CHAPTER 3

MAIN RESULTS

3.1 Main Results

In this chapter, we will show the some problem of counted ballots in the election of candidate A and candidate B. If we let some condition in this election, we will get. From [2], we have lemma 3.1.1.

Lemma 3.1.1 *In an election, candidate A receives n votes and candidate B receives (n − 1) votes number of ways may the ballots be counted so that candidate A is always ahead of candidate B is*

$$\frac{1}{n}\binom{2n-2}{n-1},$$

where n is positive integer.

In section 3.2, we propose theorem 3.2.1 which is a generalization of lemma 3.1.1.

3.2 Main theorem

Theorem 3.2.1 *In an election, if candidate A receives n votes and candidate B receives (n − k) votes then number of ways may the ballots be counted so that candidate A is always ahead of candidate B is*

$$\frac{k}{n}\binom{2n-k-1}{n-1},$$

where $k \leq n$ are positive integer.

Proof.

We want the number of strings of n A's and $(n-1)$ B's. The number of A's in an initial substring always exceeds the number of B's. We can generalize and count the routes in Pascal's triangle. The votes for A are tabulated vertically and the votes for B are tabulated horizontally. The first vote counted must be for A because of candidate A always ahead of candidate B. Therefore the routes can be considered as beginning at $\binom{1}{1}$. A route meeting the diagonal from $\binom{0}{0}$ to $\binom{2n}{n}$ would have A tied with B at this point and a route crossing the diagonal would have A lose with B at point under diagonal. A route not meeting the diagonal has A always ahead of B. Then, we will count the routes that do not meet the diagonal from $\binom{1}{1}$ to $\binom{2n-1}{n}$. We will consider a route from $\binom{1}{1}$ to $\binom{2n-1}{n}$ which meets the diagonal has a first point of intersection with the diagonal. Take the portion of the route to the first point of intersection and reflect it in the diagonal. Therefore, the result is a route from $\binom{1}{0}$ to $\binom{2n-1}{n}$.

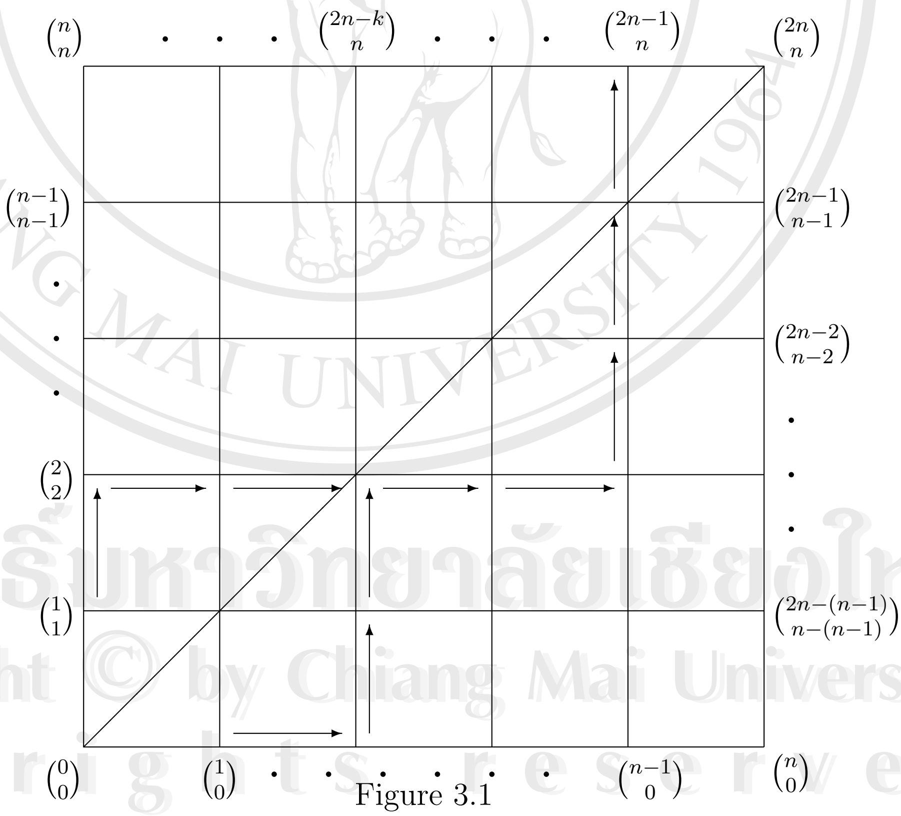


Figure 3.1

Every route from $\binom{1}{0}$ to $\binom{2n-1}{n}$ crosses the diagonal so every route from $\binom{1}{0}$ to $\binom{2n-1}{n}$ is the reflection of some route from $\binom{1}{1}$ to $\binom{2n-1}{n}$ that meets the diagonal. It follows that the number of routes from $\binom{1}{1}$ to $\binom{2n-1}{n}$ which meet the diagonal is the total number of routes from $\binom{1}{0}$ to $\binom{2n-1}{n}$. We have

$$\begin{aligned}
\binom{2n-2}{n} &= \frac{(2n-2)!}{(n-2)!n!} \\
&= \frac{(2n-2)(2n-3)...n(n-1)(n-2)!}{(n-2)!n!} \\
&= \frac{(2n-2)(2n-3)...n(n-1)}{n(n-1)(n-2)...2\cdot 1} \\
&= \frac{(n-1)}{n}\cdot\frac{(2n-2)(2n-3)...n}{(n-1)(n-2)...2\cdot 1}\cdot\frac{(n-1)!}{(n-1)!} \\
&= \frac{(n-1)}{n}\cdot\frac{(2n-2)(2n-3)...n(n-1)...2\cdot 1}{(n-1)!(n-1)!} \\
&= \frac{(n-1)}{n}\cdot\frac{(2n-2)!}{(n-1)!(n-1)!} \\
&= \frac{(n-1)}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

