CHAPTER 3

MAIN RESULTS

3.1 Forcing linearity number for $R^{(\mathbb{N})}$

In this section we want to find the forcing linearity numbers for a free module, V $= R^{(\mathbb{N})}$, over a local ring R. By a local ring we mean a commutative Noetherian ring with identity $1_R \neq 0$, not an integral domain such that R has a unique maximal ideal M consisting of all the nonunits of R.

Definition 3.1.1. *For each $i \in \mathbb{N}$ we define*

$$e_i = (u_i)_{i \in A} \quad where \quad u_j \begin{cases} 1 & if \quad j = i, \\ 0 & if \quad j \neq i. \end{cases}$$

Thus by Definition 3.1.1. we get $e_1 = (1, 0, 0, 0, \dots)$ and $e_2 = (0, 1, 0, 0, \dots)$.

Lemma 3.1.2. *If $f \in M_R(R^{(\mathbb{N})})$ the set of all homogeneous functions on $R^{(\mathbb{N})}$ and $f(\sum_{i=1}^m r_i e_i) = \sum_{i=1}^m f(r_i e_i)$ for all $r_i \in R$ and $m \in \mathbb{N}$, then $f \in End_R(R^{(\mathbb{N})})$.*

Proof. Suppose that $f \in M_R(R^{(\mathbb{N})})$ and $f(\sum_{i=1}^m r_i e_i) = \sum_{i=1}^m f(r_i e_i)$ for all $r_i \in R$ and $m \in \mathbb{N}$.

Let $b = (b_i)_{i \in \mathbb{N}}$ and $c = (c_i)_{i \in \mathbb{N}}$ be arbitrary in $R^{(\mathbb{N})}$.

Then there exists $n \in \mathbb{N}$ such that $b = \sum_{i=1}^n b_i e_i$ and $c = \sum_{i=1}^n c_i e_i$.

Thus

$$\begin{aligned} f(b + c) &= f(\textstyle\sum_{i=1}^n (b_i + c_i) e_i) \\ &= \textstyle\sum_{i=1}^n f((b_i + c_i) e_i). \end{aligned}$$

But $f \in M_R(R^{(\mathbb{N})})$, then

$$\begin{aligned} f((b_i + c_i)e_i) &= (b_i + c_i)f(e_i) \\ &= b_i f(e_i) + c_i f(e_i) \\ &= f(b_i e_i) + f(c_i e_i) \qquad \text{for all} \quad i = 1, 2, 3, \dots, n. \end{aligned}$$

Hence

$$\begin{aligned} f(b + c) &= \sum_{i=1}^{n} f((b_i + c_i)e_i) \\ &= \sum_{i=1}^{n} (f(b_i e_i) + f(c_i e_i)) \\ &= \sum_{i=1}^{n} f(b_i e_i) + \sum_{i=1}^{n} f(c_i e_i) \\ &= f(\sum_{i=1}^{n} b_i e_i) + f(\sum_{i=1}^{n} c_i e_i) \\ &= f(b) + f(c). \end{aligned}$$

Thus $f \in End_R(R^{(\mathbb{N})})$. □

Proposition 3.1.3. *Let R be a local ring and M a unique maximal ideal of R with $Ann_R(M) = \{0\}$ and let $\mathrm{V} = R^{(\mathbb{N})}$, then $fln(\mathrm{V}) \leq 1$.*

Proof. Let $\mathrm{W} := \langle Me_1 \cup Me_2 \cup \dots \rangle$ and $f \in M_R(\mathrm{V})$ such that f is linear on W. Let $f(e_i) = (\alpha_{i1}, \alpha_{i2}, \dots)$.

Let $(r_1, r_2, \dots) = r_1 e_1 + r_2 e_2 + \dots + r_m e_m$ in V, and $f((r_1, r_2, \dots)) = (s_1, s_2, \dots)$. Then for all $a \in M$, we have

$$\begin{aligned} f(a(r_1, r_2, \dots)) = af((r_1, r_2, \dots)) &= a(s_1, s_2, \dots) \\ &= as_1 e_1 + as_2 e_2 + \dots + as_k e_k \quad \text{for some } k \in \mathbb{N} \\ &= (as_1, as_2, \dots). \end{aligned}$$

Since $a(r_1, r_2, \dots) = ar_1 e_1 + ar_2 e_2 + \dots + ar_m e_m \in \mathrm{W}$, we have

$$\begin{aligned} f(a(r_1, r_2, \dots)) &= ar_1 f(e_1) + ar_2 f(e_2) + \dots + ar_m f(e_m) \\ &= ar_1(\alpha_{11}, \alpha_{12}, \dots) + ar_2(\alpha_{21}, \alpha_{22}, \dots) + \dots + ar_m(\alpha_{m1}, \alpha_{m2}, \dots). \end{aligned}$$

Thus we get

$$(as_1, as_2, \dots) = ar_1(\alpha_{11}, \alpha_{12}, \dots) + ar_2(\alpha_{21}, \alpha_{22}, \dots) + \cdots + ar_m(\alpha_{m1}, \alpha_{m2}, \dots)$$
$$= (ar_1\alpha_{11}, ar_1\alpha_{12}, \dots) + (ar_2\alpha_{21}, ar_2\alpha_{22}, \dots) + \cdots + (ar_m\alpha_{m1}, ar_m\alpha_{m2}, \dots).$$

So $as_j = ar_1\alpha_{1j} + ar_2\alpha_{2j} + \cdots + ar_m\alpha_{mj}$,

and hence

$$as_j - (ar_1\alpha_{1j} + ar_2\alpha_{2j} + \cdots + ar_m\alpha_{mj}) = 0,$$

then

$$(s_j - (r_1\alpha_{1j} + r_2\alpha_{2j} + \cdots + r_m\alpha_{mj}))a = 0.$$

This means that $s_j - (r_1\alpha_{1j} + r_2\alpha_{2j} + \cdots + r_m\alpha_{mj}) = 0$ since $Ann_R(M) = \{0\}$.

Thus $s_j = r_1\alpha_{1j} + r_2\alpha_{2j} + r_3\alpha_{3j} + \cdots + r_m\alpha_{mj}$, $j = 1, 2, \dots, k$.

Then

$$f(\sum_{i=1}^{m} r_i e_i) = f(r_1, r_2, \dots)$$
$$= (r_1\alpha_{11} + r_2\alpha_{21} + \cdots + r_m\alpha_{m1})e_1 + (r_1\alpha_{12} + r_2\alpha_{22} + \cdots + r_m\alpha_{m2})e_2 + \cdots + (r_1\alpha_{1k} + r_2\alpha_{2k} + \cdots + r_m\alpha_{mk})e_k$$
$$= r_1(\alpha_{11}, \alpha_{12}, \dots) + r_2(\alpha_{21}, \alpha_{22}, \dots) + \cdots + r_m(\alpha_{m1}, \alpha_{m2}, \dots)$$
$$= r_1 f(e_1) + r_2 f(e_2) + \cdots + r_m f(e_m) = f(r_1e_1) + f(r_2e_2) + \cdots + f(r_me_m)$$
$$= \sum_{i=1}^{m} f(r_ie_i).$$

By Lemma 3.1.2. we get $f \in End_R(\mathrm{V})$.

Therefore we have $fln(\mathrm{V}) \leq 1$. □

Proposition 3.1.4. *Let R be a local ring and M a unique maximal ideal of R with $Ann_R(M) \neq 0$. If $\mathrm{V} = R^{(\mathbb{N})}$, then $fln(\mathrm{V}) \neq 0$.*

Proof. Let $0 \neq a_o \in Ann_R(M)$ and $v = r_1e_1 + r_2e_2 + \cdots + r_me_m \in \mathrm{V}$.

Define a function $f : \mathrm{V} \to \mathrm{V}$ by

$$f(v) = \begin{cases} r_1 a_o e_1 & \text{if} \quad r_2 \in M, \\ 0 & \text{if} \quad r_2 \in R \backslash M. \end{cases}$$

It is easy to see that f is well-defined.

Next, we show that $f \in \mathrm{M}_R(\mathrm{V})$.

Let $s \in R$.

If $r_2 \in R\backslash M$, then $f(v) = f(r_1e_1 + r_2e_2 + \cdots + r_me_m) = 0$.

So if $s \in R\backslash M$, then $sr_2 \in R\backslash M$, $f(sv) = f(s(r_1e_1 + r_2e_2 + \cdots + r_me_m)) = f(sr_1e_1 + sr_2e_2 + \cdots + sr_me_m) = 0$. But if $s \in M$, then $sr_2 \in M$ since M is an ideal of R. Thus $f(sv) = f(sr_1e_1 + sr_2e_2 + \cdots + sr_me_m) = sr_1a_oe_1 = 0$ since $sr_1 \in M$.

If $r_2 \in M$, then $f(v) = f(r_1e_1 + r_2e_2 + \cdots + r_me_m) = r_1a_oe_1$ and that $sf(v) = sf(r_1e_1 + r_2e_2 + \cdots + r_me_m) = s(r_1a_oe_1) = sr_1a_oe_1$.

Since $sr_2 \in M$, we get

$$f(sv) = f(s(r_1e_1 + r_2e_2 + \cdots + r_me_m)) = f(sr_1e_1 + sr_2e_2 + \cdots + sr_me_m) = sr_1a_oe_1.$$

Hence $f \in M_R(\mathrm{V})$.

Next, we show that f is not linear on V.

Let $v_1 = e_1, v_2 = e_2 \in \mathrm{V}$, then $f(v_1 + v_2) = f(e_1 + e_2) = 0$ since $1 \in R\backslash M$.

Since $v_1 = e_1 = 1e_1 + 0e_2 + \cdots + 0e_m$ and $v_2 = e_2 = 0e_1 + 1e_2 + \cdots + 0e_m$, then $f(v_1) = (1)a_oe_1$ since $0 \in M$ and $f(v_2) = 0$ since $1 \in R\backslash M$.

Thus $f(v_1 + v_2) = 0 \neq a_oe_1 = a_oe_1 + 0 = f(v_1) + f(v_2)$.

Hence f is not linear on V.

Thus $f \notin End_R(V)$ and that $fln(V) \neq 0$. □

Lemma 3.1.5. *Let R be a finite local ring and $\mathrm{V} = R^{(\mathbb{N})}$. If $\mathrm{T} \subseteq \mathrm{V}$, then $\mathcal{A} = \{Rw \mid w \in \mathrm{T}\}$ has a maximal element.*

Proof. Let $\mid R \mid = q$ and suppose that $\mathcal{A} = \{Rw \mid w \in \mathrm{T}\}$ where $\mathrm{T} \subseteq \mathrm{V}$ does not have a maximal element.

We note that for each $w \in \mathrm{T}$, $|Rw| \leq q$.

Let $Rw_1 \in \mathcal{A}$, then there exists $Rw_2 \in \mathcal{A}$ such that

$$Rw_1 \subsetneqq Rw_2.$$

Since $Rw_2 \in \mathcal{A}$, then there exists $Rw_3 \in \mathcal{A}$ such that

$$Rw_1 \subsetneqq Rw_2 \subsetneqq Rw_3.$$

By continuing in this way, we obtain an ascending chain of submodules of V

$$Rw_1 \subsetneqq Rw_2 \subsetneqq Rw_3 \subsetneqq \cdots \subsetneqq Rw_q \subsetneqq Rw_{q+1} \ldots.$$

Since $w_1 \in Rw_1$, then we get $|Rw_1| \geq 1$.

Thus $|Rw_{q+1}| > q$, it is a contradiction and the prove is complete. □

Proposition 3.1.6. *If R is a finite local ring, M a unique maximal ideal of R with $Ann(M) \neq 0$ and $\Im = \{\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_t\}$ is a collection of proper submodules of $\mathrm{V} = R^{(\mathbb{N})}$ which forces linearity, then $\cup_{i=1}^{t} \mathrm{S}_i = \mathrm{V}$.*

Proof. We suppose that $\cup_{i=1}^{t} \mathrm{S}_i \subsetneqq \mathrm{V}$ and show that $\Im$ does not force linearity on V.

We note that R is finite, then the set $\{Rw \mid w \notin \cup_{i=1}^{t} \mathrm{S_i}\}$ has a maximal element by Lemma 3.1.5, say Rw_o. Also $Rw_o \subsetneqq \mathrm{V}$ since V is not cyclic.

Now let $0 \neq a \in Ann(M)$ and define $f : \mathrm{V} \rightarrow \mathrm{V}$ by

$$f(x) = \begin{cases} rae_1 & \text{if } \quad x = rw_o \in Rw_o, \\ 0 & \text{otherwise.} \end{cases}$$

We show that f is well-defined.

Suppose that $s_1 w_o = s_2 w_o$.

Then $(s_1 - s_2)w_o = 0$, so $(s_1 - s_2) \in M$, because if $(s_1 - s_2) \notin M$, then $(s_1 - s_2)$ is a unit, so there exist $(s_1 - s_2)^{-1}$ such that $(s_1 - s_2)(s_1 - s_2)^{-1} = 1$.

Thus $w_o = 1w_o = (s_1 - s_2)^{-1}(s_1 - s_2)w_o = (s_1 - s_2)^{-1}0 = 0$, a contradiction.

Hence $(s_1 - s_2)ae_1 = ((s_1 - s_2)a)e_1 = 0e_1 = 0$, i.e., $s_1 ae_1 = s_2 ae_1$, thus f is well-defined.

Next, we show that $f \in \mathrm{M}_R(\mathrm{V})$.

Let $v \in \mathrm{V}, r \in R$. We consider in two cases:

Case 1: $v \in Rw_0$.

Then $v = sw_0$ for some $s \in R$, so $rv = rsw_0$, $f(rv) = rsae_1 = rf(v)$.

Case 2: $v \notin Rw_0$. We consider in two subcases.

(1) If $rv \notin Rw_o$, then $rf(v) = 0 = f(rv)$.

(2) If $rv \in Rw_0$, then $rv = sw_o \in Rw_o$ for some $s \in R$. We consider as follow. If $s \in M$ and $f(rv) = sae_1 = 0 = r(0) = rf(v)$. The case $s \notin M$ can not occur since $w_0 = s^{-1}sw_0 = s^{-1}rv \in Rv$ which implies $Rw_0 \subseteq Rv$.

If $v \in \cup_{i=1}^{t}\mathrm{S}_i$, then $w_0 \in Rv \subseteq S_i$ for some i which is a contradiction.

If $v \notin \cup_{i=1}^{t}\mathrm{S}_i$, then $v \in Rv = Rw_0$ since Rw_0 is maximal which contradicts to our assumption.

Thus we have $f \in \mathrm{M}_R(\mathrm{V})$.

Now, we show that f is linear on each S_i for all $i = 1, 2, \ldots, t$.

Let $v \in \cup_{i=1}^{t}\mathrm{S}_i$.

If $v \notin Rw_o$, then $f(v) = 0$.

If $v \in Rw_o$, say $v = sw_o$, then $s \in M$, for if $s \in R \setminus M$, hence s is a unit, thus there exists s^{-1}, then $w_o = 1w_o = s^{-1}sw_o = s^{-1}v \in \cup_{i=1}^{t}\mathrm{S}_i$, a contradiction.

Thus $f(v) = sae_1 = 0$.

Consequently, $f(\cup_{i=1}^{t}\mathrm{S_i}) = 0$, then $f(x+y) = 0 = 0+0 = f(x)+f(y)$ for all $x, y \in \mathrm{S}_i, \forall i = 1, 2, \ldots, t$ and hence f is linear on each $\mathrm{S}_i \in \Im$.

Finally let $\hat{w} \in (\mathrm{V} \setminus Rw_o)$.

Then $\hat{w} + w_o \notin Rw_o$, so $f(\hat{w} + w_o) = 0 \neq ae_1 = f(w_o) = f(\hat{w}) + f(w_o)$.

Therefore $\Im$ does not force linearity on V. □

Lemma 3.1.7. *Let R be a local ring with unique maximal ideal M. Then* V/MV *is a vector space over the field R/M under addition and scalar multiplication defined by*

$$(v_1 + \mathrm{MV}) + (v_2 + \mathrm{MV}) = (v_1 + v_2) + \mathrm{MV}.$$

and

$$(r + \mathrm{M})(v + \mathrm{MV}) = rv + \mathrm{MV}.$$

Proof. Since R is a commutative ring with identity and M is a maximal ideal of R, R/M is a field.

