CHAPTER 2

PRELIMINARIES

In this chapter, we introduce some definitions and theorems those will be used in our thesis.

2.1 Local rings and Noetherian rings

A ring R is a *local ring* in case it has a unique maximal ideal.

Proposition 2.1.1. *For a ring R the following statements are equivalent :*

(*a*) *R is a local ring;*

(*b*) *If M is a maximal left ideal of R, then $M = \{x \in R | x$ is not invertible $\}$.*

Proof. See [1] page 170.

A set $\Im$ of ideals of R satisfies the *ascending chain condition* in case for every chain

$$L_1 \subseteq L_2 \subseteq \ldots L_n \subseteq \ldots$$

in $\Im$, there is an n with $L_{n+i} = L_n$ for all $i = 1, 2, 3, \ldots$.

A ring R is *Noetherian* in case the lattice $\Im(R)$ of all ideals of R satisfies the *ascending chain condition.*

Proposition 2.1.2. *For a ring R the following statements are equivalent:*

(*a*) *R is Noetherian;*

(*b*) *every non-empty set of ideals in R has a maximal element;*

(*c*) *every ideal in R is finitely generated.*

Proof. See [2] page 74-75.

Theorem 2.1.3. *Let M be an ideal in a ring R with identity $1_R \neq 0$.*

(i) If M is maximal and R is commutative, then the quotient ring R/M is a field.

(ii) If the quotient ring R/M is a division ring, then M is maximal.

Proof. See [3] page 129.

2.2 Forcing linearity numbers

Definition 2.2.1. A near-ring is a nonempty set R together with two binary operations (usually denote as addition (+) and multiplication) such that:

(i) $(R, +)$ is a group.

(ii) $(ab)c = a(bc)$ for all $a, b, c \in R$.

(iii) $(b + c)a = ba + ca$ for all $a, b, c \in R$.

Let V be a left R-module.

The set $M_R(V) = \{f : V \to V \mid f(rv) = rf(v), r \in R, v \in V\}$ is the collection of homogeneous functions determined by the R-module V. Under the operations addition and composition of functions, $\mathrm{M}_R(V)$ is a near-ring.

Note that $M_R(V)$ contains $End_R(V)$, the ring of R-endomorphisms of V.

If $M_R(V) = End_R(V)$, that is, if every R-homogeneous function from V to V is an endomorphism, then V is said to be *endomorphal*.

Let $\Im = \{W_\alpha \mid \alpha \in \mathcal{A}\}$ be a collection of proper submodules of V. We say that $\Im$ *forces linearity on* V if whenever $f \in M_R(V)$ and f is linear on each $W_\alpha \in \Im$, then $f \in End_R(V)$. Now we give the definition of forcing linearity number of V.

Definition 2.2.2. To each non-zero R-module V we assign a number

$fln(V) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, call the *forcing linearity number of V*, as follows:

(i) If $M_R(V) = End_R(V)$, then $fln(V) = 0$.

(ii) If $M_R(V) \neq End_R(V)$ and there is some finite collection $\Im$ of proper submodules of V which forces linearity with (say) $|\Im| = s$, but no collection $\Im'$ of proper submodules of V with $|\Im'| < s$ that forces linearity, then we say $fln(V) = s$.

(iii) If neither of the above conditions holds, we say $fln(V) = \infty$.

It is easy to see that if V is a cyclic R-module, then $M_R(V) = End_R(V)$, so $fln(V) = 0$ for any cyclic R-module.

2.3 Modules

Definition 2.3.1. Let V be a left R-module and N be a submodule of V. N is said to be a *maximal submodule* in case $N \neq$ V and if M is a submodule of V such that $N \subseteq M \subseteq$ V, then $N = M$ or $N =$ V.

Lemma 2.3.2. *Let* V *be a left R-module and M be a maximal submodule of* V*. Then* $M + Rm = \mathrm{V}$ *where* $m \in \mathrm{V} \backslash M$*.*

Proof. Since M is maximal and $m \in 0 + 1m \in M + Rm$, $M \subsetneqq M + Rm$ and thus $M + Rm = \mathrm{V}$.