The total number of routes from $\binom{1}{1}$ to $\binom{2n-1}{n}$ is $\binom{2n-1}{n-1}$. The number of routes from $\binom{1}{1}$ to $\binom{2n-1}{n}$ do not meet the diagonal which is

$$\binom{2n-2}{n-1} - \frac{n-1}{n}\binom{2n-2}{n-1} = \frac{1}{n}\binom{2n-2}{n-1}.$$

Next, we consider the case when candidate A receives n votes and candidate B receives $(n-2)$ votes so that candidate A is always ahead of candidate B. We find the routes from $\binom{1}{1}$ to $\binom{2n-1}{n}$ by lemma 3.1.1. Thus, candidate A receives n votes and candidate B receives $(n-2)$. We shall find the routes from $\binom{1}{1}$ to $\binom{2n-2}{n}$.

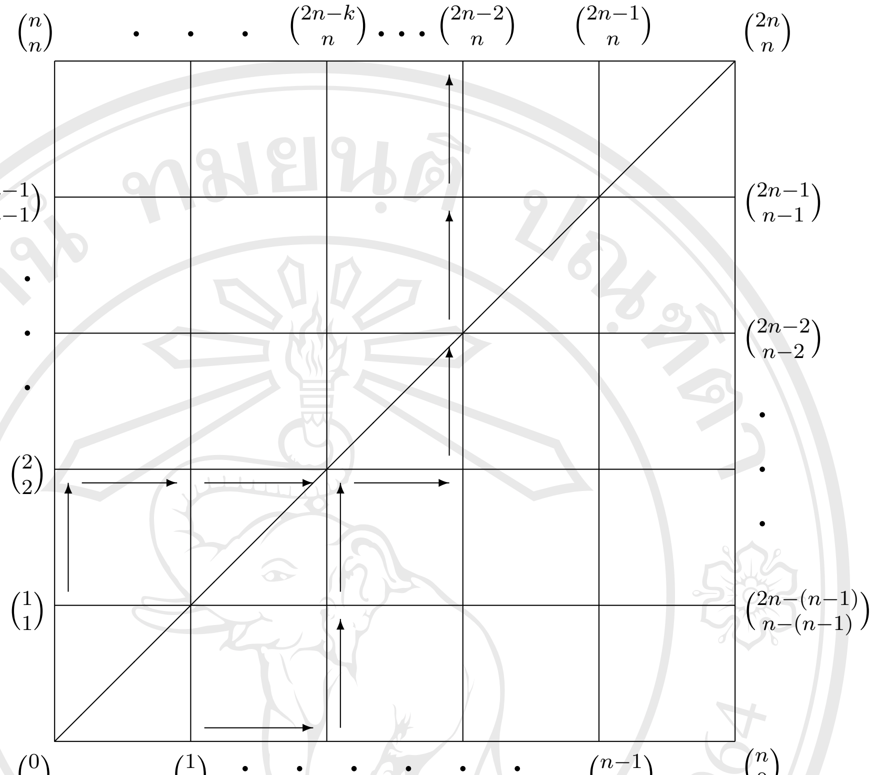


Figure 3.2

The routes from $\binom{1}{1}$ to $\binom{2n-2}{n}$ which meets the diagonal has a first point of intersection with the diagonal. Take the portion of the route to the first point of intersection and reflect it in the diagonal. The result is a route from $\binom{1}{0}$ to $\binom{2n-2}{n}$ and every route from $\binom{1}{0}$ to $\binom{2n-2}{n}$ crosses the diagonal, so every route from $\binom{1}{0}$ to $\binom{2n-2}{n}$ is the reflection of some route from $\binom{1}{1}$ to $\binom{2n-2}{n}$ that meets the diagonal. It follows that the number of routes from $\binom{1}{1}$ to $\binom{2n-2}{n}$ which meet the diagonal is the total number of routes from $\binom{1}{0}$ to $\binom{2n-2}{n}$, which is

$$\begin{aligned}
\binom{2n-3}{n} &= \frac{(2n-3)!}{(n-3)!n!} \\
&= \frac{(2n-3)(2n-4)(2n-5)...n(n-1)(n-2)(n-3)!}{(n-3)!n!} \\
&= \frac{(2n-3)(2n-4)...n(n-1)(n-2)}{n(n-1)(n-2)...2\cdot 1} \\
&= \frac{(n-2)}{n}\cdot\frac{(2n-3)(2n-4)...n(n-1)}{(n-1)(n-2)...2\cdot 1}\cdot\frac{(n-2)!}{(n-2)!} \\
&= \frac{(n-2)}{n}\cdot\frac{(2n-3)(2n-4)...n(n-1)...2\cdot 1}{(n-1)!(n-2)!} \\
&= \frac{(n-2)}{n}\cdot\frac{(2n-3)!}{(n-1)!(n-2)!} \\
&= \frac{(n-2)}{n}\binom{2n-3}{n-1}.
\end{aligned}$$

The total number of routes from $\binom{1}{1}$ to $\binom{2n-2}{n}$ is $\binom{2n-3}{n-1}$. Therefore the number of routes from $\binom{1}{1}$ to $\binom{2n-2}{n}$ which do not meet the diagonal is

$$\binom{2n-3}{n-1} - \frac{(n-2)}{n}\binom{2n-3}{n-1} = \frac{2}{n}\binom{2n-3}{n-1}.$$

Next, we shall consider the case when candidate A receives n votes and candidate B receives $(n-3)$ votes.

