max(k_1, k_2)$,

$$k_1 = min_\mu \left(c + \frac{(\mu a + S)^2}{4a\mu}\right), \quad k_2 = min_\mu \left(c + \frac{b(\mu a + S)^2}{4ab\mu - (S + \left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right|\Big|_S)^2}\right).$$

The parameter μ is chosen such that $4\mu ab - (S + \left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right|\Big|_S)^2 > 0$. Therefore, $\dot{V}$ is negative semi definite which implies that the origin of the error system (4.15) is stable, By the same argument in the proof of Theorem (4.1.1), we obtain the errors system (4.15) tend to zero as t tends to $+\infty$. Therefore, the response system (4.14) is synchronizing with the drive system (4.11) provided that the condition (4.16) are satisfied. $\square$

Numerical Simulations

The numerical simulations are carried out using the fourth-order Runge-Kutta method. The initial conditions of the drive and response systems are $(16, -4, 10)$ and $(-20, 8, 15)$. The parameters of the drive system are $a = 36$, $b = 3$ and $c = 20$. $f(x) = x^2$. Fig. 4.9-4.11 shows the time response of the states (x_1, y_1, z_1) for the drive system (4.11) and the time response of the states (x_2, y_2, z_2) of the response system (4.14) where the feedback control $k = 35$ is activated at $t = 10$.

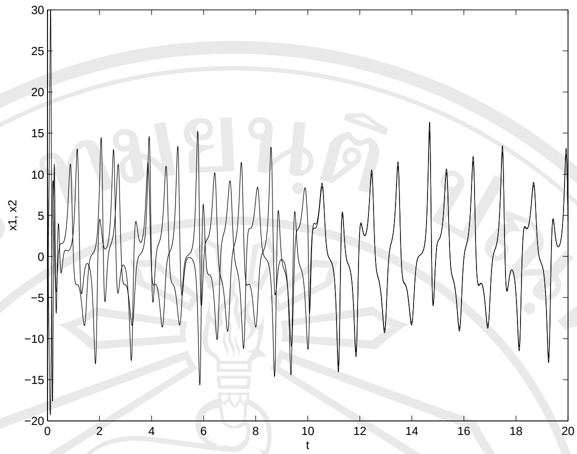

Figure 4.9: The states x_1, x_2 of the coupled perturbed Lü system of equations with the one-way coupling is activated at the time $t = 10$.

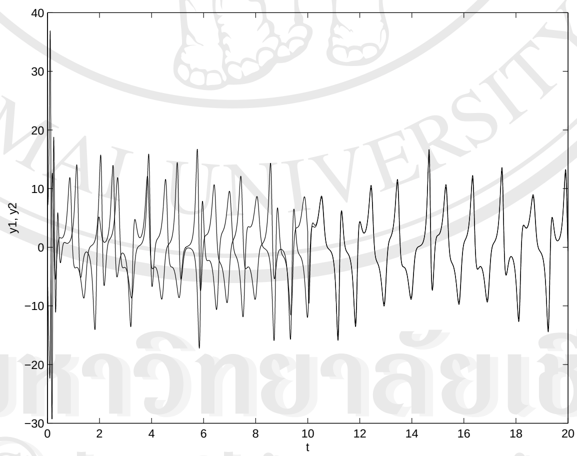

Figure 4.10: The states y_1, y_2 of the coupled perturbed Lü system of equations with the one-way coupling is activated at the time $t = 10$.

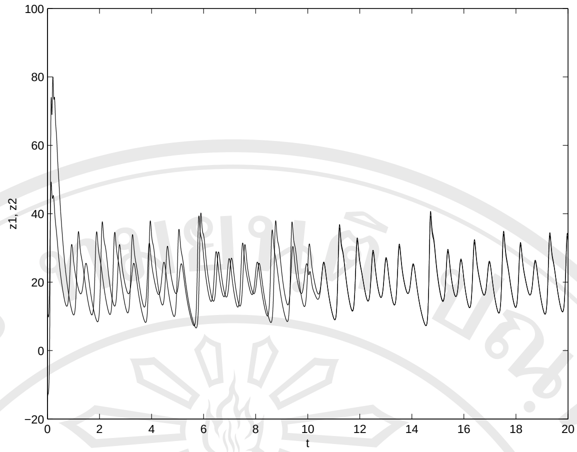

Figure 4.11: The states z_1, z_2 of the coupled perturbed Lü system of equations with the one-way coupling is activated at the time $t = 10$.