We will show that addition and scalar multiplication are well-defined.

Let $x_1 + \text{MV} = y_1 + \text{MV}$ and $x_2 + \text{MV} = y_2 + \text{MV}$ where $x_1, x_2, y_1, y_2 \in \text{V}$.
Then $x_1 - y_1 \in \text{MV}$ and $x_2 - y_2 \in \text{MV}$, so $(x_1 - y_1) + (x_2 - y_2) \in \text{MV}$.
Thus $(x_1 + x_2) - (y_1 + y_2) \in \text{MV}$.
This means that $(x_1 + x_2) + \text{MV} = (y_1 + y_2) + \text{MV}$.
Next, let $r_1 + \text{M} = r_2 + \text{M}$ and $v_1 + \text{MV} = v_2 + \text{MV}$ where $r_1, r_2 \in R$ and $v_1, v_2 \in \text{V}$.
Then $r_1 - r_2 \in \text{M}$ and $v_1 - v_2 \in \text{MV}$.
Since MV is an $R-$module, then $(r_1 - r_2)v_1 \in \text{MV}$ and $r_2(v_1 - v_2) \in \text{MV}$, then $r_1v_1 - r_2v_1 \in \text{MV}$ and $r_2v_1 - r_2v_2 \in \text{MV}$.
Thus $(r_1v_1 - r_2v_1) + (r_2v_1 - r_2v_2) \in \text{MV}$, so $(r_1v_1 - r_2v_2) \in \text{MV}$.
Hence $r_1v_1 + \text{MV} = r_2v_2 + \text{MV}$.
Therefore $(r_1 + \text{M})(v_1 + \text{MV}) = (r_2 + \text{M})(v_2 + \text{MV})$.
Next, we show that V/MV is a vector space over the field R/M under addition and scalar multiplication defined above.
Let $v_1 + \text{MV}, v_2 + \text{MV} \in \text{V/MV}$ and $r_1 + M, r_2 + M \in R/M$ where $v_1, v_2 \in \text{V}$ and $r_1, r_2 \in R$.
Then

$$\begin{aligned}(r_1 + \text{M})((v_1 + \text{MV}) + (v_2 + \text{MV})) &= (r_1 + \text{M})((v_1 + v_2) + \text{MV}) \\ &= r_1(v_1 + v_2) + \text{MV} \\ &= (r_1v_1 + r_1v_2) + \text{MV} \\ &= (r_1v_1 + \text{MV}) + (r_1v_2 + \text{MV}) \\ &= (r_1 + \text{M})(v_1 + \text{MV}) + (r_1 + \text{M})(v_2 + \text{MV})\end{aligned}$$

and

$$\begin{aligned}((r_1 + \text{M}) + (r_2 + \text{M}))(v_1 + \text{MV}) &= ((r_1 + r_2) + \text{M})(v_1 + \text{MV}) \\ &= (r_1 + r_2)v_1 + \text{MV} \\ &= (r_1v_1 + r_2v_1) + \text{MV} \\ &= (r_1v_1 + \text{MV}) + (r_2v_1 + \text{MV}) \\ &= (r_1 + \text{M})(v_1 + \text{MV}) + (r_2 + \text{M})(v_1 + \text{MV})\end{aligned}$$

and

$$\begin{aligned}(r_1 + \mathrm{M})((r_2 + \mathrm{M})(v_1 + \mathrm{MV})) &= (r_1 + \mathrm{M})(r_2 v_1 + \mathrm{MV}) \\ &= r_1(r_2 v_1) + \mathrm{MV} \\ &= (r_1 r_2) v_1 + \mathrm{MV} \\ &= (r_1 r_2)(v_1 + \mathrm{MV}) \\ &= ((r_1 r_2) + M)(v_1 + \mathrm{MV}) \\ &= ((r_1 + M)(r_2 + M))(v_1 + \mathrm{MV})\end{aligned}$$

and

$$(1 + M)(v_1 + \mathrm{MV}) \;=\; 1v_1 + \mathrm{MV} \;=\; v_1 + \mathrm{MV}$$

Thus we get V/MV is a vector space over the field R/M. □

Lemma 3.1.8. *Let V be a vector space over the field F and $S_1, S_2, \ldots, S_k$ be finitely many subspaces of* V *with $k < |F| + 1$. Then $S_1 \cup S_2 \cup \cdots \cup S_k$ is a subspace if and only if some S_i contains the others.*

Proof. See [7] page 128.

Theorem 3.1.9. *Let R be a local ring with unique maximal ideal M and* $\mathrm{V} = R^{(\mathbb{N})}$ *and $Ann_R(M) \neq 0$. If every proper submodule of* V *contains* MV *and R/M is infinite, then $fln(\mathrm{V}) = \infty$.*

Proof. Suppose that there exists a finite set $\Im = \{S_1, S_2, \ldots, S_t\}$ of proper submodules of V which forces linearity, then by Proposition 3.1.6. we get $\mathrm{V} = \cup_{i=1}^{t} S_i$. By Lemma 3.1.7, we get V/MV is a vector space over R/M and for each $1 \leq i \leq t$ we get S_i/MV is a proper subspace of V/MV since $\mathrm{MV} \leq S_i$.

Since $\cup_{i=1}^{t}(S_i/\mathrm{MV}) = (\cup_{i=1}^{t} S_i)/\mathrm{MV} = \mathrm{V}/\mathrm{MV}$, $\cup_{i=1}^{t}(S_i/\mathrm{MV})$ is a vector space on R/M.

Because $|R/M|$ is infinite, we get $t < |R/M| + 1$.

Thus by Lemma 3.1.8. there exists $1 \leq j \leq t$ such that $S_i/\mathrm{MV} \subseteq S_j/\mathrm{MV}$ for all $i \neq j$, so $S_j/\mathrm{MV} = \cup_{i=1}^{t}(S_i/\mathrm{MV}) = \mathrm{V}/\mathrm{MV}$, which contradicts to S_j/MV is a proper subspace of V/MV. Hence $fln(\mathrm{V}) = \infty$. □

Theorem 3.1.10. *Let R be local ring with maximal ideal M such that R/M is a field of cardinality q. Suppose that* $\mathrm{V} = R^{(\mathbb{N})}$, *then* $fln(\mathrm{V}) \leq q+2$.

Proof. Since R is a commutative Noetherian ring, M is finitely generated.
Let $M = \langle m_1, m_2, \ldots, m_k \rangle$ and let $(R/M)^* = \{u_1 + M, u_2 + M, \ldots, u_{q-1} + M\}$; i.e.,$u_1, u_2, \ldots, u_{q-1}$ is a system of representatives for $(R/M)^*$.
Define $S^{'} = \langle m_1e_1, \ldots, m_ke_1, e_2, e_3, \ldots \rangle$, $S^{''} = \langle e_1, m_1e_2, \ldots, m_ke_2, e_3, \ldots \rangle$ and $S_i = \langle e_1 + u_ie_2, m_1e_2, \ldots, m_ke_2, e_3, \ldots \rangle$ for $i = 1, 2, \ldots, q-1$.
Thus $\mathrm{V} = S^{'} \cup S^{''} \cup (\cup_{i=1}^{q-1} S_i)$.
Let $S = \langle e_1, e_2 \rangle$ and consider $\Im = \{S^{'}, S^{''}, S_1, \ldots, S_{q-1}, S\}$. Then all elements in $\Im$ are distinct, so $|\Im| = q+2$.
Suppose that $f \in \mathrm{M}_R(\mathrm{V})$ is linear on each submodule in $\Im$ and let $v = a_1e_1 + a_2e_2 + \cdots + a_me_m$ be arbitrary in V.

If a_1 or $a_2 \in M$, then $v \in S^{'}$ or $S^{''}$ and $f(a_1e_1 + a_2e_2 + \cdots + a_me_m) = f(a_1e_1) + f(a_2e_2) + \cdots + f(a_me_m)$.

If both $a_1, a_2 \in R \backslash M$, then $v = a_1(e_1 + a_1^{-1}a_2e_2) + a_3e_3 + \cdots + a_me_m$.
Since $a_1, a_2 \notin M$, then $a_1^{-1}a_2 \notin M$, so there exist u_l with $1 \leq l \leq q-1$ such that $a_1^{-1}a_2 + M = u_l + M$, this means that $(a_1^{-1}a_2 - u_l) \in M$, then $a_1^{-1}a_2 - u_l = \hat{m}$ where $\hat{m} \in M$, i.e., $a_1^{-1}a_2 = u_l + \hat{m},\ \hat{m} \in M$. From $\hat{m} = r_1m_1 + r_2m_2 + \cdots + r_km_k$ where $r_i \in R$, we get $a_1\hat{m}e_2 = a_1r_1m_1e_2 + a_1r_2m_2e_2 + \cdots + a_1r_km_ke_2$.
Thus $v = a_1(e_1 + u_le_2) + a_1\hat{m}e_2 + a_3e_3 + \cdots + a_me_m \in S_l$. Since f is linear on S_l,

$$\begin{aligned} f(a_1e_1 + a_2e_2 + \cdots + a_me_m) &= f(a_1(e_1 + u_le_2) + a_1\hat{m}e_2 + a_3e_3 + \cdots + a_me_m) \\ &= f(a_1e_1 + a_1u_le_2) + f(a_1\hat{m}e_2) + f(a_3e_3) + \cdots + f(a_me_m). \end{aligned}$$

Since f is linear on S, then $f(a_1e_1 + a_1u_le_2) = f(a_1e_1) + f(a_1u_le_2)$.
Hence

$$\begin{aligned} f(v) &= f(a_1e_1) + f(a_1u_le_2) + f(a_1\hat{m}e_2) + f(a_3e_3) + \cdots + f(a_me_m) \\ &= f(a_1e_1) + f([a_1u_l + a_1\hat{m}]e_2) + f(a_3e_3) + \cdots + f(a_me_m) \quad \text{since } f \text{ is linear on } S^{'} \\ &= f(a_1e_1) + f(a_2e_2) + \cdots + f(a_me_m). \end{aligned}$$

Thus from Lemma 3.1.2., we get f is linear on V and $\Im$ forces linearity on V. $\square$

3.2 General properties of $\mathbb{Z}_{p^k}^{(\mathbb{N})}$

In this section we want to find the forcing linearity numbers of a free module $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$, over a local ring $\mathbb{Z}_{p^k}$. It is easy to see that $\mathbb{Z}_{p^k}$ is a commutative local ring with identity, not an integral domain such that $\langle \bar{p} \rangle$ is a unique maximal ideal of $\mathbb{Z}_{p^k}$. Moreover $\langle \bar{p} \rangle$ consisting of all nonunits of $\mathbb{Z}_{p^k}$.

Let $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$ and u , v be fixed positive integers with $u < v$.

We let $\mathrm{M}_i = \langle e_1, e_2, \ldots, e_{i-1}, pe_i, e_{i+1}, \ldots \rangle$ and

$\mathrm{K}_{uv}(t) = \langle e_1, e_2, ..., e_{u-1}, e_u + te_v, e_{u+1}, ..., e_{v-1}, pe_v, e_{v+1}, ... \rangle$ where $1 \leq t \leq p-1$

$= \{(\bar{x_1}, \bar{x_2}, \bar{x_3}, ...) \in \mathrm{V} | (\bar{x_u}, \bar{x_v}) \in \langle (\bar{1}, \bar{t}) \rangle \oplus \langle (\bar{0}, \bar{p}) \rangle, 1 \leq t \leq p-1\}$.

Lemma 3.2.1. $Ann(\langle \bar{p} \rangle) = \{\bar{0}, \overline{p^{k-1}}, 2\overline{p^{k-1}}, ..., (p-1)\overline{p^{k-1}}\}$.

Proof. Let $\bar{a} \in Ann(\langle \bar{p} \rangle)$. Then $ax \equiv 0 (\mathrm{mod}\ p^k)$ for all $\bar{x} \in \langle \bar{p} \rangle$.

By taking $x = p$, we get $p^k | ap$ which implies $p^{k-1} | a$.

Thus $a = mp^{k-1}$ for some $m \in \mathbb{Z}$.

Since $0 \leq a < p^k$, then $0 \leq mp^{k-1} < p^k$, so $0 \leq m < p$.

Therefore $a \in \{\bar{0}, \overline{p^{k-1}}, 2\overline{p^{k-1}}, \ldots, (p-1)\overline{p^{k-1}}\}$.

Clearly $\{\bar{0}, \overline{p^{k-1}}, 2\overline{p^{k-1}}, \ldots, (p-1)\overline{p^{k-1}}\} \subseteq Ann(\langle \bar{p} \rangle)$.

Hence $Ann(\langle \bar{p} \rangle) = \{\bar{0}, \overline{p^{k-1}}, 2\overline{p^{k-1}}, \ldots, (p-1)\overline{p^{k-1}}\}$. □

Lemma 3.2.2. *If* $\Im = \{\mathrm{S}_1, \mathrm{S}_2, ..., \mathrm{S}_t\}$ *is a collection of proper submodules of* $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$ *which forces linearity, then* $\cup_{i=1}^{t} \mathrm{S}_i = \mathrm{V}$.

Proof. Let $\Im = \{\mathrm{S}_1, \mathrm{S}_2, ..., \mathrm{S}_t\}$ be a collection of proper submodules of $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$ which forces linearity.

Since $\mathbb{Z}_{p^k}$ is a finite local ring which has $\langle \bar{p} \rangle$ as a unique maximal ideal of $\mathbb{Z}_{p^k}$ with $Ann(\langle \bar{p} \rangle) \neq \bar{0}$, then by Proposition 3.1.6. we get $\cup_{i=1}^{t} \mathrm{S}_i = \mathrm{V}$. □

Theorem 3.2.3.(First Sylow Theorem) *Let G be a group of order $p^n m$, with $n \geq 1$, p prime, and* $\gcd(p, m) = 1$. *Then G contains a subgroup of order p^i for each $1 \leq i \leq n$ and every subgroup of G of order $p^i (i < n)$ is normal in some subgroup* *of order* p^{i+1}.

Proof. See [3] page 94. □

Lemma 3.2.4. *If* $\text{V} = \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ *and* M *is a maximal submodule of* V*, then* $|M| = p^{2k-1}$.

Proof. Let M be a maximal submodule of V. Then M is a subgroup of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$. Suppose $|M| < p^{2k-1}$. By First Sylow Theorem, M is contained in some subgroup K of order p^{2k-1}. We prove that K is a submodule of V.

Let $\bar{s} \in \mathbb{Z}_{p^k}$ and $x \in K$.

If $\bar{s} = \bar{0}$, $\bar{s}x = \bar{0}x = 0 \in K$.

If $\bar{1} \leq \bar{s} < \overline{p^k}$, then $\bar{s}x = (\underbrace{\bar{1} + \bar{1} + \cdots + \bar{1}}_{s\ times})x = \underbrace{x + x + \cdots + x}_{s\ times}$.

Because $x \in K$ and K is a group, $\underbrace{x + x + \cdots + x}_{s\ times} \in K$. Thus $\bar{s}x \in K$ and get K is a submodule of V.

Thus $M \subsetneqq K \subsetneqq \text{V}$, which contradicts to the maximality of V. □

Proposition 3.2.5. (i) *If* $\bar{x} \in \mathbb{Z}_{p^k}$ *and* $s \in \mathbb{Z}$*, then* $s\bar{x} = \overline{sx}$.

(ii) $\mathbb{Z}_{p^k}(\bar{x}, \bar{y}) = \mathbb{Z}(\bar{x}, \bar{y})$ *for all* $\bar{x}, \bar{y} \in \mathbb{Z}_{p^k}$.

Proof. (i) Let $\bar{x} \in \mathbb{Z}_{p^k}$ and $s \in \mathbb{Z}$, we consider in three cases:

Case $s = 0$: We get $0\bar{x} = \bar{0} = \bar{0}\bar{x} = \overline{0x}$.