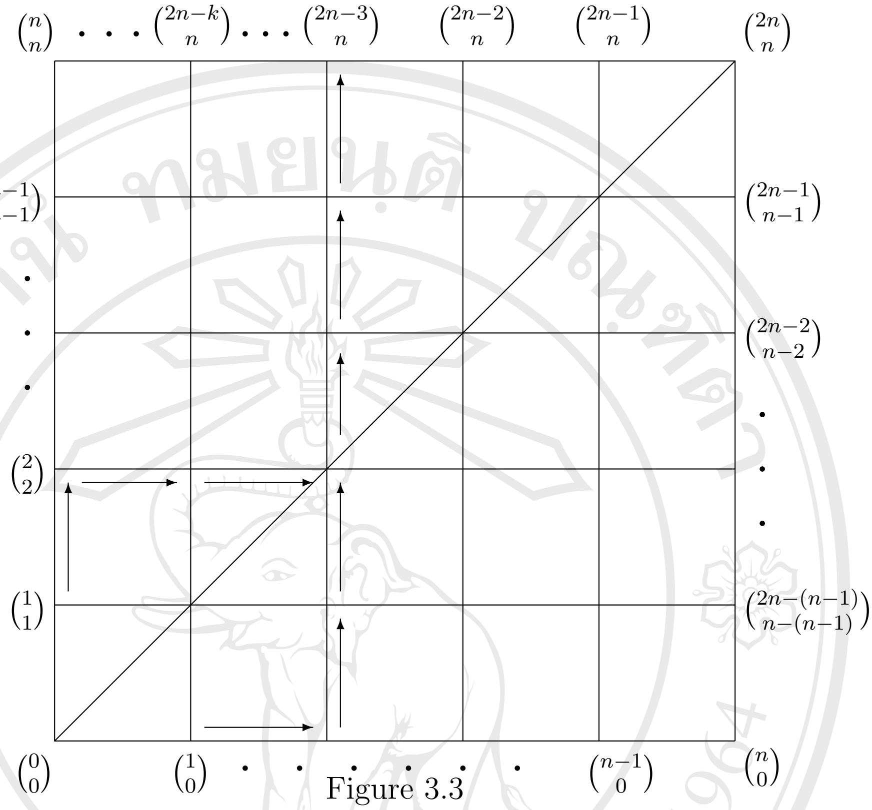


Figure 3.3

As in lemma 3.1.1, routes from $\binom{1}{1}$ to $\binom{2n-3}{n}$ which meets the diagonal has a first point of intersection with the diagonal. Take the portion of the route to the first point of intersection and reflect it in the diagonal. Therefore, the result is a route from $\binom{1}{0}$ to $\binom{2n-3}{n}$ and every route from $\binom{1}{0}$ to $\binom{2n-3}{n}$ crosses the diagonal so every route from $\binom{1}{0}$ to $\binom{2n-3}{n}$ is the reflection of some route from $\binom{1}{1}$ to $\binom{2n-3}{n}$ that meets the diagonal. It follows that the number of routes from $\binom{1}{1}$ to $\binom{2n-3}{n}$

which meet the diagonal is the total number of routes from $\binom{1}{0}$ to $\binom{2n-3}{n}$, which is

$$\begin{aligned}\binom{2n-4}{n} &= \frac{(2n-4)!}{(n-4)!n!} \\ &= \frac{(2n-4)(2n-5)(2n-6)...n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!n!} \\ &= \frac{(2n-4)(2n-5)...n(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)...2\cdot 1} \\ &= \frac{(n-3)}{n}\cdot\frac{(2n-4)(2n-5)...n(n-1)(n-2)}{(n-1)(n-2)(n-3)...2\cdot 1}\cdot\frac{(n-3)!}{(n-3)!} \\ &= \frac{(n-3)}{n}\cdot\frac{(2n-4)(2n-5)...n(n-1)...2\cdot 1}{(n-1)!(n-3)!} \\ &= \frac{(n-3)}{n}\cdot\frac{(2n-4)!}{(n-1)!(n-3)!} \\ &= \frac{(n-3)}{n}\binom{2n-4}{n-1}.\end{aligned}$$

The total number of routes from $\binom{1}{1}$ to $\binom{2n-3}{n}$ is $\binom{2n-4}{n-1}$, so the number of routes from $\binom{1}{1}$ to $\binom{2n-3}{n}$ which do not meet the diagonal is

$$\binom{2n-4}{n-1} - \frac{(n-3)}{n}\binom{2n-4}{n-1} = \frac{3}{n}\binom{2n-4}{n-1},$$

For the case when candidate A receives n votes, and candidate B receives $(n-4)$ votes so that candidate A is always ahead of candidate B, it is not difficult to show that the result is

