# CHAPTER 2

# PRELIMINARIES

In this chapter, we give some notations and definitions that will be used in the later chapters.

## 2.1 Some basic concept of combinatorics

### 2.1.1 Permutations

An *r-permutation* of $n$ distinct objects is an ordered arrangement of $r$ of the $n$ object. The number of $r$-permutations of $n$ objects is $P(n,r) = n(n-1)...(n-(r-1))$.

**Example** *The 2-permutations of the three letters A, B, and C are AB,BA,AC,CA,BC and CB. Thus* $P(3,2) = 6$.

We can choose any of the $n$ objects as the first object in the permutation, any of the remaining $n-1$ objects as the second object of permutation, etc. There are $r$ factors in the expression for $P(n,r)$, each one less than the preceeding. The first is $n$, so the last is $n-(r-1)$.

An $n$-permutation of $n$ distinct objects is a *permutation* of the objects. There are

$$P(n,n) = n(n-1)...1 = n!$$

permutations of $n$ objects.

### 2.1.2 Combinations

An *r-combination* of $n$ distinct objects is an unordered selection of $r$ of the $n$ objects. More simply, it is an $r$-element subset of the set of $n$ objects. The number of $r$-combinations of $n$ objects is sometime written $C(n, r)$, but we prefer the notation $\binom{n}{r}$ which is read *n choose r*.

**Example** *The 3-element set* $\{a,\ b,\ c\}$ *has three 2-element subsets* $\{a,\ b\}$,$\{a,\ c\}$*and* $\{b,\ c\}$. *Thus* $\binom{3}{2}$=*3.*

To obtain the $r$-permutation of $n$ objects, we may order each of the $r$-element subsets in all possible ways. Each $r$-element subset may be ordered in $p(r, r)$ ways, giving the relation

$$P(n, r) = \binom{n}{r} P(r, r).$$

Dividing by $P(r, r)$,

$$\binom{n}{r} = \frac{P(n, r)}{P(r, r)} = \frac{n(n-1)...(n-(r-1))}{r!}.$$

Multiplyig numerator and denominator by $(n-r)!$ gives the neater result

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

### 2.1.3 Binomial Coefficients

The *binomial coefficients* are the number $\binom{n}{r}$ of *combinations*. The name derives the *binomial expansion*

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r.$$

**Example** *The binomial expansion of* $(x+y)^4$ *is*

$$\begin{aligned}(x+y)^4 &= \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

*The reason that the coefficient of $x^2y^2$ in the expansion is $\binom{4}{2}$=6 is this: if we multiply out*

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

*by choosing an x or y from each factor, we get six terms with two x's and two y's, the terms xxyy, xyxy, xyyx, yxxy, yxyx, and yyxx. The term xxyy, for example, results by choosing x from the first two factors and y from the last two. The number of terms is the number of ways to choose two of the four factors, the two from which to take the y's.*

To verify the binomial expansion in general, we may reason exactly as in the above example. The coefficient of $x^{n-r}y^r$ in the expansion of $(x+y)^n$ is $\binom{n}{r}$ because $\binom{n}{r}$ is the number of ways to choose $r$ of factors of $(x+y)^n$, those $r$ from which to take the $y's$.

There are many relations between the binomial coefficients. One which is immediate is

$$\binom{n}{r} = \binom{n}{n-r},$$

which says that the binomial expansion is symmetric in $x$ and $y$. It says also that the number of ways of choosing $r$ elements from an $n$-element set is the number of ways of choosing $n-r$ elements. Another relation is

$$\binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} = 2^n.$$

This relation may be derived from the binomial expansion by putting $x = y = 1$. It says that the total number of subsets of an $n$-element set $S$, counting the empty set $S$ itself, is $2^n$.

The most important relation between the binomial coefficients is *Pascal's reletion*, which states that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r},$$

for $n$ and $r$ greater than zero. It is of course possible to verify to Pascal's relation algebraically, but it is more in the spirit of combinatorics to reason as follows: given an $n$-element set $S$, let $s$ be one of the element of $S$. Of the $\binom{n}{r}$ $r$-element subsets of $S$, there are $\binom{n-1}{r-1}$ which do contain $s$, and $\binom{n-1}{r}$ which do not contain $s$.

Pascal's relation leads to *Pascal's triangle.* Writing the binomial coefficients in the triangular array

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
 & & & . & . & . & & &
\end{array}$$

they are the number

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & . & . & . & & &
\end{array}$$

and Pascal's relation states that each number inside the triangle is the sum of the two numbers immediately it.

Rotating Pascal's triangle, and regarding it as a square grid, we see it as a block-walker or a road-runner might see it:

The point on the grid with coordinates $(n-r, r)$ is labeled by the binomial coefficient $\binom{n}{r} = \binom{n-r}{n-r,r}$ The number of ways to get from the point $\binom{0}{0}$ to the point $\binom{n}{r}$ always proceeding upward and to the right, is $\binom{n}{r}$ The directions Up, Right, Up, Right give one route $\binom{0}{0}$ to $\binom{4}{2}$ as shown above. There are $\binom{4}{2}$ routes to $\binom{4}{2}$ because there are $\binom{4}{2}$ strings of U's and R's with two U's and two R's.