Case $s > 0$: Then $s\bar{x} = \underbrace{\bar{x} + \bar{x} \cdots + \bar{x}}_{s\ times} = \overline{\underbrace{x + x + \cdots + x}_{s\ times}} = \overline{sx}$.

Case $s < 0$: Then $-s > 0$ and $s\bar{x} = -(-s)\bar{x} = -(-s\bar{x}) = -(\overline{-sx}) = \overline{sx}$ since $\overline{sx}$ is an inverse of $\overline{-sx}$.

Thus by three cases we get $s\bar{x} = \overline{sx}$ for $\bar{x} \in \mathbb{Z}_{p^k}, s \in \mathbb{Z}$.

(ii) ($\Rightarrow$) Let $a \in \mathbb{Z}_{p^k}(\bar{x}, \bar{y})$. Then $a = \bar{s}(\bar{x}, \bar{y})$ for some $0 \leq s < p^k$. Since $a = \bar{s}(\bar{x}, \bar{y}) = (\bar{s}\bar{x}, \bar{s}\bar{y}) = (\overline{sx}, \overline{sy}) = (s\bar{x}, s\bar{y}) = s(\bar{x}, \bar{y})$ where $s \in \mathbb{Z}$ by (i), we get $a \in \mathbb{Z}(\bar{x}, \bar{y})$.

Thus $\mathbb{Z}_{p^k}(\bar{x}, \bar{y}) \subseteq \mathbb{Z}(\bar{x}, \bar{y})$.

($\Leftarrow$) Let $b \in \mathbb{Z}(\bar{x}, \bar{y})$. Then $b = r(\bar{x}, \bar{y})$ for some $r \in \mathbb{Z}$.

Since $b = r(\bar{x}, \bar{y}) = (r\bar{x}, r\bar{y}) = (\overline{rx}, \overline{ry}) = (\bar{r}\bar{x}, \bar{r}\bar{y}) = \bar{r}(\bar{x}, \bar{y})$ where $\bar{r} \in \mathbb{Z}_{p^k}$ by (i), we get $a \in \mathbb{Z}_{p^k}(\bar{x}, \bar{y})$.

Thus $\mathbb{Z}(\bar{x}, \bar{y}) \subseteq \mathbb{Z}_{p^k}(\bar{x}, \bar{y})$.

Hence $\mathbb{Z}_{p^k}(\bar{x}, \bar{y}) = \mathbb{Z}(\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in \mathbb{Z}_{p^k}$. □

Lemma 3.2.6. *If* V *is a finitely generated abelian group generated by* n *elements, then every subgroup* M *of* V *can be generated by* m *elements with* $m \leq n$.

Proof. See[3] page 74. □

Proposition 3.2.7. *Let* $\mathrm{V} = \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ *and* M *a maximal submodule of* V *over* $\mathbb{Z}_{p^k}$. *Then there exist* $(\bar{1}, \bar{g})$ *or* $(\overline{g'}, \bar{1}) \in M$ *for some* $g, g' \in \mathbb{Z}_{p^k}$.

Proof. Suppose $(\bar{1}, \bar{g})$ and $(\overline{g'}, \bar{1}) \notin M$ for all g and $g' \in \mathbb{Z}_{p^k}$.

If there is an element $(\bar{x}, \bar{y}) \in M$ such that $|(\bar{x}, \bar{y})| = p^k$. Then $\gcd(x, p) = 1$ or $\gcd(y, p) = 1$

If $\gcd(x, p) = 1$, then there is $\bar{z} \in \mathbb{Z}_{p^k}$ such that $\bar{z}(\bar{x}, \bar{y}) = (\bar{z}\bar{x}, \bar{z}\bar{y}) = (\bar{1}, \bar{z}\bar{y}) \in M$, which is a contradiction.

If $\gcd(y, p) = 1$, then there is $\bar{s} \in \mathbb{Z}_{p^k}$ such that $\bar{s}(\bar{x}, \bar{y}) = (\bar{s}\bar{x}, \bar{s}\bar{y}) = (\bar{s}\bar{x}, \bar{1}) \in M$, which is a contradiction.

Thus every element $(\bar{x}, \bar{y}) \in M$, $|(\bar{x}, \bar{y})| < p^k$.

Since V is generated by two elements, M can be generated by at most two elements by Lemma 3.2.6, that is $M = \mathbb{Z}(\bar{x}, \bar{y}) + \mathbb{Z}(\bar{u}, \bar{v})$ for some $(\bar{x}, \bar{y}), (\bar{u}, \bar{v}) \in M$.

Thus $|M| < p^{2k-1}$.

By Proposition 3.2.5, we have $\mathbb{Z}(\bar{x}, \bar{y}) + \mathbb{Z}(\bar{u}, \bar{v}) = \mathbb{Z}_{p^k}(\bar{x}, \bar{y}) + \mathbb{Z}_{p^k}(\bar{u}, \bar{v})$.

Thus we get $M = \mathbb{Z}_{p^k}(\bar{x}, \bar{y}) + \mathbb{Z}_{p^k}(\bar{u}, \bar{v})$. And that $|M| < p^{2k-1}$.

But M is a maximal submodule of V, so $|M| = p^{2k-1}$, which is a contradiction.

Therefore, there exists $(\bar{1}, \bar{g})$ or $(\overline{g'}, \bar{1}) \in M$. □

Lemma 3.2.8. *Let* $(\bar{0},\bar{p}),(\bar{p},\bar{0}) \in \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$*, then*

$$(1)\ \mathbb{Z}_{p^k}(\bar{0},\bar{p}) = \{(\bar{0},\bar{x}) \mid x = up\ ;\ \ where\ \ 0 \le u \le p^{k-1}-1\}$$

and $$(2)\ \mathbb{Z}_{p^k}(\bar{p},\bar{0}) = \{(\bar{y},\bar{0}) \mid y = vp\ ;\ \ where\ \ 0 \le v \le p^{k-1}-1\}.$$

Moreover $|\mathbb{Z}_{p^k}(\bar{0},\bar{p})| = p^{k-1} = |\mathbb{Z}_{p^k}(\bar{p},\bar{0})|$.

Proof. (1) $(\subseteq)$ Let $a \in \mathbb{Z}_{p^k}(\bar{0},\bar{p})$, then $a = \bar{t}(\bar{0},\bar{p}) = (\bar{0},\overline{tp})$ where $0 \le t < p^k$.

If $0 \le t \le p^{k-1}-1$, we get $a \in \{(\bar{0},\bar{x}) \mid x = up\ ;\ \text{where}\ \ 0 \le u \le p^{k-1}-1\}$.

If $t > p^{k-1}-1$, then by division algorithm there exists $q,s \in \mathbb{Z}$ such that $t = p^{k-1}q + s$ where $0 \le s < p^{k-1}$.

Then $\bar{t}\bar{p} = \bar{p}\bar{t} = \bar{p}(\overline{p^{k-1}}\bar{q} + \bar{s}) = \overline{p^k q} + \overline{ps} = \bar{0} + \overline{ps} = \overline{ps} = \overline{sp}$.

Since $0 \le s < p^{k-1}$, this means that $0 \le s \le p^{k-1}-1$.

Then $a = (\bar{0},\overline{tp}) \in \{(\bar{0},\bar{x}) \mid x = up\ ;\ \text{where}\ \ 0 \le u \le p^{k-1}-1\}$.

Thus $\mathbb{Z}_{p^k}(\bar{0},\bar{p}) \subseteq \{(\bar{0},\bar{x}) \mid x = up\ ;\ \text{where}\ \ 0 \le u \le p^{k-1}-1\}$.

$(\supseteq)$ Let $a \in \{(\bar{0},\bar{x}) \mid x = up\ ;\ \text{where}\ \ 0 \le u \le p^{k-1}-1\}$.

Then there exists $0 \le b \le p^{k-1}-1$ such that $a = (\bar{0},\overline{bp}) = \bar{b}(\bar{0},\bar{p})$.

Since $0 \le b \le p^{k-1}-1$, then $\bar{b} \in \mathbb{Z}_{p^k}$.

Thus $a \in \mathbb{Z}_{p^k}(\bar{0},\bar{p})$ and $\{(\bar{0},\bar{x}) \mid x = up\ ;\ \text{where}\ \ 0 \le u \le p^{k-1}-1\} \subseteq \mathbb{Z}_{p^k}(\bar{0},\bar{p})$.

Thus we get $\mathbb{Z}_{p^k}(\bar{0},\bar{p}) = \{(\bar{0},\bar{x}) \mid x = up\ ;\ \text{where}\ \ 0 \le u \le p^{k-1}-1\}$.

Next, we show that $|\mathbb{Z}_{p^k}(\bar{0},\bar{p})| = p^{k-1}$.

Let $(\bar{0},\bar{x_1}) = (\bar{0},\bar{x_2})$, then there is $0 \le y_1, y_2 \le p^{k-1}-1$ such that $x_1 = y_1 p$ and $x_2 = y_2 p$.

Suppose that $y_1 \ge y_2$. Then $0 \le y_1 - y_2 \le p^{k-1}-1$.

Thus $(\bar{0},\overline{y_1 p}) = (\bar{0},\overline{y_2 p})$, so $\overline{y_1 p} = \overline{y_2 p}$ and $\bar{0} = \overline{y_1 p} - \overline{y_2 p} = (\bar{y_1} - \bar{y_2})p$.

Hence $\bar{y_1} - \bar{y_2} = \bar{0}$ since $0 \le y_1 - y_2 \le p^{k-1}-1$.

Thus $y_1 = y_2$.

Hence $\{(\bar{0},\bar{x}) \mid x = up\ ;\ \text{where}\ \ 0 \le u \le p^{k-1}-1\}$ has order p^{k-1}.

This means that $|\mathbb{Z}_{p^k}(\bar{0},\bar{p})| = p^{k-1}$.

(2) By using the same proof as given in (1), we get

$\mathbb{Z}_{p^k}(\bar{p},\bar{0}) = \{(\bar{y},\bar{0}) \mid y = vp\ ;\ \text{where}\ \ 0 \le v \le p^{k-1}-1\}$ and $|\mathbb{Z}_{p^k}(\bar{p},\bar{0})| = p^{k-1}$. □

Lemma 3.2.9. *Let* $\mathrm{V} = \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ *and* M *a maximal submodule of* V*. Then* M *contains elements* x, y *such that* $|x| = p^k$ *,* $|y| = p^{k-1}$ *and* $M = \langle x\rangle \oplus \langle y\rangle$*.*

Proof. Let M be a maximal submodule of V.

By Proposition 3.2.7. we get $(\bar{1}, \bar{g})$ or $(\bar{g'}, \bar{1}) \in M$ where $0 \le g, g' < p^k$.

We consider in two cases.

Case i: $(\bar{1}, \bar{g}) \in M$ where $0 \le g < p^k$.

We defined $f : M \to M$ by

$$f((\bar{x}, \bar{y})) = (\bar{x}, \overline{xg})$$

Let $(\bar{x_1}, \bar{y_1}), (\bar{x_2}, \bar{y_2}) \in M$ and $(\bar{x_1}, \bar{y_1}) = (\bar{x_2}, \bar{y_2})$.

So $\bar{x_1} = \bar{x_2}, \bar{y_1} = \bar{y_2}$ and $\overline{x_1 g} = \overline{x_2 g}$.

We get, $f((\bar{x_1}, \bar{y_1})) = (\bar{x_1}, \overline{x_1 g}) = (\bar{x_2}, \overline{x_2 g}) = f((\bar{x_2}, \bar{y_2}))$.

Thus f is well-defined.

Let $K = \{\bar{y} \mid (\bar{0}, \bar{y}) \in Ker f \text{ and } 0 \le y < p^k\}$ and $K' = \{y \mid \bar{y} \in K\}$.

Thus $K' \subseteq \mathbb{N} \cup \{0\}$.

Let d be the smallest positive element in K'.

We claim that $Ker f = \{(\bar{0}, \bar{y}) | \bar{y} \in \mathbb{Z}_{p^k}\} \cap M = \mathbb{Z}_{p^k}(\bar{0}, \bar{d})$.

Since $f((\bar{0}, \bar{y})) = (\bar{0}, \bar{0}\bar{g}) = (\bar{0}, \bar{0})$, it follows that $\{(\bar{0}, \bar{y}) | \bar{y} \in \mathbb{Z}_{p^k}\} \cap M \subseteq Ker f$.

Let $(\bar{a}, \bar{b}) \in Ker f$.

Thus $(\bar{a}, \overline{ag}) = f((\bar{a}, \bar{b})) = (\bar{0}, \bar{0})$, so $\bar{a} = \bar{0}$ and

$(\bar{a}, \bar{b}) = (\bar{0}, \bar{b}) \in \{(\bar{0}, \bar{y}) | \bar{y} \in \mathbb{Z}_{p^k}\} \cap M$.

Thus $Ker f = \{(\bar{0}, \bar{y}) | \bar{y} \in \mathbb{Z}_{p^k}\} \cap M$.

Let $a \in \mathbb{Z}_{p^k}(\bar{0}, \bar{d})$.

Then $a = e(\bar{0}, \bar{d}) = (\bar{0}, \overline{ed})$ where $e \in \mathbb{Z}_{p^k}$, so $f(a) = f(\bar{0}, \overline{ed}) = (\bar{0}, \overline{0g}) = (\bar{0}, \bar{0})$.

Thus $a \in Ker f$ and $\mathbb{Z}_{p^k}(\bar{0}, \bar{d}) \subseteq Ker f$.

Let $(\bar{0}, \bar{m}) \in Ker f$.

Consider m and d, by division algorithm we get $m = xd + r$ for some $x \in \mathbb{Z}$, $0 \le r < d$.

Since $(\bar{0}, \overline{xd})$ and $(\bar{0}, \bar{m}) \in Ker f$, then $(\bar{0}, \bar{r}) \in Ker f$.

If $r \ne 0$, then $r \in K'$ and $r < d$ which is a contradiction.

This implies that $\bar{m} = \overline{xd} \in \mathbb{Z}_{p^k}$ and so $(\bar{0}, \bar{m}) = x(\bar{0}, \bar{d}) \in \mathbb{Z}(\bar{0}, \bar{d})$.

But by Proposition 3.2.5. $\mathbb{Z}(\bar{0},\bar{d}) = \mathbb{Z}_{p^k}(\bar{0},\bar{d})$, so $(\bar{0},\bar{m}) \in \mathbb{Z}_{p^k}(\bar{0},\bar{d})$.

Thus $Kerf \subseteq \mathbb{Z}_{p^k}(\bar{0},\bar{d})$.

Therefor $Kerf = \{(\bar{0},\bar{y}) | \bar{y} \in \mathbb{Z}_{p^k}\} \cap M = \mathbb{Z}_{p^k}(\bar{0},\bar{d})$ where d is the smallest positive element in $K^{'}$.

Next, we show that f is an idempotent endomorphism of M.

Let $(\bar{x},\bar{y}) \in M$. We get

$ff((\bar{x},\bar{y})) = f((\bar{x},\overline{xg})) = (\bar{x},\overline{xg}) = f((\bar{x},\bar{y}))$.

Thus f is an idempotent.

Let $\bar{n} \in \mathbb{Z}_{p^k}$, $(\bar{x},\bar{y}) \in M$. We have,

$\bar{n}f((\bar{x},\bar{y})) = \bar{n}(\bar{x},\overline{xg}) = (\overline{nx},\overline{nxg})$ and

$f(\bar{n}(\bar{x},\bar{y})) = f((\overline{nx},\overline{ny})) = (\overline{nx},\overline{nxg})$.

Thus $f \in M_{\mathbb{Z}_{p^k}}(M)$.

Next, we show that $f \in End_{\mathbb{Z}_{p^k}}(M)$.

Let $(\bar{x_1},\bar{y_1}),(\bar{x_2},\bar{y_2}) \in M$. We get,

$f((\bar{x_1},\bar{y_1}) + (\bar{x_2},\bar{y_2})) = f((\bar{x_1}+\bar{x_2},\bar{y_1}+\bar{y_2})) = (\overline{x_1+x_2},\overline{(x_1+x_2)g})$ and

$f((\bar{x_1},\bar{y_1})) + f((\bar{x_2},\bar{y_2})) = (\bar{x_1}+\overline{x_1g}) + (\bar{x_2}+\overline{x_2g})) = (\overline{x_1+x_2},\overline{(x_1+x_2)g})$.