$$\frac{4}{n}\binom{2n-5}{n-1}.$$

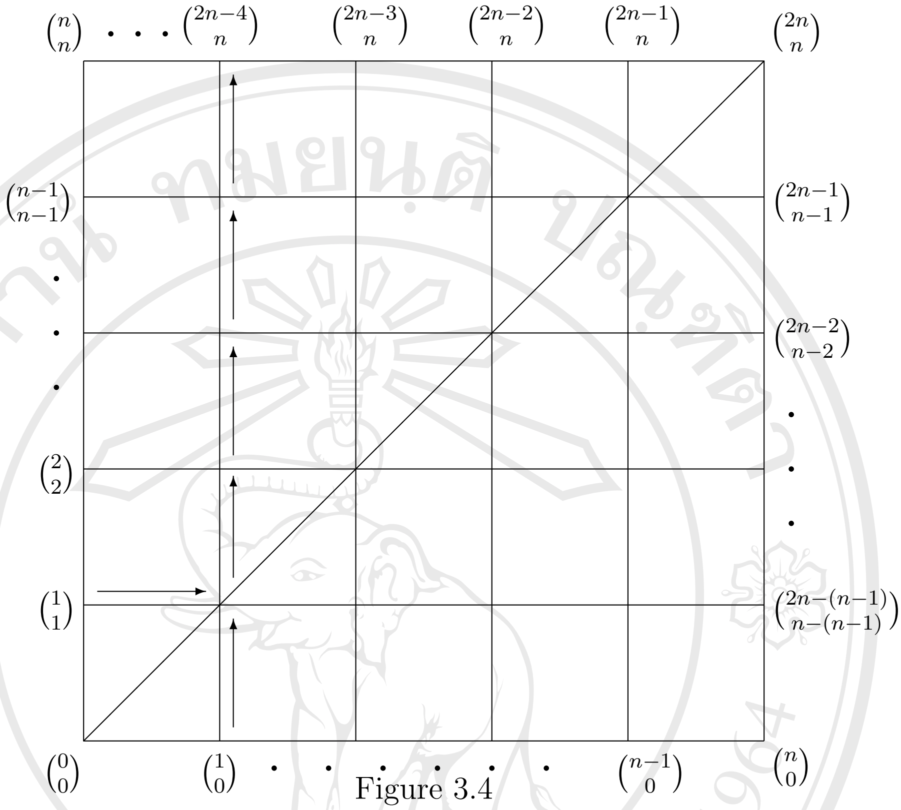


Figure 3.4

Similarly, in the case when candidate A receives n votes and candidate B receives $(n-k)$ votes so that candidate A is always ahead of candidate B, a route from $\binom{1}{1}$ to $\binom{2n-k}{n}$ which meets the diagonal has a first point of intersection with the diagonal. Take the portion of the route to the first point of intersection and reflect it in the diagonal. The result is a route from $\binom{1}{0}$ to $\binom{2n-k}{n}$:

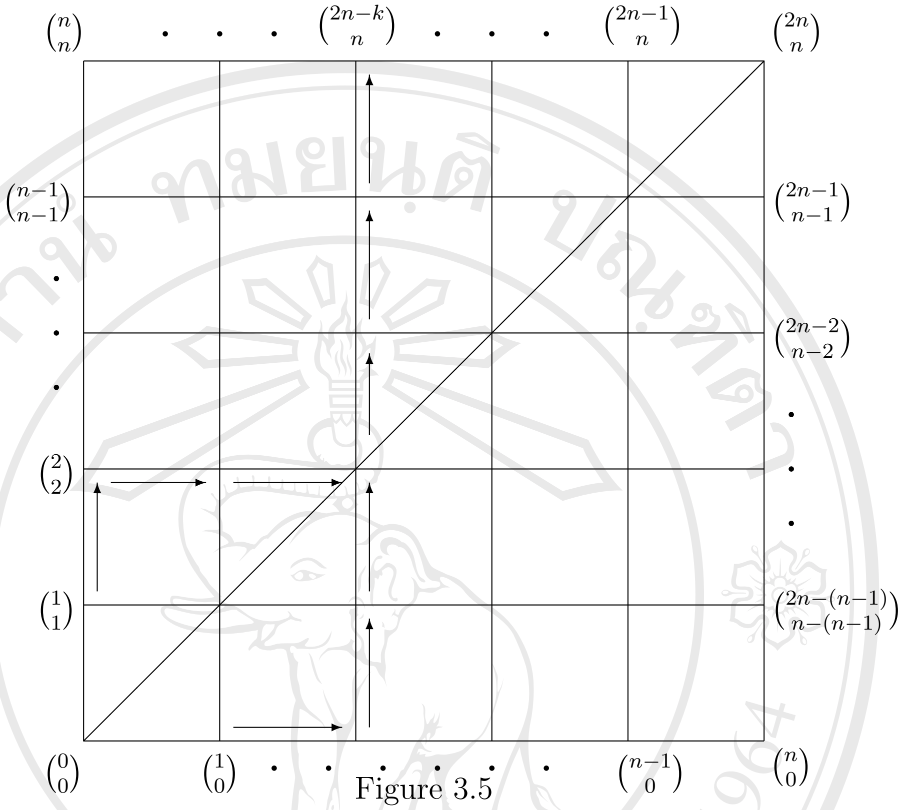


Figure 3.5

Every route from $\binom{1}{0}$ to $\binom{2n-k}{n}$ crosses the diagonal, so every route from $\binom{1}{0}$ to $\binom{2n-k}{n}$ is the reflection of some route from $\binom{1}{1}$ to $\binom{2n-k}{n}$ that meets the diagonal. It follows that the number of routes from $\binom{1}{1}$ to $\binom{2n-k}{n}$ which meet the diagonal is the total number of routes from $\binom{1}{0}$ to $\binom{2n-k}{n}$, which is

$$\begin{aligned}
\binom{2n-k-1}{n} &= \frac{(2n-k-1)!}{(n-k-1)!n!} \\
&= \frac{(2n-k-1)(2n-k-2)...(n-k+1)(n-k)(n-k-1)!}{(n-k-1)!n!} \\
&= \frac{(2n-k-1)(2n-k-2)...(n-k+1)(n-k)}{n(n-1)(n-2)...2\cdot 1} \\
&= \frac{(n-k)}{n}\cdot\frac{(2n-k-1)(2n-k-2)...(n-k+1)}{(n-1)(n-2)...2\cdot 1}\cdot\frac{(n-k)!}{(n-k)!} \\
&= \frac{(n-k)}{n}\cdot\frac{(2n-k-1)(2n-k-2)...n(n-1)(n-2)...2\cdot 1}{(n-1)!(n-k)!} \\
&= \frac{(n-k)}{n}\cdot\frac{(2n-k-1)!}{(n-1)!(n-k)!} \\
&= \frac{(n-k)}{n}\binom{2n-k-1}{n-1}.
\end{aligned}$$