Lemma 2.3.3. *Let e be an idempotent in* $End_R(M)$*. Then* $1 - e$ *is an idempotent in* $End_R(M)$ *such that*

$$Ker e = \{x \in M \mid x = x(1-e)\} = Im(1-e),$$

$$Im e = \{x \in M \mid x = x - e\} = Ker(1-e),$$

and
$$M = Me \oplus M(1-e).$$

Proof. See [1] page 70.

2.4 Direct products and direct sums

Definition 2.4.1. Let $\{M_\alpha \mid \alpha \in \mathcal{A}\}$ be an indexed set of left $R-$modules. The cartesian product $\times_{\mathcal{A}} M_\alpha$ is a $R-$module with addition and scalar multiplication defined by

$$(x_\alpha) + (y_\alpha) = (x_\alpha + y_\alpha) \text{ and } r(x_\alpha) = (rx_\alpha).$$

The resulting module is called the *direct(or cartesian)product* of $\{M_\alpha \mid \alpha \in \mathcal{A}\}$ and will be denoted by

$$\prod_{\mathcal{A}} M_\alpha.$$

In case $\mathcal{A} = \{1, 2, \ldots, n\}$ we write $\prod_{i=1}^{n} M_i$ or $M_1 \times M_2 \times \cdots \times M_n$ for $\prod_{\mathcal{A}} M_\alpha$.

If $M_\alpha = M$ for all $\alpha \in \mathcal{A}$, we write $M^{\mathcal{A}} = \prod_{\mathcal{A}} M_\alpha$.

If $\mathcal{A} = \phi$, then $\prod_{\mathcal{A}} M_\alpha = 0$.

Definition 2.4.2. Let π_α be the projection map and $\{M_\alpha \mid \alpha \in \mathcal{A}\}$ be an indexed set of left $R-$modules. An element $x \in \prod_{\mathcal{A}} M_\alpha$ is almost always zero in case its support

$$S(x) = \{\alpha \in \mathcal{A} \mid x(\alpha) = \pi_\alpha(x) \neq 0\}$$

is finite. Since $S(0) = \phi$ and both $S(x + y) \subseteq S(x) \cup S(y)$ and $S(rx) \subseteq S(x)$, it follows that

$$\bigoplus_{\mathcal{A}} M_\alpha = \{x \in \prod_{\mathcal{A}} M_\alpha \mid S(x) \text{ is finite }\}$$

is a submodule of $\prod_{\mathcal{A}} M_\alpha$.

This submodule is the *(external)direct sum* of $\{M_\alpha \mid \alpha \in \mathcal{A}\}$.

If $\mathcal{A}$ is finite, then $\bigoplus_{\mathcal{A}} M_\alpha = \prod_{\mathcal{A}} M_\alpha$.

If $M_\alpha = M$ for all $\alpha \in \mathcal{A}$, then we write $M^{(\mathcal{A})} = \bigoplus_{\mathcal{A}} M_\alpha$.

2.5 Free modules

Definition 2.5.1. Let $\{x_\alpha \mid \alpha \in \mathcal{A}\}$ be an indexed set of elements of a left $R-$module M. The set $\{x_\alpha \mid \alpha \in \mathcal{A}\}$ is *linearly independent* in case for every finite sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of distinct elements of $\mathcal{A}$ and every $r_1, r_2, \ldots, r_n \in R$

$$r_1 x_{\alpha_1} + r_2 x_{\alpha_2} + \cdots + r_n x_{\alpha_n} = 0 \text{ implies } r_1 = r_2 = \cdots = r_n = 0.$$

An $R-$module M with linearly independent spanning set $\{x_\alpha \mid \alpha \in \mathcal{A}\}$ is called a $free\ R-module$ with basis $\{x_\alpha \mid \alpha \in \mathcal{A}\}$.

Let R be a commutative local ring with identity and $\mathrm{V} = R^{(\mathbb{N})}$. Then V is a free module with basis $\{e_i \mid i \in \mathbb{N}\}$ where $\pi_j(e_i) = 1$ if $j = i$ and $\pi_j(e_i) = 0$ if $j \neq i$.