Thus $f((\bar{x_1},\bar{y_1}) + (\bar{x_2},\bar{y_2})) = f((\bar{x_1},\bar{y_1})) + f((\bar{x_2},\bar{y_2}))$ and so $f \in End_{\mathbb{Z}_{p^k}}(M)$.

Now, we show that $Imf = \mathbb{Z}_{p^k}(\bar{1},\bar{g})$.

By the definition of f we see that Imf is a submodule of $\mathbb{Z}_{p^k}(\bar{1},\bar{g})$.

Let $(\bar{a},\bar{b}) \in \mathbb{Z}_{p^k}(\bar{1},\bar{g})$, so $(\bar{a},\bar{b}) = \bar{m}(\bar{1},\bar{g})$ for some $\bar{m} \in \mathbb{Z}_{p^k}$.

And $f((\bar{m},\overline{mg})) = (\bar{m},\overline{mg}) = (\bar{a},\bar{b})$, then $(\bar{a},\bar{b}) \in Imf$.

Thus $Imf = \mathbb{Z}_{p^k}(\bar{1},\bar{g})$.

By Lemma 2.3.3, $M = \mathbb{Z}_{p^k}(\bar{1},\bar{g}) \oplus \mathbb{Z}_{p^k}(\bar{0},\bar{d}) = \langle(\bar{1},\bar{g})\rangle \oplus \langle(\bar{0},\bar{d})\rangle$ where $0 \le g < p^k$.

Since $|M| = p^{2k-1}$ and $\gcd(1,p) = 1$, then $|(\bar{1},\bar{g})| = p^k$ and $|(\bar{0},\bar{d})| = p^{k-1}$.

Case ii: $(\bar{g^{'}},\bar{1}) \in M$ where $0 \le g^{'} < p^k$.

We defined $f : M \to M$ by

$$f((\bar{x},\bar{y})) = (\overline{yg^{'}},\bar{y})$$

Let $L = \{\bar{x} \mid (\bar{x},\bar{0}) \in Kerf \ and \ 0 \le x < p^k\}$ and $L^{'} = \{x \mid \bar{x} \in K\}$.

Let c be the smallest positive element in $L^{'}$.

Then f is an idempotent endomorphism with

$Kerf = \{(\bar{x}, \bar{0}) | \bar{x} \in \mathbb{Z}_{p^k}\} \cap M = \mathbb{Z}_{p^k}(\bar{c}, \bar{0})$ where c is the smallest positive element in L'.

By the same prove as we give in case i, we get

$M = \mathbb{Z}_{p^k}(\bar{g'}, \bar{1}) \oplus \mathbb{Z}_{p^k}(\bar{c}, \bar{0}) = \langle(\bar{g'}, \bar{1})\rangle \oplus \langle(\bar{c}, \bar{0})\rangle$ where $0 \leq g' < p^k$.

Since $|M| = p^{2k-1}$ and $\gcd(1, p) = 1$, then $|(\bar{g'}, \bar{1})| = p^k$ and $|(\bar{c}, \bar{0})| = p^{k-1}$. □

By Lemma 3.2.8. and in case i of Lemma 3.2.9, we get $\mathbb{Z}_{p^k}(\bar{0}, \bar{p}) \subseteq Kerf = \mathbb{Z}_{p^k}(\bar{0}, \bar{d})$, so $\mathbb{Z}_{p^k}(\bar{0}, \bar{p}) = \mathbb{Z}_{p^k}(\bar{0}, \bar{d})$ since they have the same order p^{k-1}.

Similarly, we get $\mathbb{Z}_{p^k}(\bar{p}, \bar{0}) = \mathbb{Z}_{p^k}(\bar{c}, \bar{0})$.

Lemma 3.2.10. *If* $\mathrm{V} = \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ *is a module over the ring* $\mathbb{Z}_{p^k}$*, then all distinct maximal submodules of* V *are of the forms* $\langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ *where* $0 \leq t \leq p-1$ *and* $\langle(\bar{0}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$.

Proof. Let M be a maximal submodule of V. Thus $|M| = p^{2k-1}$.

Then $M = \langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ or $\langle(\bar{t'}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$ where $\bar{t}, \bar{t'} \in \mathbb{Z}_{p^k}$.

We claim that $\langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle = \langle(\bar{1}, \bar{s})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ where $0 \leq s \leq p-1$.

Consider $\langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$, we have

$t = ap + s$ for some $a \in \mathbb{Z}, 0 \leq s \leq p-1$ by division algorithm.

$(\bar{1}, \bar{t}) = (\bar{1}, \overline{ap+s}) = (\bar{1}, \bar{s}) + \bar{a}(\bar{0}, \bar{p}) \in \langle(\bar{1}, \bar{s})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$. So $\langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle \subseteq \langle(\bar{1}, \bar{s})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$.

Since $\langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ and $\langle(\bar{1}, \bar{s})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ have order p^{2k-1},

thus $\langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle = \langle(\bar{1}, \bar{s})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ where $0 \leq s \leq p-1$.

For the case $M = \langle(\bar{t'}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$,

we claim that $\langle(\bar{t'}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle = \langle(\bar{r}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$ where $0 \leq r \leq p-1$.

Consider $\langle(\bar{t'}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$, we have

$t' = bp + r$ for some $b \in \mathbb{Z}, 0 \leq r \leq p-1$ by division algorithm.

$(\bar{t'}, \bar{1}) = (\overline{bp+r}, \bar{1}) = (\bar{r}, \bar{1}) + \bar{b}(\bar{p}, \bar{0}) \in \langle(\bar{r}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$. Hence $\langle(\bar{t'}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle \subseteq \langle(\bar{r}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$.

Since $\langle(\bar{t'}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$ and $\langle(\bar{r}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$ have order p^{2k-1},

thus $\langle(\bar{t'}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle = \langle(\bar{r}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$ where $0 \leq r \leq p-1$.

Consider $\langle(\bar{r},\bar{1})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$ where $0 \leq r \leq p-1$.

If $r = 0$, then $\langle(\bar{r},\bar{1})\rangle \oplus \langle(\bar{p},\bar{0})\rangle = \langle(\bar{0},\bar{1})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$.

If $r \neq 0$, then $\gcd(r,p) = 1$, there exists $\bar{c} \in \mathbb{Z}_{p^k}$ such that $\bar{c}\bar{r} = \bar{1} \in \mathbb{Z}_{p^k}$ and $(\bar{r},\bar{1}) = \bar{r}(\bar{1},\bar{c}) \in \langle(\bar{1},\bar{c})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$.

Since $\langle(\bar{r},\bar{1})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$ and $\langle(\bar{1},\bar{c})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$ have order p^{2k-1},

thus $\langle(\bar{r},\bar{1})\rangle \oplus \langle(\bar{p},\bar{0})\rangle = \langle(\bar{1},\bar{c})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$.

Next, consider $\langle(\bar{1},\bar{c})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$, we get

$c = dp + g$ for some $d \in \mathbb{Z}, 0 \leq g \leq p-1$.

Then $(\bar{1},\bar{c}) = (\bar{1},\overline{dp+g}) = (\bar{1},\bar{g}) + \bar{d}(\bar{0},\bar{p})$ and we have $(\bar{p},\bar{0}) = \bar{p}(\bar{1},\bar{g}) + (\overline{-g})(\bar{0},\bar{p})$.

Since $\langle(\bar{1},\bar{c})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$ and $\langle(\bar{1},\bar{g})\rangle \oplus \langle(\bar{0},\bar{p})\rangle$ have order p^{2k-1},

we have $\langle(\bar{1},\bar{c})\rangle \oplus \langle(\bar{p},\bar{0})\rangle = \langle(\bar{1},\bar{g})\rangle \oplus \langle(\bar{0},\bar{p})\rangle$ where $0 \leq g \leq p-1$.

Hence maximal submodules of V are of the form

$\langle(\bar{1},\bar{t})\rangle \oplus \langle(\bar{0},\bar{p})\rangle$ where $0 \leq t \leq p-1$

and $\langle(\bar{0},\bar{1})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$. □

Note that if $\mathrm{V} = \mathbb{Z}_p \oplus \mathbb{Z}_p$, then M are of the forms $\langle(\bar{1},\bar{t})\rangle \oplus \langle(\bar{0},\bar{0})\rangle$ where $0 \leq t \leq p-1$ or $\langle(\bar{0},\bar{1})\rangle \oplus \langle(\bar{0},\bar{0})\rangle$. □

Lemma 3.2.11. *For each $i,u,v \in \mathbb{N}$, M_i and $\mathrm{K}_{uv}(t)$ where $1 \leq t \leq p-1$ are maximal submodules of $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$.*

Proof. Let $\pi_j : \mathbb{Z}_{p^k}^{(\mathbb{N})} \to \mathbb{Z}_{p^k}$ be the projection map.

Let $\mathrm{M}_i \subseteq G \subseteq \mathrm{V}$ with $G \neq \mathrm{M}_i$.

We show that $G = \mathrm{V}$.

Since $\pi_j(\mathrm{M}_i) = \mathbb{Z}_{p^k}$ for all $j \neq i$, then $\pi_j(G) = \mathbb{Z}_{p^k}$ for all $j \neq i$ since $\mathrm{M}_i \subseteq G$.

At the component i we get $\pi_j(M_i) = \langle\bar{p}\rangle$. But $\langle\bar{p}\rangle = \pi_j(\mathrm{M}_i) \subsetneqq \pi_j(G)$, then $\pi_j(G) = \mathbb{Z}_{p^k}$ since $\langle\bar{p}\rangle$ is maximal ideal of $\mathbb{Z}_{p^k}$.

Thus we get $\pi_j(G) = \mathbb{Z}_{p^k}$ for all $j \in \mathbb{N}$, then $G = \mathrm{V}$.

Therefore M_i is a maximal submodule of V.

Let $\mathrm{K}_{uv}(t) \subseteq S \subseteq \mathrm{V}$ with $S \neq \mathrm{V}, 1 \leq t \leq p-1$.

Next, we show that $S = \mathrm{K}_{uv}(t)$

Since $K_{uv}(t) = \{(\bar{x_1}, \bar{x_2}, \bar{x_3}, \dots) | (\bar{x_u}, \bar{x_v}) \in \langle(\bar{1}, \bar{t}) \oplus (\bar{0}, \bar{p})\rangle\}$.

Then $\pi_j(K_{uv}(t)) = \mathbb{Z}_{p^k}$ for all $j \neq u, v$.

Thus we get $\pi_j(S) = \mathbb{Z}_{p^k}$ for all $j \neq u, v$ since $K_{uv}(t) \subseteq S$.

For each $(\bar{s_k}) \in S$.

Consider at components u, v of $(\bar{s_k}) \in S$.

If $(\bar{s_u}, \bar{s_v}) \notin \langle(\bar{1}, \bar{t}) \oplus (\bar{0}, \bar{p})\rangle$, then we get $\langle(\bar{1}, \bar{t}) \oplus (\bar{0}, \bar{p})\rangle + \langle(\bar{s_u}, \bar{s_v})\rangle \subseteq \pi_u(S) \oplus \pi_v(S)$ since $K_{uv}(t) \subseteq S$.

By the maximality of $\langle(\bar{1}, \bar{t}) \oplus (\bar{0}, \bar{p})\rangle$, we get $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k} = \langle(\bar{1}, \bar{t}) \oplus (\bar{0}, \bar{p})\rangle + \langle(\bar{s_u}, \bar{s_v})\rangle \subseteq \pi_u(S) \oplus \pi_v(S)$, then $\pi_u(S) \oplus \pi_v(S) = \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$, so $S = V$, a contradiction.

Then $(\bar{s_u}, \bar{s_v}) \in \langle(\bar{1}, \bar{t}) \oplus (0, p)\rangle$, $S \subseteq K_{uv}(t)$.

Thus $K_{uv}(t)$ where $1 \leq t \leq p - 1$ are maximal submodules of V. □

Proposition 3.2.12. *Let* $V = \mathbb{Z}_{p^k}^{(\mathbb{N})}$. *Then maximal submodules of* V *are of the form* M_i *or* $K_{uv}(t)$ *where* $1 \leq t \leq p - 1$.

Proof. Let W be a maximal submodule of V.

For each $u, v \in \mathbb{N}$, let

$W_{uv} = \{(\bar{x_u}, \bar{x_v}) \in \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k} |$ there is $(\bar{a_i}) \in W$ such that $\bar{x_u} = \bar{a_u}$ and $\bar{x_v} = \bar{a_v}\}$.

We prove that W_{uv} is a submodule of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$.

Let $(\bar{b_u}, \bar{b_v}), (\bar{c_u}, \bar{c_v}) \in W_{uv}$, then there exists $(-\bar{c_u}, -\bar{c_v}) \in W_{uv}$ such that

$\bar{c_u} + (-\bar{c_u}) = \bar{0}$ and $\bar{c_v} + (-\bar{c_v}) = \bar{0}$ since W is subgroup of V.

Then $(\bar{b_u}, \bar{b_v}) + (-\bar{c_u}, -\bar{c_v}) = (\bar{b_u} - \bar{c_u}, \bar{b_v} - \bar{c_v}) \in W_{uv}$ because W is subgroup of V.

Hence W_{uv} is a subgroup of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$.

Let $\bar{s} \in \mathbb{Z}_{p^k}$ and $(\bar{x_u}, \bar{x_v}) \in W_{uv}$.

If $\bar{s} = \bar{0}$, $\bar{s}(\bar{x_u}, \bar{x_v}) = \bar{0}(\bar{x_u}, \bar{x_v}) = (\bar{0}, \bar{0}) \in W_{uv}$.

If $\bar{1} \leq \bar{s} < \overline{p^k}$, then $\bar{s}(\bar{x_u}, \bar{x_v}) = s(\bar{x_u}, \bar{x_v}) = \underbrace{(\bar{x_u}, \bar{x_v}) + (\bar{x_u}, \bar{x_v}) + \cdots + (\bar{x_u}, \bar{x_v})}_{s\ times}$.

Since $(\bar{x_u}, \bar{x_v}) \in W_{uv}$ and W_{uv} is a group, $(\bar{x_u}, \bar{x_v}) + (\bar{x_u}, \bar{x_v}) + \cdots + (\bar{x_u}, \bar{x_v}) \in W_{uv}$.

Thus $\bar{s}(\bar{x_u}, \bar{x_v}) \in W_{uv}$ for all $\bar{s} \in \mathbb{Z}_{p^k}$ and $(\bar{x_u}, \bar{x_v}) \in W_{uv}$.

Therefore W_{uv} is a submodule of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$.

Since $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ is finitely generated, every submodule of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ is contained in a maximal submodule. Thus $W_{uv} \subseteq \langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ where $1 \leq t \leq p - 1$

or $W_{uv} \subseteq \langle(\bar{0},\bar{1})\rangle \oplus \langle(\bar{p},\bar{0})\rangle$, by Lemma 3.2.10. Therefore $W \subseteq K_{uv}(t)$ where $1 \leq t \leq p-1$ or $W \subseteq M_u$. Because W is maximal, so $W = K_{uv}(t)$ or $W = M_u$ where $1 \leq t \leq p-1$. □

Proposition 3.2.13. *If* $V = \mathbb{Z}_{p^k}^{(\mathbb{N})}$, *then* $fln(V) \neq 0$.

Proof. Since $\mathbb{Z}_{p^k}$ is a local ring having $\langle\bar{p}\rangle$ as a unique maximal ideal with $Ann(\langle\bar{p}\rangle) \neq 0$, then $fln(V) \neq 0$ by Proposition 3.1.4. □

Lemma 3.2.14. *Let* $V = \mathbb{Z}_{p^k}^{(\mathbb{N})}$. *Then*

(i) $V = M_i \cup M_j \cup (\cup_{t=1}^{p-1} K_{ij}(t))$.

(ii) $M_i \cap M_j \cap (\cap_{t=1}^{p-1} K_{ij}(t)) = \langle e_1, e_2, ..., e_{i-1}, pe_i, e_{i+1}, ..., e_{j-1}, pe_j, e_{j+1}, ...\rangle$.