The total number of routes from $\binom{1}{1}$ to $\binom{2n-k}{n}$ is $\binom{2n-k-1}{n-1}$, so the number of routes from $\binom{1}{1}$ to $\binom{2n-k}{n}$ which do not meet the diagonal is

$$\binom{2n-k-1}{n-1} - \frac{n-k}{n}\binom{2n-k-1}{n-1} = \frac{k}{n}\binom{2n-k-1}{n-1},$$

where $k \leq n$ are positive integer.

Lastty, we prove that the result is true by mathematical induction. From lemma 3.1.1, we have

$$\binom{2n-2}{n-1} - \binom{2n-2}{n} = \frac{1}{n}\binom{2n-2}{n-1}.$$

Suppose that if candidate A receives n votes and candidate B receives $(n-(k-1))$ votes, Therefore

$$\begin{aligned}\binom{2n-(k-1)-1}{n-1} - \binom{2n-(k-1)-1}{n} &= \frac{k-1}{n}\binom{2n-(k-1)-1}{n-1} \\ &= \frac{k-1}{n}\binom{2n-k}{n-1}.\end{aligned}$$

Consider

$$\binom{2n-k-1}{n-1} - \binom{2n-k-1}{n}$$

from

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}.$$

Hence,

$$\begin{aligned}
\binom{2n-k-1}{n-1} - \binom{2n-k-1}{n} &= \binom{2n-k}{n-1} - \binom{2n-k-1}{n-2} - \binom{2n-k}{n} \\
&\quad + \binom{2n-k-1}{n-1} \\
&= \frac{(k-1)}{n}\binom{2n-k}{n-1} - \binom{2n-k-1}{n-2} \\
&\quad + \binom{2n-k-1}{n-1} \\
&= \frac{(k-1)(2n-k)}{n(n-k+1)}\binom{2n-k-1}{n-1} \\
&\quad - \frac{(n-1)}{(n-k+1)}\binom{2n-k-1}{n-1} \\
&\quad + \binom{2n-k-1}{n-1} \\
&= \frac{(k-1)(2n-k)}{n(n-k+1)}\binom{2n-k-1}{n-1} \\
&\quad - \frac{n(n-1)}{n(n-k+1)}\binom{2n-k-1}{n-1} \\
&\quad + \frac{n(n-k+1)}{n(n-k+1)}\binom{2n-k-1}{n-1} \\
&= \frac{k}{n}\binom{2n-k-1}{n-1}.
\end{aligned}$$

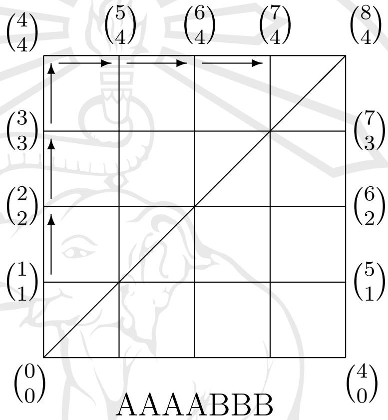
By induction, the theorem is true for all $k \leq n$.

Hence we have completed the proof.

Example 3.2.1 In an election, if candidate A receives 4 votes and candidate B receives 3 votes then number of ways may the ballots be counted so that candidate A is always ahead of candidate B is

AAAABBB,AABAABB,AAABABB,AABABAB and AAABBAB

In pascal's triangle



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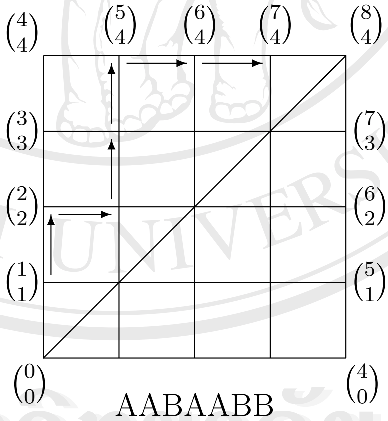

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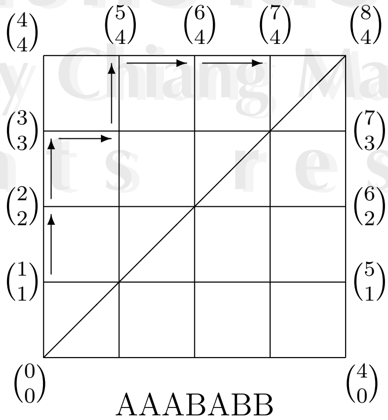

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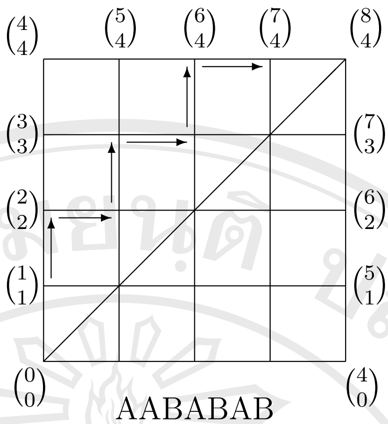

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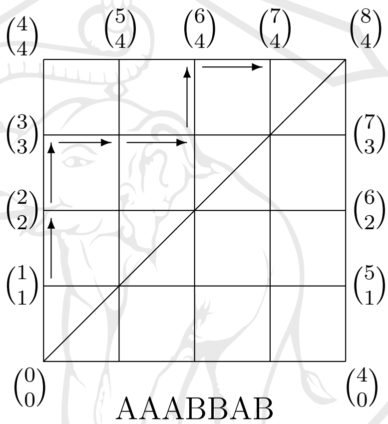


AAABBAB

From theorem 3.2.1 number of ways may the ballots be counted are

$$\begin{aligned}
\frac{k}{n}\binom{2n-k-1}{n-1} &= \frac{1}{4}\binom{2(4)-1-1}{4-1} \\
&= \frac{1}{4}\binom{6}{3} \\
&= \frac{1}{4}\cdot\frac{6!}{3!3!} \\
&= \frac{1}{4}\cdot(20) \\
&= 5.
\end{aligned}$$

Next, we have a more generalization.

Theorem 3.2.2 *In an election, if candidate A receives n votes and candidate B receives (n − k) votes then number of ways may the ballots be counted so that candidate A is always ahead of candidate B at least r votes is*

$$\frac{(k-r+1)}{(n-r+1)}\binom{2n-k-r}{n-r},$$

where $r \leq k \leq n$ are positive integer.

Proof. From theorem 3.2.1 show that case candidate A receives n votes and candidate B receives $(n-k)$ votes so that number of ways may the ballots be counted candidate A is always ahead of candidate B at least 1 votes. Next, we shall show that case candidate A receives n votes and candidate B receives $(n-k)$ votes so that candidate A is always ahead of candidate B at least 2 votes. Therefore the routes can be considered as beginning at $\binom{2}{2}$. Then, we will count the routes that do not meet the diagonal from $\binom{1}{1}$ to $\binom{2n-1}{n}$. We will consider a route from $\binom{2}{2}$ to $\binom{2n-k}{n}$ which meets the diagonal has a first point of intersection with the diagonal. Take the portion of the route to the first point of intersection and reflect it in the diagonal. Therefore, the result is a route from $\binom{2}{1}$ to $\binom{2n-k}{n}$ and every route from $\binom{2}{1}$ to $\binom{2n-k}{n}$ crosses the diagonal so every route from $\binom{2}{1}$ to $\binom{2n-k}{n}$ is the reflection of some route from $\binom{2}{2}$ to $\binom{2n-k}{n}$ that meets the diagonal. It follows that the number of routes from $\binom{2}{2}$ to $\binom{2n-k}{n}$ which meet the diagonal is the total number of routes from $\binom{2}{1}$ to $\binom{2n-k}{n}$, which is

$$\begin{aligned}
\binom{2n-k-2}{n-1} &= \frac{(2n-k-2)!}{(n-k-1)!(n-1)!} \\
&= \frac{(2n-k-2)(2n-k-3)...n(n-1)...(n-k)(n-k-1)!}{(n-k-1)!(n-1)!} \\
&= \frac{(2n-k-2)(2n-k-3)...n(n-1)...(n-k)}{(n-1)(n-2)(n-3)...2\cdot 1} \\
&= \frac{(n-k)}{(n-1)}\cdot\frac{(2n-k-2)(2n-k-3)...n(n-1)...(n-k+1)}{((n-2)(n-3)...2\cdot 1}\cdot\frac{(n-k)!}{(n-k)!} \\
&= \frac{(n-k)}{(n-1)}\cdot\frac{(2n-k-2)(2n-k-3)...n(n-1)...2\cdot 1}{(n-k)!(n-2)!} \\
&= \frac{(n-k)}{(n-1)}\cdot\frac{(2n-k-2)!}{(n-k)!(n-2)!} \\
&= \frac{(n-k)}{(n-1)}\binom{2n-k-2}{n-2}.
\end{aligned}$$

The total number of routes from $\binom{2}{2}$ to $\binom{2n-k}{n}$ is $\binom{2n-k-2}{n-2}$, so the number of routes from $\binom{2}{2}$ to $\binom{2n-k}{n}$ which do not meet the diagonal is

$$\binom{2n-k-2}{n-2}-\frac{(n-k)}{(n-1)}\binom{2n-k-2}{n-2}=\frac{(k-1)}{(n-1)}\binom{2n-k-2}{n-2},$$