Proof. (i) Let $v \in V$.

Thus $v = v_1e_1 + v_2e_2 + ... + v_me_m$, we consider in two cases :

Case i: $\bar{v_i} \in \langle\bar{p}\rangle$ or $\bar{v_j} \in \langle\bar{p}\rangle$. Then we get $v \in M_i \cup M_j$.

Case ii: $\bar{v_i} \notin \langle\bar{p}\rangle$ and $\bar{v_j} \notin \langle\bar{p}\rangle$.

Then $\gcd(p, v_i) = 1$. This means that $\langle\bar{v_i}\rangle = \mathbb{Z}_{p^k}$.

Then there exists $\bar{n} \in \mathbb{Z}_{p^k}$ such that $\bar{v_i}\bar{n} = \bar{v_j}$.

Thus

$$\begin{aligned} v &= v_1e_1 + v_2e_2 + \cdots + v_{i-1}e_{i-1} + v_ie_i + v_{i+1}e_{i+1} + \cdots + v_{j-1}e_{j-1} \\ &\quad + (v_in)e_j + v_{j+1}e_{j+1} + \cdots + v_me_m \quad (\text{ since } \bar{v_j} = \bar{v_i}\bar{n}) \\ &= v_1e_1 + v_2e_2 + \cdots + v_{i-1}e_{i-1} + v_i(e_i + ne_j) + v_{i+1}e_{i+1} + \cdots + \\ &\quad v_{j-1}e_{j-1} + v_{j+1}e_{j+1} + \cdots + v_me_m. \end{aligned}$$

By division algorithm, there exist, $s, l \in \mathbb{Z}$ such that $1 \leq s \leq p-1$ and $n = lp+s$ since $p \nmid v_j$.

Thus we have

$$
\begin{aligned}
v &= v_1e_1 + v_2e_2 + \cdots + v_{i-1}e_{i-1} + v_i(e_i + (s+lp)e_j) + v_{i+1}e_{i+1} + \dots \\
&\quad + v_{j-1}e_{j-1} + v_{j+1}e_{j+1} + \cdots + v_me_m \\
&= v_1e_1 + v_2e_2 + \cdots + v_{i-1}e_{i-1} + v_i(e_i + se_j) + v_{i+1}e_{i+1} + \cdots + v_{j-1}e_{j-1} \\
&\quad + v_i(lpe_j) + v_{j+1}e_{j+1} + \cdots + v_me_m,
\end{aligned}
$$

so that $v \in \mathrm{K}_{ij}(s)$ where $1 \leq s \leq p-1$.

Thus from both cases we get $\mathrm{V} \subseteq \mathrm{M}_i \cup \mathrm{M}_j \cup (\cup_{t=1}^{p-1}\mathrm{K}_{ij}(t))$.

Therefore $\mathrm{V} = \mathrm{M}_i \cup \mathrm{M}_j \cup (\cup_{t=1}^{p-1}\mathrm{K}_{ij}(t))$.

(ii) Let $a \in \langle e_1, e_2, \dots, e_{i-1}, pe_i, e_{i+1}, \dots, e_{j-1}, pe_j, e_{j+1}, \dots \rangle$ and $1 \leq t \leq p-1$.

Then

$$
\begin{aligned}
a &= a_1e_1 + a_2e_2 + \cdots + a_{i-1}e_{i-1} + a_i(pe_i) + a_{i+1}e_{i+1} + \cdots + a_{j-1}e_{j-1} \\
&\quad + a_j(pe_j) + a_{j+1}e_{j+1} + \cdots + a_me_m \\
&= (\bar{a_1}, \bar{a_2}, \dots, \overline{a_{i-1}}, \overline{a_ip}, \overline{a_{i+1}}, \dots, \overline{a_{j-1}}, \overline{a_jp}, \overline{a_{j+1}}, \dots).
\end{aligned}
$$

Thus $a \in \mathrm{M}_i \cap \mathrm{M}_j$.

We have $p|(a_jp - a_ipt)$ for all $1 \leq t \leq p-1$ since $p|a_ipt$ and $p|a_jp$.

Then there exists $a_i^{'} \in \mathbb{Z}$ for all $1 \leq t \leq p-1$ such that $a_i^{'}p = a_jp - a_ipt$.

Hence $a_jp = (a_it + a_i^{'})p$.

Thus

$$
\begin{aligned}
a &= (\bar{a_1}, \bar{a_2}, \dots, \overline{a_{i-1}}, \overline{a_ip}, \overline{a_{i+1}}, \dots, \overline{a_{j-1}}, \overline{(a_it + a_i^{'})p}, \overline{a_{j+1}}, \dots) \\
&= a_1e_1 + a_2e_2 + \cdots + a_{i-1}e_{i-1} + a_ipe_i + a_{i+1}e_{i+1} + \cdots + a_{j-1}e_{j-1} \\
&\quad + ((a_it + a_i^{'})p)e_j + a_{j+1}e_{j+1} + \cdots + a_me_m \\
&= a_1e_1 + a_2e_2 + \cdots + a_{i-1}e_{i-1} + a_ip(e_i + te_j) + a_{i+1}e_{i+1} + \cdots + a_{j-1}e_{j-1} \\
&\quad + a_i^{'}(pe_j) + a_{j+1}e_{j+1} + \cdots + a_me_m.
\end{aligned}
$$

Then $a \in \mathrm{K}_{ij}(t)$ for all $1 \leq t \leq p-1$. So

$\langle e_1, e_2, \dots, e_{i-1}, pe_i, e_{i+1}, \dots, e_{j-1}, pe_j, e_{j+1}, \dots \rangle \subseteq \mathrm{M}_i \cap \mathrm{M}_j \cap (\cap_{t=1}^{p-1}\mathrm{K}_{ij}(t))$.

It is clear that $\mathrm{M}_i \cap \mathrm{M}_j \subseteq \langle e_1, e_2, \ldots, e_{i-1}, pe_i, e_{i+1}, \ldots, e_{j-1}, pe_j, e_{j+1}, \ldots \rangle$.

Next, we show that $\cap_{t=1}^{p-1}\mathrm{K}_{ij}(t) \subseteq \langle e_1, e_2, \ldots, e_{i-1}, pe_i, e_{i+1}, \ldots, e_{j-1}, pe_j, e_{j+1}, \ldots \rangle$.

Let $x = (\bar{x_i}) \in \cap_{t=1}^{p-1}\mathrm{K}_{ij}(t)$.

Then $x \in \mathrm{K}_{ij}(t)$ for all $1 \leq t \leq p-1$.

Consider at components i, j.

Let $\bar{x_i} = \bar{a}$, then $\bar{x_j} = \overline{at} + \overline{bp}$ where $\bar{a}$ and $\bar{b} \in \mathbb{Z}_{p^k}$.

Thus $\bar{x_j} = \bar{a} + \overline{b_1 p} = \overline{2a} + \overline{b_2 p} = \overline{3a} + \overline{b_3 p} = \cdots = \overline{(p-1)a} + \overline{b_{p-1}p}$ where $\bar{b_1}, \bar{b_2}, \bar{b_3}, \ldots, \overline{b_{p-1}} \in \mathbb{Z}_{p^k}$ since $1 \leq t \leq p-1$.

This means that $\bar{a} + \overline{b_1 p} = \overline{2a} + \overline{b_2 p}$, so $\overline{b_1 p} - \overline{b_2 p} = \bar{a}$.

Since $\overline{b_1 p} - \overline{b_2 p} \in \mathbb{Z}_{p^k}p = \langle \bar{p} \rangle$, then $\bar{a} \in \langle \bar{p} \rangle$ and that $\overline{at} \in \langle \bar{p} \rangle$ where $1 \leq t \leq p-1$ since $\langle \bar{p} \rangle$ is an ideal of $\mathbb{Z}_{p^k}$.

Thus $\bar{x_j} = \overline{at} + \overline{bp} \in \langle \bar{p} \rangle$.

Hence $\bar{x_i}, \bar{x_j} \in \langle \bar{p} \rangle$.

Consequently $x \in \langle e_1, e_2, \ldots, e_{i-1}, pe_i, e_{i+1}, \ldots, e_{j-1}, pe_j, e_{j+1}, \ldots \rangle$.

Thus $\cap_{t=1}^{p-1}\mathrm{K}_{ij}(t) \subseteq \langle e_1, e_2, \ldots, e_{i-1}, pe_i, e_{i+1}, \ldots, e_{j-1}, pe_j, e_{j+1}, \ldots \rangle$.

Therefore $\mathrm{M}_i \cap \mathrm{M}_j \cap (\cap_{t=1}^{p-1}\mathrm{K}_{ij}(t)) = \langle e_1, e_2, ..., e_{i-1}, pe_i, e_{i+1}, ..., e_{j-1}, pe_j, e_{j+1}, ... \rangle$.□

Let $\mathrm{M}_i \cap \mathrm{M}_j \cap (\cap_{t=1}^{p-1}\mathrm{K}_{ij}(t)) = \langle e_1, e_2, ..., e_{i-1}, pe_i, e_{i+1}, ..., e_{j-1}, pe_j, e_{j+1}, ... \rangle :=$ B and $g_t = e_i + te_j$ where $1 \leq t \leq p-1$. Then $\mathrm{M}_i = \langle B, e_j \rangle$, $e_j \notin \mathrm{M}_j \cup (\cup_{t=1}^{p-1}\mathrm{K}_{ij}(t))$ and $\mathrm{M}_j = \langle B, e_i \rangle$, $e_i \notin \mathrm{M}_i \cup (\cup_{t=1}^{p-1}\mathrm{K}_{ij}(t))$.

Lemma 3.2.15. (i) $\mathrm{K}_{ij}(t) = \langle B, g_t \rangle$ *for all* $1 \leq t \leq p-1$.

(ii) $g_t \notin \mathrm{M}_i \cup \mathrm{M}_j \cup (\cup_{s \neq t}\mathrm{K}_{ij}(s))$ *where* $1 \leq s \neq t \leq p-1$.

Proof. (i) Let $1 \leq t \leq p-1$ and $a \in \mathrm{K}_{ij}(t)$.

Then $a = (\bar{a_1}, \bar{a_2}, \ldots, \overline{a_{i-1}}, \bar{a_i}, \overline{a_{i+1}}, \ldots, \overline{a_{j-1}}, \overline{a_i t} + \overline{a_j p}, \overline{a_{j+1}}, \ldots)$

$= (\bar{0}, \bar{0}, \ldots, \bar{0}, \bar{a_i}, \bar{0}, \ldots, \bar{0}, \overline{a_i t}, \bar{0}, \ldots) + (\bar{a_1}, \bar{a_2}, \ldots, \overline{a_{i-1}}, \bar{0}, \overline{a_{i+1}}, \ldots, \overline{a_{j-1}}, \overline{a_j p}, \overline{a_{j+1}}, \ldots)$

$= a_i g_t + (\bar{a_1}, \bar{a_2}, \ldots, \overline{a_{i-1}}, \bar{0}, \overline{a_{i+1}}, \ldots, \overline{a_{j-1}}, \overline{a_j p}, \overline{a_{j+1}}, \ldots)$.

Then $a \in \langle B, g_t \rangle$ since $(\bar{a_1}, \bar{a_2}, \ldots, \overline{a_{i-1}}, \bar{0}, \overline{a_{i+1}}, \ldots, \overline{a_{j-1}}, \overline{a_j p}, \overline{a_{j+1}}, \ldots) \in \mathrm{B}$.

Conversely, by above lemma we have $B \subseteq \mathrm{K}_{ij}(t)$ and $g_t \in \mathrm{K}_{ij}(t)$, so $\langle B, g_t \rangle \subseteq \mathrm{K}_{ij}(t)$.

Thus $\mathrm{K}_{ij}(t) = \langle B, g_t \rangle$ for all $1 \leq t \leq p-1$.

(ii) Obviously, $g_t \notin \mathrm{M}_i \cup \mathrm{M}_j$ since $g_t = e_i + te_j$ and $1 \leq t \leq p-1$.
Suppose that there exists s such that $1 \leq s \neq t \leq p-1$ and $g_t \in \mathrm{K}_{ij}(s)$.
Then there exists $a \in \mathrm{K}_{ij}(s)$ such that

$$g_t = a = (\bar{a_1}, \bar{a_2}, ..., \overline{a_{i-1}}, \bar{a_i}, \overline{a_{i+1}}, ..., \overline{a_{j-1}}, \overline{a_i s} + \overline{a_j p}, \overline{a_{j+1}}, ...).$$

Consider components i and j of g_t, we get $\bar{a_i} = \bar{1}$ and $\bar{t} = \overline{a_i s} + \overline{a_j p} = \bar{s} + \overline{a_j p}$ where $\bar{a_j} \in \mathbb{Z}_{p^k}$.

If $\bar{a_j} = \bar{0}$, then $\bar{t} = \bar{s}$ which implies $s = t$, a contradiction.

If $\bar{a_j} \neq \bar{0}$, then $\bar{t} = \bar{s} + \overline{a_j p} \notin \{\bar{1}, \bar{2}, \dots, \overline{p-1}\}$, a contradiction.

Thus we must have $g_t \notin \mathrm{K}_{ij}(s)$ for all s distinct from t.
Hence $g_t \notin \mathrm{M}_i \cup \mathrm{M}_j \cup (\cup_{s \neq t} \mathrm{K}_{ij}(s))$ where $1 \leq s \neq t \leq p-1$. □

Proposition 3.2.16. *Let* $\Im = \{\mathrm{M}_i, \mathrm{M}_j, \mathrm{K}_{ij}(1), \mathrm{K}_{ij}(2), ..., \mathrm{K}_{ij}(p-2), \mathrm{K}_{ij}(p-1)\}$. *Then* $\Im$ *does not force linearity on* $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$.

Proof. Let $\mathrm{S}_1 = \mathrm{K}_{ij}(1), \mathrm{S}_2 = \mathrm{K}_{ij}(2), ..., \mathrm{S}_{p-1} = \mathrm{K}_{ij}(p-1), \mathrm{S}_p = \mathrm{M}_i, \mathrm{S}_{p+1} = \mathrm{M}_j$ and $\bar{0} \neq a \in Ann(\langle \bar{p} \rangle)$, then $ag_1 \neq 0$ where $g_1 = e_i + e_j$. We have by Lemma 3.2.15. that $\mathrm{K}_{ij}(1) = \langle B, g_1 \rangle$, so for convenience let $B = \langle b_1, b_2, b_3, \dots \rangle$.
We choose $\overline{p^{k-1}} \in Ann(\langle \bar{p} \rangle)$ and define $f : \mathrm{V} \to \mathrm{V}$ by

$$f(v) = \begin{cases} \beta p^{k-1} g_1 & , \quad v \in \mathrm{S}_1 = \mathrm{K}_{ij}(1) \text{ and } v = \sum_{y=1} \beta_y b_y + \beta g_1, \\ 0 & , \quad v \in \mathrm{S}_x, x \neq 1. \end{cases}$$

We first show that f is well-defined.
If $v \in \mathrm{S}_1$ is also represented as $v = \sum_{y=1} \gamma_y b_y + \gamma g_1$, then we have $(\beta - \gamma) g_1 \in B$. This means that $\overline{\beta - \gamma} \in \langle \bar{p} \rangle$, since if $\overline{\beta - \gamma} \notin \langle \bar{p} \rangle$, then $\gcd(\beta - \gamma, p) = 1$, we must have $\langle \overline{\beta - \gamma} \rangle = \mathbb{Z}_{p^k}$, so there exists $m \in \mathbb{Z}_{p^k}$ such that $m(\beta - \gamma) = \bar{1}$, then $g_1 = 1g_1 = m(\beta - \gamma) g_1 \in B$, a contradiction.
Since $p^{k-1} \in Ann(\langle \bar{p} \rangle)$, then $(\beta - \gamma) p^{k-1} g_1 = 0$, i.e., $\beta p^{k-1} g_1 = \gamma p^{k-1} g_1$.
Suppose that $v \in \mathrm{S}_1 \cap \mathrm{S}_x$, $x \neq 1$.
Thus v also has representation, $v = \sum_{y=1} \delta_y b_y + \delta g_1$ since $v \in \mathrm{S}_1$, then
$v - \sum_{y=1} \delta_y b_y = \delta g_1$.