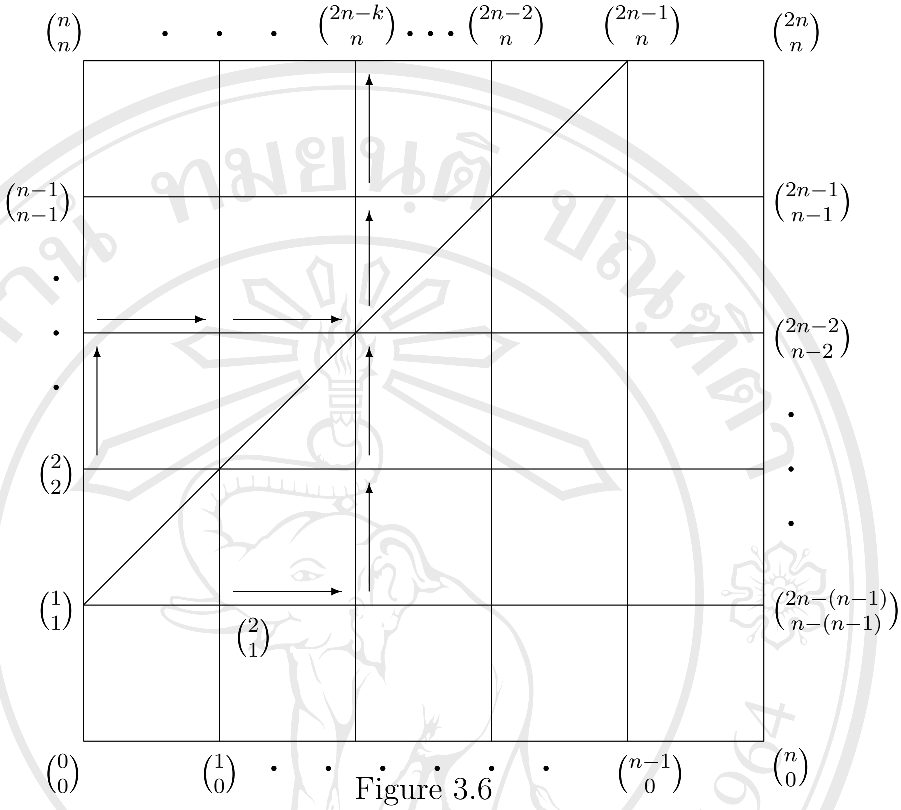

$\binom{n}{n}$
$\binom{2n-k}{n}$
$\binom{2n-2}{n}$
$\binom{2n-1}{n}$
$\binom{2n}{n}$
$\binom{n-1}{n-1}$
$\binom{2n-1}{n-1}$
$\binom{2n-2}{n-2}$
$\binom{2}{2}$
$\binom{1}{1}$
$\binom{2n-(n-1)}{n-(n-1)}$
$\binom{2}{1}$
$\binom{0}{0}$
$\binom{1}{0}$
$\binom{n-1}{0}$
$\binom{n}{0}$

Figure 3.6

Similarly, consider the case when candidate A receives n votes and candidate B receives $(n-k)$ votes then number of ways may the ballots be counted so that candidate A is always ahead of candidate B at least r votes. A route from $\binom{r}{r}$ to $\binom{2n-k}{n}$ which meets the diagonal from $\binom{r-1}{r-1}$ to $\binom{2n-r+1}{n}$ has a first point of intersection with the diagonal. Take the portion of the route to the first point of intersection and reflect it in the diagonal. The result is a route from $\binom{r}{r-1}$ to $\binom{2n-k}{n}$:

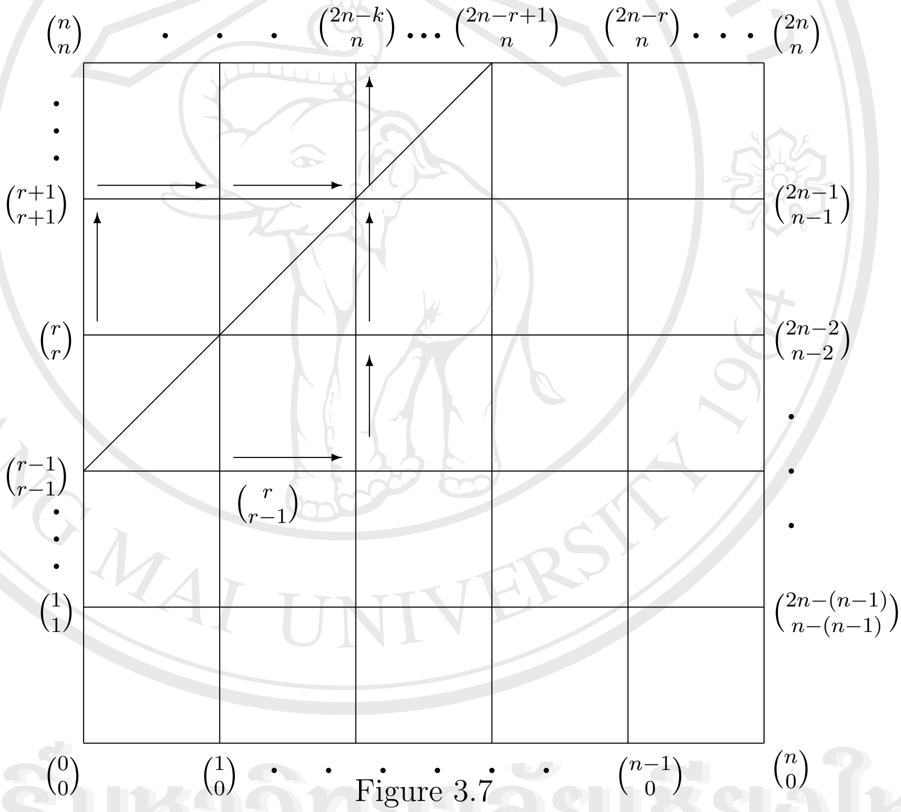


Figure 3.7

Every route from $\binom{r}{r-1}$ to $\binom{2n-k}{n}$ crosses the diagonal, so every route from $\binom{r}{r-1}$ to $\binom{2n-k}{n}$ is the reflection of some route from $\binom{r}{r}$ to $\binom{2n-k}{n}$ that meets the diagonal. It follows that the number of routes from $\binom{r}{r}$ to $\binom{2n-k}{n}$ which meet the diagonal is the total number of routes from $\binom{r}{r-1}$ to $\binom{2n-k}{n}$, which is

$$\begin{aligned}\binom{2n-k-r}{n-r+1} &= \frac{(2n-k-r)!}{(n-k-1)!(n-r+1)!}\\ &= \frac{(2n-k-r)(2n-k-r-1)...n(n-1)...(n-k)(n-k-1)!}{(n-k-1)!(n-r+1)!}\\ &= \frac{(2n-k-r)(2n-k-r-1)...n(n-1)...(n-k)}{(n-r+1)(n-r)...2\cdot 1}\\ &= \frac{(n-k)}{(n-r+1)}\cdot\frac{(2n-k-r)...n(n-1)...(n-k+1)}{(n-r)!}\cdot\frac{(n-k)!}{(n-k)!}\\ &= \frac{(n-k)}{(n-r+1)}\cdot\frac{(2n-k-r)(2n-k-r-1)...n(n-1)(n-2)...2\cdot 1}{(n-r)!(n-k)!}\\ &= \frac{(n-k)}{(n-r+1)}\cdot\frac{(2n-k-r)!}{(n-r)!(n-k)!}\\ &= \frac{(n-k)}{(n-r+1)}\binom{2n-k-r}{n-r}.\end{aligned}$$

The total number of routes from $\binom{r}{r}$ to $\binom{2n-k}{n}$ is $\binom{2n-k-r}{n-r}$, so the number of routes from $\binom{r}{r}$ to $\binom{2n-k}{n}$ which do not meet the diagonal is

$$\binom{2n-k-r}{n-r} - \frac{n-k}{(n-r+1)}\binom{2n-k-r}{n-r} = \frac{(k-r+1)}{(n-r+1)}\binom{2n-k-r}{n-r},$$

where $r \leq k \leq n$ are positive integer.

Lastty, we prove that the equation is true by mathematical induction. From theorem 3.2.1, we have

$$\binom{2n-k-1}{n-1} - \binom{2n-k-1}{n} = \frac{k}{n}\binom{2n-k-1}{n-1}.$$

Suppose that if candidate A receives n votes and candidate B receives $(n-k)$ votes so that candidate A is always ahead of candidate B at least $(r-1)$ votes. Therefore

$$\begin{aligned}\binom{2n-k-(r-1)}{n-(r-1)} - \binom{2n-k-(r-1)}{n-(r-1)+1} &= \frac{(k-(r-1)+1)}{(n-(r-1)+1)}\binom{2n-k-(r-1)}{n-(r-1)}\\ &= \frac{(k-r+2)}{(n-r+2)}\binom{2n-k-r+1}{n-r+1}.\end{aligned}$$

Consider

$$\binom{2n-k-r}{n-r} - \binom{2n-k-r}{n-r+1}$$

from

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}.$$

Hence,

$$\begin{aligned}
\binom{2n-k-r}{n-r} - \binom{2n-k-r}{n-r+1} &= \binom{2n-k-r+1}{n-r+1} - \binom{2n-k-r}{n-r+1} \\
&\quad - \binom{2n-k-r+1}{n-r+2} + \binom{2n-k-r}{n-r+2} \\
&= \frac{(k-r+2)}{(n-r+2)}\binom{2n-k-r+1}{n-r+1} - \binom{2n-k-r}{n-r+1} \\
&\quad + \binom{2n-k-r}{n-r+2} \\
&= \frac{(k-r+2)(2n-k-r+1)}{(n-r+2)(n-r+1)}\binom{2n-k-r}{n-r} \\
&\quad - \frac{(n-k)}{(n-r+1)}\binom{2n-k-r}{n-r} \\
&\quad \frac{(n-k)(n-k-1)}{(n-r+2)(n-r+1)}\binom{2n-k-r}{n-r} \\
&= \frac{(k-r+2)(2n-k-r+1)}{(n-r+2)(n-r+1)}\binom{2n-k-r}{n-r} \\
&\quad - \frac{(n-k)(n-r+2)}{(n-r+1)(n-r+2)}\binom{2n-k-r}{n-r} \\
&\quad + \frac{(n-k)(n-k-1)}{(n-r+2)(n-r+1)}\binom{2n-k-r}{n-r} \\
&= \frac{(k-r+1)}{(n-r+1)}\binom{2n-k-r}{n-r}.
\end{aligned}$$

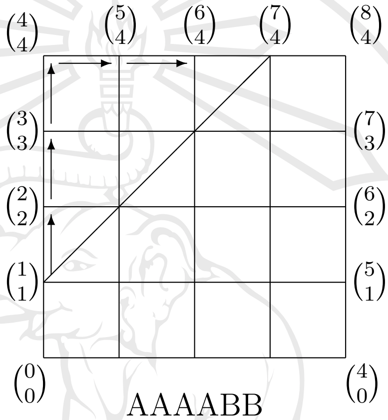
By induction, the theorem is true for all $r \leq k \leq n$.

Hence we have completed the proof.

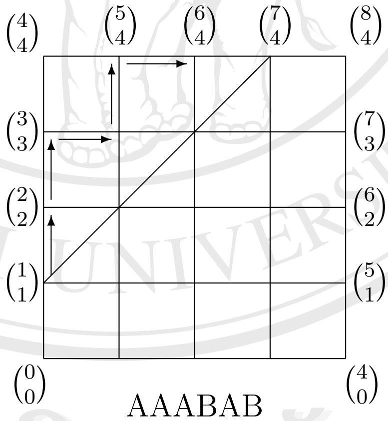
Example 3.2.2 In an election, if candidate A receives 4 votes and candidate B receives 2 votes then number of ways may the ballots be counted so that candidate A is always ahead of candidate B less than 2 votes is

AAAABB,AAABAB

In pascal's triangle



AAAABB



AAABAB

From theorem 3.2.2 number of ways may the ballots be counted are

$$
\begin{aligned}
\frac{(k-r+1)}{(n-r+1)}\binom{2n-k-r}{n-r} &= \frac{2-2+1}{4-2+1}\binom{2(4)-2-2}{4-2} \\
&= \frac{1}{3}\binom{4}{2} \\
&= \frac{1}{3}\cdot\frac{4!}{2!2!} \\
&= \frac{1}{3}\cdot(6) \\
&= 2.
\end{aligned}
$$