Since $\sum_{y=1}\delta_y b_y \in \mathrm{S}_x$ by Lemma 3.2.14, then $\delta g_1 \in \mathrm{S}_x$, which implies $\bar{\delta} \in \langle \bar{p} \rangle$, since if $\bar{\delta} \notin \langle \bar{p} \rangle$, then $\gcd(\delta, p) = 1$, we must have $\langle \bar{\delta} \rangle = \mathbb{Z}_{p^k}$, so there exists $\bar{n} \in \mathbb{Z}_{p^k}$ such that $\bar{n}(\bar{\delta}) = \bar{1}$, then $g_1 = 1g_1 = n(\delta)g_1 \in S_x$, a contradiction by Lemma 3.2.15.

Consequently $f(v) = \delta p^{k-1} e_j = 0$.

Thus f is well-defined on V.

We note that f is not the zero function since $f(g_1) = p^{k-1}g_1 \neq 0$.

Next, we show that $f \in \mathrm{M}_{\mathbb{Z}_{p^k}}(\mathbb{Z}_{p^k}^{(\mathbb{N})})$.

For $v \in \mathrm{S}_1$, say $v = \sum_{y=1}\beta_y b_y + \beta g_1$ and for any $\bar{r} \in \mathbb{Z}_{p^k}$, we have $rv \in \mathrm{S}_1$, so $rf(v) = r\beta p^{k-1} g_1 = f(rv)$.

Now suppose $v \in \mathrm{S}_x, x \neq 1$.

Then $f(v) = 0$ and for any $\bar{r} \in \mathbb{Z}_{p^k}$, $rf(v) = 0$.

Moreover, $rv \in \mathrm{S}_x$, which implies $f(rv) = 0$.

Thus $rf(v) = f(rv)$ for all $\bar{r} \in \mathbb{Z}_{p^k}$ and $v \in \mathrm{V}$.

Hence $f \in \mathrm{M}_{\mathbb{Z}_{p^k}}(\mathbb{Z}_{p^k}^{(\mathbb{N})})$.

Next, we show that f is linear on each S_x in $\Im$.

For each $v_1 = \sum_{y=1}\beta_y b_y + \beta g_1, v_2 = \sum_{y=1}\delta_y b_y + \delta g_1 \in \mathrm{S}_1$, we have $f(v_1) = \beta p^{k-1} g_1$ and $f(v_2) = \delta p^{k-1} g_1$. Since S_1 is a module, $v_1 + v_2 \in \mathrm{S}_1$, so $f(v_1 + v_2) = f(\sum_{y=1}(\delta_y + \beta_y)b_y + (\delta + \beta)g_1) = (\delta + \beta)p^{k-1}g_1 = \delta p^{k-1}g_1 + \beta p^{k-1}g_1 = f(v_1) + f(v_2)$.

Then f is linear on S_1.

Since $f(\mathrm{S}_x) = 0$ for all $x \neq 1$, then f is linear on S_x, $x \neq 1$.

Hence f is linear on each S_x in $\Im$.

Because $g_1 + g_2 \notin \mathrm{S}_1$ since if $g_1 + g_2 \in \mathrm{S}_1$, we get $g_2 = (g_1 + g_2) - g_1 \in \mathrm{S}_1$, a contradiction by Lemma 3.2.15.

However, $f(g_1) + f(g_2) = p^{k-1}g_1 + 0 \neq 0 = f(g_1 + g_2)$.

This shows that $f \notin End_{\mathbb{Z}_{p^k}}(\mathrm{V})$.

Thus $\Im$ does not force linearity on V. □

Next, we show that $fln(\mathrm{V}) \leq p + 2$ where $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$.

Proposition 3.2.17. *Let* $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$, *then* $fln(\mathrm{V}) \leq p + 2$.

Proof. Since $V = \mathbb{Z}_{p^k}^{(\mathbb{N})}$ where $\mathbb{Z}_{p^k}$ is a local ring and has $\langle \bar{p} \rangle$ as a unique maximal ideal of $\mathbb{Z}_{p^k}$. Moreover $\mathbb{Z}_{p^k}/\langle p \rangle$ is a field of cardinality p. Then by Theorem 3.1.10. we get $fln(V) \leq p+2$. □

Lemma 3.2.18. *Let $\cup_{i=1}^{\ell} S_i = V$ where S_i is a maximal submodule of V and $\ell \leq p$, then*

(i) *$\langle \bar{p} \rangle V$ is a submodule of S_i.*

(ii) *$S_i/\langle \bar{p} \rangle V$ is a proper subspace of $V/\langle \bar{p} \rangle V$ over the field $\mathbb{Z}_{p^k}/\langle \bar{p} \rangle$.*

(iii) *$\cup_{i=1}^{\ell}[S_i/\langle \bar{p} \rangle V]$ is a subspace of $V/\langle \bar{p} \rangle V$ over the field $\mathbb{Z}_{p^k}/\langle \bar{p} \rangle$.*

Proof. (i) Clearly $\langle \bar{p} \rangle V$ is a submodule of V.
Next, we will show that $\langle \bar{p} \rangle V \subseteq S_i$ where S_i is maximal submodule of V.
Let S_i be a maximal submodule of V and $(\bar{a_k}) \in \langle \bar{p} \rangle V$, then we consider in two case:

Case i: $S_i = M_j$.
Let $(\bar{a_k}) \in \langle \bar{p} \rangle V = \langle pe_1, pe_2, pe_3, \ldots \rangle$.
Then

$$\begin{aligned}(\bar{a_k}) &= a_1pe_1 + a_2pe_2 + \cdots + a_{j-1}pe_{j-1} + a_jpe_j + a_{j+1}pe_{j+1} + \cdots + a_mpe_m \\ &= (a_1p)e_1 + (a_2p)e_2 + \cdots + (a_{j-1}p)e_{j-1} + a_j(pe_j) + (a_{j+1}p)e_{j+1} + \cdots + (a_mp)e_m\end{aligned}$$

Thus $(\bar{a_k}) \in \langle e_1, e_2, \ldots, e_{j-1}, pe_j, e_{j+1}, \ldots \rangle = M_j$.
Hence $\langle \bar{p} \rangle V \subseteq M_j$.

Case ii: $S_i = K_{uv}(t)$ where $1 \leq t \leq p-1$.
Let $(\bar{a_k}) \in \langle \bar{p} \rangle V = \langle pe_1, pe_2, pe_3, \ldots \rangle$.
Then

$$\begin{aligned}(\bar{a_k}) =& a_1pe_1 + a_2pe_2 + \cdots + a_{u-1}pe_{u-1} + a_upe_u + a_{u+1}pe_{u+1} + \ldots \\ &+ a_{v-1}pe_{v-1} + a_vpe_v + a_{v+1}pe_{v+1} + \cdots + a_mpe_m \\ =& a_1pe_1 + a_2pe_2 + \cdots + a_{u-1}pe_{u-1} + a_up(e_u + te_v) + a_{u+1}pe_{u+1} + \ldots \\ &+ a_{v-1}pe_{v-1} + (a_v - a_ut)pe_v + a_{v+1}pe_{v+1} + \cdots + a_mpe_m.\end{aligned}$$

Thus $(\bar{a_k}) \in \langle e_1, e_2, \ldots, e_{u-1}, (e_u + te_v), e_{u+1}, \ldots, e_{v-1}, pe_v, e_{v+1}, \ldots \rangle$.
Hence $(\bar{a_k}) \in K_{uv}(t)$ where $1 \leq t \leq p-1$.

Thus from both cases we get $\langle\bar{p}\rangle\mathrm{V} \subseteq \mathrm{S}_i$.

Therefore $\langle\bar{p}\rangle\mathrm{V}$ is submodule of S_i.

(ii) By (i) we have $\langle\bar{p}\rangle\mathrm{V}$ is a submodule of S_i, then $\mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V}$ is defined, so we get $\mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V}$ is a subspace of $\mathrm{V}/\langle\bar{p}\rangle\mathrm{V}$ over the field $\mathbb{Z}_{p^k}/\langle\bar{p}\rangle$ see Theorem 3.1.7. and proper since S_i is maximal submodule of V.

(iii) Since $\cup_{i=1}^{\ell}[\mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V}] = (\cup_{i=1}^{\ell}\mathrm{S}_i)/\langle\bar{p}\rangle\mathrm{V} = \mathrm{V}/\langle\bar{p}\rangle\mathrm{V}$. □

Proposition 3.2.19. *If* $\Im = \{\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_\ell\}$, $\ell \leq p$ *is a collection of maximal submodules of* $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$, *then* $\Im$ *does not force linearity on* V.

Proof. Suppose that $\Im$ forces linearity on V, then by Lemma 3.2.2. we get $\cup_{i=1}^{\ell}\mathrm{S}_i = \mathrm{V}$.

We have $\mathrm{V}/\langle\bar{p}\rangle\mathrm{V}$ is a vector space over the field $\mathbb{Z}_{p^k}/\langle\bar{p}\rangle$ under addition and scalar multiplication defined by

$$(v_1 + \langle\bar{p}\rangle\mathrm{V}) + (v_2 + \langle\bar{p}\rangle\mathrm{V}) = (v_1 + v_2) + \langle\bar{p}\rangle\mathrm{V}.$$

and
$$(r + \langle\bar{p}\rangle)(v + \langle\bar{p}\rangle\mathrm{V}) = rv + \langle\bar{p}\rangle\mathrm{V}.$$

Since $\Im = \{\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_\ell\}$ is a collection of maximal submodules of V, we must have $\mathrm{V}/\langle\bar{p}\rangle\mathrm{V} = \cup_{i=1}^{\ell}\mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V} = \cup_{i=1}^{\ell}(\mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V})$.

We have $\mathrm{S}_1/\langle\bar{p}\rangle\mathrm{V}$, $\mathrm{S}_2/\langle\bar{p}\rangle\mathrm{V}, \ldots, \mathrm{S}_\ell/\langle\bar{p}\rangle\mathrm{V}$ are finitely many subspaces of $\mathrm{V}/\langle\bar{p}\rangle\mathrm{V}$ over $\mathbb{Z}_{p^k}/\langle\bar{p}\rangle$ with $\ell \leq p = |\mathbb{Z}_{p^k}/\langle\bar{p}\rangle|$ and $\cup_{i=1}^{\ell}(\mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V})$ is a subspace of $\mathrm{V}/\langle\bar{p}\rangle\mathrm{V}$.

By Lemma 3.1.8. we get there exists j such that $1 \leq j \leq \ell$ and $\mathrm{S}_j/\langle\bar{p}\rangle\mathrm{V} \supseteq \mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V}$ for all $i \neq j$.

We prove that $\mathrm{S}_i \subseteq \mathrm{S}_j$ for all $i \neq j$.

Suppose not, then there exists $i \neq j$ and $\mathrm{S}_i \nsubseteq \mathrm{S}_j$. So there exists $x \in \mathrm{S}_i$ and $x \notin \mathrm{S}_j$.

This means that $x + \langle\bar{p}\rangle\mathrm{V} \in \mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V}$ and $x + \langle\bar{p}\rangle\mathrm{V} \notin \mathrm{S}_j/\langle\bar{p}\rangle\mathrm{V}$.

Hence $\mathrm{S}_i/\langle\bar{p}\rangle\mathrm{V} \nsubseteq \mathrm{S}_j/\langle\bar{p}\rangle\mathrm{V}$, a contradiction.

Thus $\mathrm{S}_i \subseteq \mathrm{S}_j$ for all $i \neq j$.

Therefore $\mathrm{V} = \cup_{i=1}^{\ell}\mathrm{S}_i \subseteq \mathrm{S}_j$, so $\mathrm{V} = \mathrm{S}_j$ which contradicts to the maximality of S_j.

Hence $\Im = \{\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_\ell\}$ does not force linearity on V. □

Lemma 3.2.20. *Every proper submodule of* $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$ *is contained in a maximal submodule.*

Proof. Let W be a proper submodule of V.

For each $u, v \in \mathbb{N}$, let

$\mathrm{W}_{uv} = \{(\bar{x_u}, \bar{x_v}) \in \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k} |$ there is $(\bar{a_i}) \in \mathrm{W}$ such that $x_u = a_u$ and $x_v = a_v\}$.

We prove that W_{uv} is a submodule of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$.

Let $(\bar{b_u}, \bar{b_v}), (\bar{c_u}, \bar{c_v}) \in \mathrm{W}_{uv}$, then there exists $(-\bar{c_u}, -\bar{c_v}) \in \mathrm{W}_{uv}$ such that

$\bar{c_u} + (-\bar{c_u}) = \bar{0}$ and $\bar{c_v} + (-\bar{c_v}) = \bar{0}$ since W is subgroup of V.

Then $(\bar{b_u}, \bar{b_v}) + (-\bar{c_u}, -\bar{c_v}) = (\bar{b_u} - \bar{c_u}, \bar{b_v} - \bar{c_v}) \in \mathrm{W}_{uv}$ because W is subgroup of V.

Hence W_{uv} is a subgroup of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$.

Let $\bar{s} \in \mathbb{Z}_{p^k}$ and $(\bar{x_u}, \bar{x_v}) \in \mathrm{W}_{uv}$.

If $\bar{s} = \bar{0}$, $\bar{s}(\bar{x_u}, \bar{x_v}) = \bar{0}(\bar{x_u}, \bar{x_v}) = (\bar{0}, \bar{0}) \in \mathrm{W}_{uv}$.

If $\bar{1} \leq \bar{s} < \overline{p^k}$, then $\bar{s}(\bar{x_u}, \bar{x_v}) = s(\bar{x_u}, \bar{x_v}) = \underbrace{(\bar{x_u}, \bar{x_v}) + (\bar{x_u}, \bar{x_v}) + \cdots + (\bar{x_u}, \bar{x_v})}_{s \text{ times}}$.

Since $(\bar{x_u}, \bar{x_v}) \in \mathrm{W}_{uv}$ and W_{uv} is a group, $(\bar{x_u}, \bar{x_v}) + (\bar{x_u}, \bar{x_v}) + \cdots + (\bar{x_u}, \bar{x_v}) \in \mathrm{W}_{uv}$.

Thus $\bar{s}(\bar{x_u}, \bar{x_v}) \in \mathrm{W}_{uv}$ for all $\bar{s} \in \mathbb{Z}_{p^k}$ and $(\bar{x_u}, \bar{x_v}) \in \mathrm{W}_{uv}$.

Therefore W_{uv} is a submodule of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$.

Since $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ is finitely generated, every submodule of $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ is contained in a maximal submodule. Thus $\mathrm{W}_{uv} \subseteq \langle(\bar{1}, \bar{t})\rangle \oplus \langle(\bar{0}, \bar{p})\rangle$ where $1 \leq t \leq p-1$ or $\mathrm{W}_{uv} \subseteq \langle(\bar{0}, \bar{1})\rangle \oplus \langle(\bar{p}, \bar{0})\rangle$, by Lemma 3.2.10. Therefore $\mathrm{W} \subseteq \mathrm{K}_{uv}(t)$ where $1 \leq t \leq p-1$ or $\mathrm{W} \subseteq \mathrm{M}_u$.

By Proposition 3.2.12, maximal submodules of V are of the form M_i or $\mathrm{K}_{uv}(t)$ where $1 \leq t \leq p-1$, $i, u, v \in \mathbb{N}$.

Thus the prove is complete. □

Corollary 3.2.21. *If* $\Im = \{\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_\ell\}$, $\ell \leq p$ *is a collection of proper submodule of* $\mathrm{V} = \mathbb{Z}_{p^k}^{(\mathbb{N})}$, *then* $\Im$ *does not force linearity on* V.

Proof. Let $\Im = \{\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_\ell\}$, $\ell \leq p$ be a collection of proper submodules of V, then by Lemma 3.2.20. we get $\Im' = \{\mathrm{S}'_1, \mathrm{S}'_2, \ldots, \mathrm{S}'_t\}$, $1 \leq t \leq l$ the collection of maximal submodules of V such that for each $1 \leq i \leq t$ there exists $1 \leq j \leq t$ with

$S_i \subseteq S'_j$.

Thus by Proposition 3.2.19, we get $\Im'$ does not force linearity on V, this means that there exists a homogeneous function f such that $f_{|_{S'_j}}$ are linear but f is not linear on V where $1 \leq j \leq t$.

Thus f is linear on each S_i in $\Im$ but not linear on V.

So $\Im$ does not force linearity on V. □

By Proposition 3.2.17, we have $fln(V) \leq p+2$ and Proposition 3.2.13 and Corollary 3.2.21 show that $fln(V) \notin \{0,1,2,\ldots,p\}$. Hence $fln(V) = p+1$ or $p+2$.

3.3 Forcing linearity number for $\mathbb{Z}_2^{(\mathbb{N})}$

Let $V = \mathbb{Z}_2^{(\mathbb{N})}$ and u, v be fixed positive integers such that $u < v$.

Then $M_i = \langle e_1, e_2, \ldots, e_{i-1}, 0, e_{i+1}, \ldots \rangle$ and

$$\begin{aligned} K_{uv}(1) &= \langle e_1, e_2, ..., e_{u-1}, e_u + e_v, e_{u+1}, ..., e_{v-1}, 0, e_{v+1}, ... \rangle \\ &= \{(x_1, x_2, x_3, ...) \in V | (x_u, x_v) \in \langle (1,1) \rangle \oplus \langle (0,0) \rangle \}. \\ &= \{(x_1, x_2, x_3, \ldots) \mid (x_u, x_v) \in \langle (1,1) \rangle \} \end{aligned}$$

since $p = 2$ and $k = 1$.

Lemma 3.3.1. *Let* M_i, M_j, M_k *where* $i < j < k$ *be maximal submodules of* V, *then* $M_i \cup M_j \cup M_k \subsetneq V$.

Proof. Let $U = M_i \cup M_j \cup M_k$, then we choose $(a_t) \in V$ be such that

$$a_t = \begin{cases} 1 & , \quad t \in \{i, j, k\}, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus it is obvious that $(a_t) \notin U$. □

We aim to prove that

$K_{i_1 j_1}(1) \cup K_{i_2 j_2}(1) \cup K_{i_3 j_3}(1) = V$ where $i_1 \leq i_2 \leq i_3$ iff $i_1 = i_2$, $j_2 = j_3$ and $j_1 = i_3$.

Let $S = \{i_1, i_2, i_3\}$, $T = \{j_1, j_2, j_3\}$ and $X = S \cup T$.

For each $x, y \in \mathrm{X}$ define

$$x \sim y \Leftrightarrow x = y.$$

Then $\sim$ is an equivalence relation on X . Let $\{ \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, ..., \mathrm{P}_n \}$ be the partition of X with respect to $\sim$. Let $\mathrm{C} = \bigcup_{k=1}^{s_1} \mathrm{P}_{i_k}$ where $\mathrm{P}_{i_k} \cap \mathrm{T} = \emptyset$ and $\mathrm{C}' = \bigcup_{k=1}^{s_2} \mathrm{P}_{j_k}$ where $\mathrm{P}_{j_k} \cap \mathrm{S} = \emptyset$.

Lemma 3.3.2. (i) $\mathrm{C} = \mathrm{S} \setminus \mathrm{T}$ *and* $\mathrm{C}' = \mathrm{T} \setminus \mathrm{S}$.

(ii) $i_1 \in \mathrm{C}$ *and* $j_3 \in \mathrm{C}'$.

Proof. (i) Let $x \in \mathrm{C}$, then there exist k such that $x \in \mathrm{P}_{i_k}$ where $\mathrm{P}_{i_k} \cap \mathrm{T} = \emptyset$. Thus $x \in \mathrm{P}_{i_k}$ and $x \notin$ T.Since $\mathrm{P}_{i_k} \subset \mathrm{S}$, then $x \in \mathrm{S}$ and $x \notin \mathrm{T}$, so $x \in \mathrm{S} \setminus \mathrm{T}$, consequently $\mathrm{C} \subseteq \mathrm{S} \setminus \mathrm{T}$.

Converse, let $x \in \mathrm{S} \setminus \mathrm{T}$. Then $x \in \mathrm{S}$ and $x \notin \mathrm{T}$.

Thus there exist k such that $x \in \mathrm{P}_{i_k}$ where $\mathrm{P}_{i_k} \cap \mathrm{T} = \emptyset$.

Hence $x \in \bigcup_{k=1}^{s} \mathrm{P}_{i_k} = \mathrm{C}$.Then $x \in \mathrm{C}$.

Consequently, we get $\mathrm{S} \setminus \mathrm{T} \subseteq \mathrm{C}$.

Thus complete the prove and similarly we can show that $\mathrm{C}' = \mathrm{T} \setminus \mathrm{S}$.

(ii) is obvious. □

Proposition 3.3.3. *Let* $\mathrm{K}_{i_1 j_1}(1), \mathrm{K}_{i_2 j_2}(1)$ *and* $\mathrm{K}_{i_3 j_3}(1)$ *be all distinct. Then* $\mathrm{K}_{i_1 j_1}(1) \cup \mathrm{K}_{i_2 j_2}(1) \cup \mathrm{K}_{i_3 j_3}(1) = \mathrm{V}$ *where* $i_1 \leq i_2 \leq i_3$ *if and only if* $i_1 = i_2, j_2 = j_3$ *and* $j_1 = i_3$.

Proof.($\Rightarrow$) Let $\mathrm{S} = \{i_1, i_2, i_3\}$, $\mathrm{T} = \{j_1, j_2, j_3\}$ and $\mathrm{X} = \mathrm{S} \cup \mathrm{T}$.

Then $i_1 < j_1, i_2 < j_2$ and $i_3 < j_3$.

For each $x, y \in \mathrm{X}$ define

$$x \sim y \iff x = y.$$

Then $\sim$ is an equivalence relation on X. Let $\{\mathrm{P}_1, \mathrm{P}_2, ..., \mathrm{P}_n\}$ be the partition of X with respect to $\sim$. Let $\mathrm{C} = \bigcup_{k=1}^{s_1} \mathrm{P}_{i_k}$ where $\mathrm{P}_{i_k} \cap \mathrm{T} = \emptyset$ and $\mathrm{C}' = \bigcup_{k=1}^{s_2} \mathrm{P}_{j_k}$ where

$P_{j_k} \cap S = \emptyset$.

For convenience let $U = K_{i_1j_1}(1) \cup K_{i_2j_2}(1) \cup K_{i_3j_3}(1)$.

We first show that $i_1 = i_2$.

Suppose not, so $i_2 \neq i_1$ and $i_3 \neq i_1$. Then we consider in three cases :

Case 1: If $S \cap T = \emptyset$, then we choose $(a_k) \in V$ be such that

$$a_k = \begin{cases} 1 & \text{if } k \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin K_{i_1j_1}(1)$ since $a_{i_1} = 1, a_{j_1} = 0$.

$(a_k) \notin K_{i_2j_2}(1)$ since $a_{i_2} = 1, a_{j_2} = 0$.

$(a_k) \notin K_{i_3j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin U$, a contradiction.

Case 2: If $|S \cap T| = 1$, then suppose $C_1 = S \cap T$ and consider as follow:

If $j_1 = i_2$, then we choose $(a_k) \in V$ be such that

$$a_k = \begin{cases} 1 & \text{if } k \in C_1 \cup \{i_3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin K_{i_1j_1}(1)$ since $a_{i_1} = 0, a_{j_1} = 1$.

$(a_k) \notin K_{i_2j_2}(1)$ since $a_{i_2} = 1, a_{j_2} = 0$.

$(a_k) \notin K_{i_3j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin U$, a contradiction.

If $j_1 = i_3$, then we choose $(a_k) \in V$ as follow

$$a_k = \begin{cases} 1 & \text{if } k \in C_1 \cup \{i_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin \mathrm{K}_{i_1j_1}(1)$ since $a_{i_1} = 0, a_{j_1} = 1$.

$(a_k) \notin \mathrm{K}_{i_2j_2}(1)$ since $a_{i_2} = 1, a_{j_2} = 0$.

$(a_k) \notin \mathrm{K}_{i_3j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin \mathrm{U}$, a contradiction.

If $j_2 = i_3$, then we choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & \text{if } \quad k \in \mathrm{C}_1 \cup \{i_1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get ,

$(a_k) \notin \mathrm{K}_{i_1j_1}(1)$ since $a_{i_1} = 1, a_{j_1} = 0$.

$(a_k) \notin \mathrm{K}_{i_2j_2}(1)$ since $a_{i_2} = 0, a_{j_2} = 1$.

$(a_k) \notin \mathrm{K}_{i_3j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin \mathrm{U}$, a contradiction.

Case 3: If $|\mathrm{S} \cap \mathrm{T}| = 2$, then $j_1 = i_2$ and $j_2 = i_3$. Suppose $\mathrm{C}_1 = \{j_1\}$, then we choose $(a_k) \in \mathrm{V}$ as follow

$$a_k = \begin{cases} 1 & \text{if } \quad k \in \mathrm{C}_1 \cup \mathrm{C}', \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin \mathrm{K}_{i_1j_1}(1)$ since $a_{i_1} = 0, a_{j_1} = 1$.

$(a_k) \notin \mathrm{K}_{i_2j_2}(1)$ since $a_{i_2} = 1, a_{j_2} = 0$.

$(a_k) \notin \mathrm{K}_{i_3j_3}(1)$ since $a_{i_3} = 0, a_{j_3} = 1$.

Then $(a_k) \notin \mathrm{U}$, a contradiction.

Hence by the three cases, we get $i_1 = i_2$.

So $j_1 \neq j_2$.

Suppose $j_1 < j_2$

Now, we will show that $j_2 = j_3$.

Suppose that $j_2 \neq j_3$, then we have $j_1 \neq j_2$ and $j_2 \neq j_3$.

Case 1: If $\mathrm{S} \cap \mathrm{T} = \emptyset$, then we choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & \text{if } k \in \mathrm{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin \mathrm{K}_{i_1 j_1}(1)$ since $a_{i_1} = 1, a_{j_1} = 0$.

$(a_k) \notin \mathrm{K}_{i_2 j_2}(1)$ since $a_{i_2} = 1, a_{j_2} = 0$.

$(a_k) \notin \mathrm{K}_{i_3 j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin \mathrm{U}$, a contradiction.

Case 2: If $|\mathrm{S} \cap \mathrm{T}| = 1$, then suppose $\mathrm{C}_1 = \mathrm{S} \cap \mathrm{T}$ and consider as follow:

If $j_1 = i_3$, then we choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & \text{if } k \in \mathrm{C}_1 \cup \{j_2\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin \mathrm{K}_{i_1 j_1}(1)$ since $a_{i_1} = 0, a_{j_1} = 1$.

$(a_k) \notin \mathrm{K}_{i_2 j_2}(1)$ since $a_{i_2} = 0, a_{j_2} = 1$.

$(a_k) \notin \mathrm{K}_{i_3 j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin \mathrm{U}$, a contradiction.

If $j_2 = i_3$, then we choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & \text{if } k \in \mathrm{C}_1 \cup \{j_1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin \mathrm{K}_{i_1 j_1}(1)$ since $a_{i_1} = 0, a_{j_1} = 1$.

$(a_k) \notin K_{i_2j_2}(1)$ since $a_{i_2} = 0, a_{j_2} = 1$.

$(a_k) \notin K_{i_3j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin U$, a contradiction.

Hence by both cases we must have $j_2 = j_3$.

Finally, we will show that $j_1 = i_3$. Suppose that $j_1 \neq i_3$, then we choose $(a_k) \in V$ as follow

$$a_k = \begin{cases} 1 & \text{if} \quad k \in \{i_1, i_3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get,

$(a_k) \notin K_{i_1j_1}(1)$ since $a_{i_1} = 1, a_{j_1} = 0$.

$(a_k) \notin K_{i_2j_2}(1)$ since $a_{i_2} = 1, a_{j_2} = 0$.

$(a_k) \notin K_{i_3j_3}(1)$ since $a_{i_3} = 1, a_{j_3} = 0$.

Then $(a_k) \notin U$, a contradiction.

Hence $j_1 = i_3$.

($\Leftarrow$) Let $i_1 = i_2 = i$, $j_2 = j_3 = \ell$ and $j_1 = i_3 = j$.

We show that $K_{ij}(1) \cup K_{i\ell}(1) \cup K_{j\ell}(1) = V$.

Let $(a_k) \in V$. We consider the components i, j and ℓ of (a_k).

Since $\mathbb{Z}_2 = \{0, 1\}$ there are at least two components from i, j and ℓ which have the same values.

Then $(a_k) \in K_{ij}(1)$ or $K_{i\ell}(1)$ or $K_{j\ell}(1)$.

Thus $V \subseteq K_{ij}(1) \cup K_{i\ell}(1) \cup K_{j\ell}(1)$ and the prove is complete. □

Next, we prove that, if M_ℓ , $K_{ij}(1)$ and S are maximal submodules of V with $M_\ell \cup K_{ij}(1) \cup S = V$, then $S = M_i$ or M_j and $\ell = i$ or j.

Lemma 3.3.4. $K_{i_1j_1}(1) \cup K_{i_2j_2}(1) \subsetneq V$ *where* $i_1 \leq i_2$.

Proof. We have $i_1 < j_1, i_2 < j_2$ and $i_1 \leq i_2$.

If $i_1 = i_2$, then $j_1 \neq j_2$. So choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & , \quad k \in \{j_1, j_2\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $(a_k) \notin \mathrm{K}_{i_1 j_1}(1) \cup \mathrm{K}_{i_2 j_2}(1)$.

If $i_1 < i_2$, then we consider in two cases:

$j_1 = j_2$: Let $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & , \quad k \in \{i_1, i_2\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $(a_k) \notin \mathrm{K}_{i_1 j_1}(1) \cup \mathrm{K}_{i_2 j_2}(1)$.

$j_1 \neq j_2$: Let $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & , \quad k \in \{i_2, j_1\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus $(a_k) \notin \mathrm{K}_{i_1 j_1}(1) \cup \mathrm{K}_{i_2 j_2}(1)$. □

Proposition 3.3.5. $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{S} = \mathrm{V}$ *where* S *is a maximal submodule of* V *if and only if* $\ell = i$ *and* $\mathrm{S} = \mathrm{M}_j$*; or* $\ell = j$ *and* $\mathrm{S} = \mathrm{M}_i$.

Proof. $(\Rightarrow)$ Let $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{S} = \mathrm{V}$ such that S is a maximal submodule of V.

We show that $\mathrm{S} = \mathrm{M}_t$ or $\mathrm{S} = \mathrm{K}_{uv}(1)$.

If $\mathrm{S} = \mathrm{K}_{uv}(1)$, we consider in two cases:

Case 1: If $\ell \in \{i, j, u, v\}$, then $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{K}_{uv}(1) \subseteq \mathrm{K}_{ij}(1) \cup \mathrm{K}_{uv}(1) \subsetneqq \mathrm{V}$.

Case 2: If $\ell \notin \{i, j, u, v\}$, then there is $(a_k) \in \mathrm{V} - \mathrm{K}_{ij}(1) \cup \mathrm{K}_{uv}(1)$. Let $(b_k) \in \mathrm{V}$ be such that

$$b_k = \begin{cases} 1 & , \quad k = \ell \\ a_k & , \quad k \neq \ell. \end{cases}$$

So $(b_k) \notin \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{K}_{uv}(1)$ and thus $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{K}_{uv}(1) \subsetneqq \mathrm{V}$.

Thus $\mathrm{S} = \mathrm{K}_{uv}(1)$ can not be happen and hence $\mathrm{S} = \mathrm{M}_t$.

Next, we prove that $\{\ell, t\} = \{i, j\}$.

If $\ell \notin \{i, j\}$, then we consider in two cases.

Case 1: If $t \notin \{i, j\}$, then choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & , \quad k \in \{\ell, t, i\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $(a_k) \notin \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{M}_t$.

Case 2: If $t \in \{i, j\}$, in this case we consider in two subcases.

(1) If $t = i$, then choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & , \quad k \in \{\ell, i\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $(a_k) \notin \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{M}_t$.

(2) If $t = j$, then choose $(a_k) \in \mathrm{V}$ be such that

$$a_k = \begin{cases} 1 & , \quad k \in \{\ell, j\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then $(a_k) \notin \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{M}_t$.

Thus, we see that if $\ell \notin \{i, j\}$, then $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{M}_t \subsetneq \mathrm{V}$, which contradics to our assumption.

Similarly, if $t \notin \{i, j\}$, also leads to a contradiction.

Thus $\{\ell, t\} \subseteq \{i, j\}$.

Now, we prove that $\ell \neq t$.

Suppose that $\ell = t$, then $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{S} = \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{M}_\ell = \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \subsetneq \mathrm{V}$.

Since, if $\ell = i$ we choose $(a_k) \in \mathrm{V}$ be such that $a_\ell = 1 = a_i$, $a_j = 0$, then $(a_k) \notin \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1)$, which is a contradiction.

If $\ell = j$ we choose $(a_k) \in \mathrm{V}$ be such that $a_i = 0$, $a_\ell = 1 = a_j$, then $(a_k) \notin \mathrm{M}_\ell \cup \mathrm{K}_{ij}(1)$, which is a contradiction.

Thus $\ell \neq t$ and $\{\ell, t\} = \{i, j\}$.

Therefore, $\ell = i$ and $\mathrm{S} = \mathrm{M}_j$; or $\ell = j$ and $\mathrm{S} = \mathrm{M}_i$.

($\Leftarrow$) If $\ell = i$ and $\mathrm{S} = \mathrm{M}_j$, then $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{S} = \mathrm{M}_i \cup \mathrm{K}_{ij}(1) \cup \mathrm{M}_j = \mathrm{V}$ by Lemma 3.2.14.

If $\ell = j$ and $\mathrm{S} = \mathrm{M}_i$, then $\mathrm{M}_\ell \cup \mathrm{K}_{ij}(1) \cup \mathrm{S} = \mathrm{M}_j \cup \mathrm{K}_{ij}(1) \cup \mathrm{M}_i = \mathrm{V}$ by Lemma 3.2.14. □

Lemma 3.3.6.

(i) $\mathrm{K}_{ij}(1) \cap \mathrm{K}_{i\ell}(1) \cap \mathrm{K}_{j\ell}(1) = \langle e_1, e_2, ..., e_{i-1}, (e_i + e_j + e_\ell), e_{i+1}, ..., e_{j-1}, 0, e_{j+1}, ..., e_{\ell-1}, 0, e_{\ell+1}, ...\rangle$.

(ii) $\mathrm{M}_i \cap \mathrm{M}_j \cap \mathrm{K}_{ij}(1) = \langle e_1, e_2, \dots, e_{i-1}, 0, e_{i+1}, \dots, e_{j-1}, 0, e_{j+1}, \dots\rangle$.

Proof. (i) Let $(a_k) \in \mathrm{K}_{ij}(1) \cap \mathrm{K}_{i\ell}(1) \cap \mathrm{K}_{j\ell}(1)$, consider as follow.

If $a_i = 0$, then $a_j = 0$ and $a_\ell = 0$.

$$
\begin{aligned}
\text{Thus} \quad (a_k) &= (a_1, a_2, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_{j-1}, 0, a_{j+1}, \dots, a_{\ell-1}, 0, a_{\ell+1}, \dots). \\
&= a_1e_1 + a_2e_2 + \cdots + a_{i-1}e_{i-1} + 0(e_i) + a_{i+1}e_{i+1} + \cdots + a_{j-1}e_{j-1} \\
&\quad + 0(e_j) + a_{j+1}e_{j+1} + \cdots + a_{\ell-1}e_{\ell-1} + 0(e_\ell) + a_{\ell+1}e_{\ell+1} + \dots. \\
&= a_1e_1 + a_2e_2 + \cdots + a_{i-1}e_{i-1} + 0(e_i + e_j + e_\ell) + a_{i+1}e_{i+1} + \dots \\
&\quad + a_{j-1}e_{j-1} + 0 + a_{j+1}e_{j+1} + \cdots + a_{\ell-1}e_{\ell-1} + 0 + a_{\ell+1}e_{\ell+1} + \dots.
\end{aligned}
$$

Then $(a_k) \in \langle e_1, e_2, \dots, e_{i-1}, (e_i+e_j+e_\ell), e_{i+1}, \dots, e_{j-1}, 0, e_{j+1}, \dots, e_{\ell-1}, 0, e_{\ell+1}, \dots\rangle$.

If $a_i = 1$, then $a_j = 1$ and $a_\ell = 1$.

By the same prove as above then we get

$(a_k) \in \langle e_1, e_2, ..., e_{i-1}, (e_i + e_j + e_\ell), e_{i+1}, ..., e_{j-1}, 0, e_{j+1}, ..., e_{\ell-1}, 0, e_{\ell+1}, ...\rangle$.

Conversely,

let $(a_k) \in \langle e_1, e_2, ..., e_{i-1}, (e_i + e_j + e_\ell), e_{i+1}, ..., e_{j-1}, 0, e_{j+1}, ..., e_{\ell-1}, 0, e_{\ell+1}, ...\rangle$.

$$
\begin{aligned}
\text{Then} \quad (a_k) &= a_1e_1 + a_2e_2 + \cdots + a_{i-1}e_{i-1} + a_i(e_i + e_j + e_\ell) + a_{i+1}e_{i+1} + \dots \\
&\quad + a_{j-1}e_{j-1} + 0 + a_{j+1}e_{j+1} + \cdots + a_{\ell-1}e_{\ell-1} + 0 + a_{\ell+1}e_{\ell+1} + \dots. \\
&= (a_1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{\ell-1}, a_i, a_{\ell+1}, \dots).
\end{aligned}
$$

Thus $(a_k) \in \mathrm{K}_{ij}(1) \cap \mathrm{K}_{i\ell}(1) \cap \mathrm{K}_{j\ell}(1)$ since the components i, j , ℓ are the same.

Therefore $\mathrm{K}_{ij}(1)\cap\mathrm{K}_{i\ell}(1)\cap\mathrm{K}_{j\ell}(1)=\langle e_1,e_2,...,e_{i-1},(e_i+e_j+e_\ell),e_{i+1},...,e_{j-1},0,e_{j+1},$
$...,e_{\ell-1},0,e_{\ell+1},...\rangle$

(ii) By Lemma 3.2.14. let $p=2$.

Therefore $\mathrm{M}_i\cap\mathrm{M}_j\cap\mathrm{K}_{ij}(1)=\langle e_1,e_2,\ldots,e_{i-1},0,e_{i+1},\ldots,e_{j-1},0,e_{j+1},\ldots\rangle$. □

For convenience, let

$B=\langle e_1,e_2,\ldots,e_{i-1},0,e_{i+1},\ldots,e_{j-1},0,e_{j+1},\ldots\rangle$,

$C=\langle e_1,e_2,\ldots,e_{i-1},(e_i+e_j+e_\ell),e_{i+1},\ldots,e_{j-1},0,e_{j+1},\ldots,e_{\ell-1},0,e_{\ell+1},\ldots\rangle$,

and $g_1=e_i+e_j$.

Lemma 3.3.7. (i) $\mathrm{K}_{ij}(1)=\langle C,e_\ell\rangle$ *if* $i<j<\ell$.

(ii) $\mathrm{K}_{ij}(1)=\langle B,g_1\rangle$.

Proof. (i) It is clear that $e_\ell\in\mathrm{K}_{ij}(1)$ and $C\subseteq\mathrm{K}_{ij}(1)$.

Then $\langle C,e_\ell\rangle\subseteq\mathrm{K}_{ij}(1)$.

Conversely , let $(a_k)\in\mathrm{K}_{ij}(1)$.

Thus
$$\begin{aligned}(a_k)&=a_1e_1+a_2e_2+\cdots+a_{i-1}e_{i-1}+a_i(e_i+e_j)+a_{i+1}e_{i+1}+\cdots+a_{j-1}e_{j-1}\\&\quad+0e_j+a_{j+1}e_{j+1}+\cdots+a_{\ell-1}e_{\ell-1}+a_\ell e_\ell+a_{\ell+1}e_{\ell+1}+\cdots+a_me_m\\&=(a_1,a_2,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_{j-1},a_i,a_{j+1},\ldots,a_{\ell-1},a_\ell,a_{\ell+1},\ldots)\\&=(a_1,a_2,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_{j-1},a_i,a_{j+1},\ldots,a_{\ell-1},a_i-a_i+a_\ell,a_{\ell+1},\ldots)\\&=(a_1,a_2,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_{j-1},a_i,a_{j+1},\ldots,a_{\ell-1},a_i,a_{\ell+1},\ldots)+(a_\ell-a_i)e_\ell\\&=(a_1e_1+a_2e_2+\cdots+a_{i-1}e_{i-1}+a_i(e_i+e_j+e_\ell)+a_{i+1}e_{i+1}+\cdots+a_{j-1}e_{j-1}\\&\quad+0e_j+a_{j+1}e_{j+1}+\cdots+a_{\ell-1}e_{\ell-1}+0e_\ell+a_{\ell+1}e_{\ell+1}+\cdots+a_me_m)+(a_\ell-a_i)e_\ell.\end{aligned}$$

Hence $(a_k)\in\langle C,e_\ell\rangle$ and we must have $\mathrm{K}_{ij}(1)\subseteq\langle C,e_\ell\rangle$.

Therefore $\mathrm{K}_{ij}(1)=\langle C,e_\ell\rangle$ where $i<j<\ell$.

(ii) is obvious by Lemma 3.2.15. when $t=1$. □

Similarly as above lemma we can show that

(i) $\mathrm{K}_{i\ell}(1)=\langle C,e_j\rangle$ if $i<j<\ell$.

(ii) $\mathrm{K}_{j\ell}(1)=\langle C,e_i\rangle$ if $i<j<\ell$.

(iii) $\mathrm{M}_i = \langle B, e_j\rangle$ if $i \neq j$.

(iv) $\mathrm{M}_j = \langle B, e_i\rangle$ if $i \neq j$.

Proposition 3.3.8. *Let* $\Im = \{\mathrm{K}_{ij}(1), \mathrm{K}_{i\ell}(1), \mathrm{K}_{j\ell}(1)\}$. *Then* $\Im$ *does not force linearity on* V.

Proof. Let $\mathrm{S}_1 = \mathrm{K}_{ij}(1)$, $\mathrm{S}_2 = \mathrm{K}_{i\ell}(1)$ and $\mathrm{S}_3 = \mathrm{K}_{j\ell}(1)$.

Then $\mathrm{V} = \mathrm{S}_1 \cup \mathrm{S}_2 \cup \mathrm{S}_3$ and $\mathrm{S}_1 = \mathrm{K}_{ij}(1) = \langle C, e_\ell\rangle$ where $i < j < \ell$.

For convenience, we put $C = \langle c_1, c_2, c_3, \dots\rangle$.

Since $e_\ell \neq 0$, we define $f : \mathrm{V} \rightarrow \mathrm{V}$ by

$$f(v) = \begin{cases} \beta e_\ell & , \quad v \in \mathrm{S}_1 \text{ and } v = \sum_{s=1} \beta_s c_s + \beta e_\ell, \\ 0 & , \quad v \in \mathrm{S}_2 \cup \mathrm{S}_3. \end{cases}$$

We first show that f is well-defined.

If $v \in \mathrm{S}_1$ is also represented as $v = \sum_{s=1} \gamma_s c_s + \gamma e_\ell$, then we have $(\beta - \gamma)e_\ell \in C$.

This mean that $(\beta - \gamma) = 0$, since if $(\beta - \gamma) \neq 0$, then $(\beta - \gamma) = 1$, then we get $e_\ell = (\beta - \gamma)e_\ell \in C$ which is a contradiction.

Consequently $(\beta - \gamma)e_\ell = 0e_\ell = 0$. Hence $\beta e_\ell = \gamma e_\ell$.

Suppose that $v \in \mathrm{S}_1 \cap \mathrm{S}_t$, $t \neq 1$.

Then v also has a representation, $v = \sum_{s=1} \delta_s c_s + \delta e_\ell$.

Since $\sum_{s=1} \delta_s c_s \in \mathrm{S}_t$, we obtain $\delta e_\ell \in \mathrm{S}_t$ which implies $\delta = 0$.

Consequently $f(v) = \delta e_\ell = 0 e_\ell = 0$.

Thus f is well-defined on V.

We note that f is not the zero function since $f(e_\ell) = e_\ell \neq 0$.

Next, we show that $f \in \mathrm{M}_{\mathbb{Z}_2}(\mathbb{Z}_2^{(\mathbb{N})})$.

For $v \in \mathrm{S}_1$, say $v = \sum_{s=1} \beta_s c_s + \beta e_\ell$ and for $r \in \mathbb{Z}_2$, we have $rv \in \mathrm{S}_1$, so $rf(v) = r\beta e_\ell = f(rv)$.

Now, suppose $v \in \mathrm{S}_t, t \neq 1$.

Then $f(v) = 0$ and for any $r \in \mathbb{Z}_2$, $rf(v) = 0$.

Moreover, $rv \in \mathrm{S}_t$, which implies $f(rv) = 0$.

Thus , $rf(v) = f(rv)$ for all $r \in \mathbb{Z}_2$ and $v \in \mathrm{V}$.

Hence $f \in \mathrm{M}_{\mathbb{Z}_2}(\mathbb{Z}_2^{(\mathbb{N})})$.

It is clear that f is linear on each S_t in $\Im$.

Since $e_i + e_\ell \notin S_1$, $f(e_i + e_\ell) = 0$.

But $f(e_i) + f(e_\ell) \ = \ 0 + e_\ell \ \neq \ 0$.

Then $f \notin \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2^{(\mathbb{N})})$.

This shows that $\Im$ does not force linearity on V. □

Proposition 3.3.9. *Let* $\Im = \{M_i, M_j, K_{ij}(1)\}$. *Then* $\Im$ *does not force linearity on* V.

Proof. Since $p = 2$ and $k = 1$, then by Proposition 3.2.16 we get $\Im$ does not force linearity on $V = \mathbb{Z}_2^{(\mathbb{N})}$. □

Next, we show that $fln(V) = 4$ where $V = \mathbb{Z}_2^{(\mathbb{N})}$.

Proposition 3.3.10. *The forcing linearity number of* $V = \mathbb{Z}_2^{(\mathbb{N})}$ *is* 4.

Proof. By Proposition 3.2.13, Proposition 3.2.17. and Corollary 3.2.21, we get $fin(V) \leq 4$ and $fln(V) \notin \{0, 1, 2\}$.

Now, we show that $fln(V) \neq 3$.

Let $\Im = \{S_1, S_2, S_3\}$ be a collection of proper submodules of V, then by Lemma 3.2.20. there exists $\Im' = \{S_1', S_2', S_3'\}$ a collection of maximal submodules of V such that $S_i \subseteq S_i'$ for all $i = 1, 2, 3$.

If $\cup_{i=1}^3 S_i' \subsetneq V$, then $\Im'$ does not force linearity on V by Proposition 3.1.6.

If $\cup_{i=1}^3 S_i' = V$, then by Proposition 3.3.3 and Proposition 3.3.5 we get $\Im' = \{K_{ij}(1), K_{i\ell}(1), K_{j\ell}(1)\}$ or $\Im' = \{M_i, M_j, K_{ij}(1)\}$, we consider in two cases:

Case i: If $\Im' = \{K_{ij}(1), K_{i\ell}(1), K_{j\ell}(1)\}$, then by Proposition 3.3.8 we get $\Im'$ does not force linearity on V. This means that there exists a homogeneous function f such that f is linear on each S_i' in $\Im'$ but f is not linear on V.

Thus f is linear on each S_i in $\Im$ but not linear on V where $1 \leq i \leq 3$.

Then we get $\Im$ does not force linearity on V.

Case ii : If $\Im' = \{M_i, M_j, K_{ij}(1)\}$, then by Proposition 3.3.9 we get $\Im'$ does not force linearity on V. By considering as the above case, then we get $\Im$ does not force linearity on V.

Thus from both cases we must have $fln(\mathrm{V}) \neq 3$. Therefore $fln(\mathrm{V}) = 4$.$\square$